

Chapter (8)

Estimation and Confidence Intervals Examples

Types of estimation:

i. Point estimation:

Example (1)

Consider the sample observations

17,3,25,1,18,26,16,10

$$\hat{\mu} = \bar{X} = \frac{\sum_{i=1}^8 X_i}{8} = \frac{17+3+25+1+18+26+16+10}{8} = \frac{116}{8} = 14.5$$

14.5 is a point estimate for μ using the estimator \bar{X} and the given sample observations.

ii. Interval estimation:

Constructing confidence interval : The general form of an interval estimate of a population parameter:

Point Estimate \pm Criticalvalue *Standard error

This formula generates two values called the confidence limits;

- Lower confidence limit (LCL).
- Upper confidence limit (UCL).

Another way to find the confidence interval we used the **confidence**

Confidence Interval for a Population Mean

Case1: Confidence Interval for Population Mean with known Standard Deviation (normal case):

The confidence limits are:

$$\bar{X} \pm Z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

Steps for calculating:

1. Obtain $Z_{\alpha/2}$, from the table of the area under the normal curve.

2. Calculate $Z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$.

3. $L = \bar{X} - Z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$ $U = \bar{X} + Z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$

\bar{X} : The mean estimator σ : The standard deviation of the population .

$\frac{\sigma}{\sqrt{n}}$: The standard error of the mean ($\sigma_{\bar{x}}$).

$\pm Z_{\frac{\alpha}{2}}$: **Critical value.**

Example (2)

A sample of 49 observations is taken from a normal population with a standard deviation of 10. The sample mean is 55, determine the 99 percent confidence interval for the population mean.

Solution:

$$X \sim N(\mu, \sigma^2) \quad \bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right) \quad \sigma = 10, n = 49, \bar{X} = 55, \text{Confidence level} = 0.99,$$

$$\therefore \alpha = 1 - 0.99 = 0.01$$

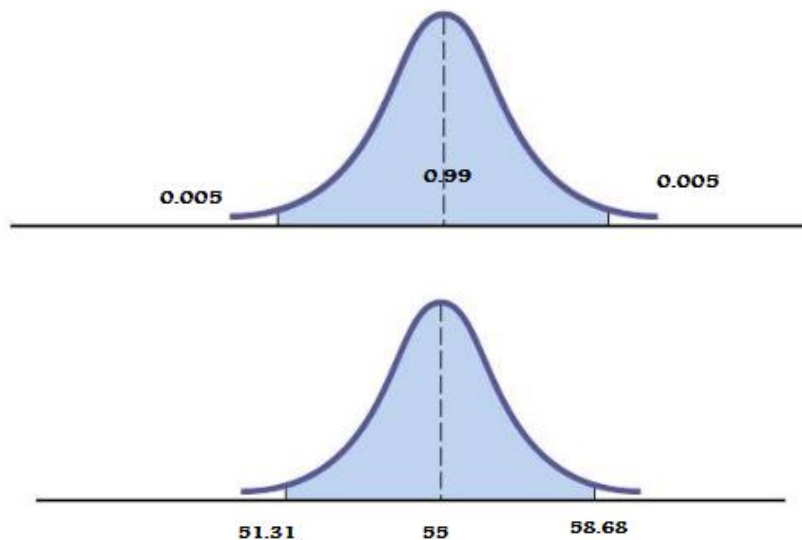
$$\therefore Z_{\frac{0.01}{2}} = Z_{0.005} = -2.58$$

The confidence limits are:

$$\bar{X} \pm Z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} = 55 \pm 2.58 \left(\frac{10}{\sqrt{49}} \right) = 55 \pm 3.6857$$

$$51.3143 \leq \hat{\mu} \leq 58.6857$$

$$(51.3143, 58.6857)$$



Example (3)

- IF you have (51.3143, 58.6857). Based on this information, you know that the best point estimate of the population mean ($\hat{\mu}$) is:

$$\hat{\mu} = \frac{\text{upper} + \text{lower}}{2} = \frac{58.6857 + 51.3143}{2} = \frac{110}{2} = 55$$

Case2: Confidence Interval for a Population Mean with unknown Standard Deviation

$$\hat{\mu} = \bar{X} \pm t_{n-1; \frac{\alpha}{2}} \frac{S}{\sqrt{n}}$$

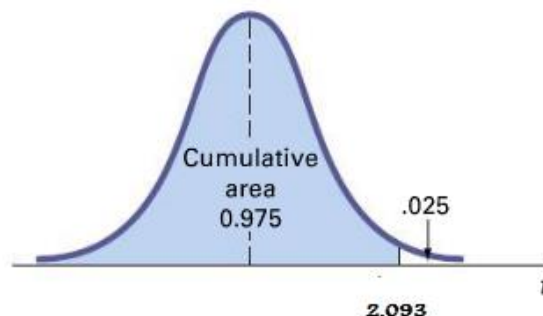
Example (4)

The owner of Britten's Egg Farm wants to estimate the mean number of eggs laid per chicken. A sample of 20 chickens shows they laid an average of 20 eggs per month with a standard deviation of 8 eggs per month (a sample is taken from a normal population).

- i. What is the value of the population mean? What is the best estimate of this value?
- ii. Explain why we need to use the t distribution. What assumption do you need to make?
- iii. For a 95 percent confidence interval, what is the value of t?
- iv. Develop the 95 percent confidence interval for the population mean.
- v. Would it be reasonable to conclude that the population mean is 21 eggs? What about 5 eggs?

Solution:

- i. the population mean is unknown, but the best estimate is 20, the sample mean
- ii. Use the t distribution as the standard deviation is unknown. However, assume the population is normally distributed.
- iii. $t_{n-1; \frac{\alpha}{2}} = t_{20-1, \frac{0.05}{2}} = t_{19, 0.025} = 2.093$
- iv. $\bar{X} \pm t_{n-1; \frac{\alpha}{2}} \frac{S}{\sqrt{n}} = 20 \pm 2.093 \left(\frac{8}{\sqrt{20}} \right) = 20 \pm 3.74$
 $16.26 \leq \hat{\mu} \leq 23.74$
 $(16.26, 23.74)$
- v. Yes, because the value of $\mu=21$ is included within the confidence interval estimate.
No, because the value of $\mu=5$ is not included within the confidence interval estimate.



Example (5)

Find a 90% confidence interval for a population mean μ for these values:
 $n=14$, $\bar{x}=1258$, $s^2 = 45796$, $X \sim N(\mu, \sigma^2)$

Solution:

$$\alpha = 1 - 0.90 = 0.10$$

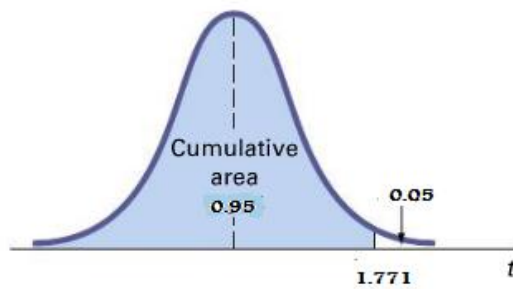
$$t_{n-1; \frac{\alpha}{2}} = t_{14-1, \frac{0.10}{2}} = t_{13, 0.05} = 1.771$$

$$\hat{\mu} = \bar{X} \pm t_{n-1; \frac{\alpha}{2}} \frac{S}{\sqrt{n}}$$
$$= 1258 \pm 1.771 \left(\frac{214}{\sqrt{14}} \right)$$

$$= 1258 \pm 101.29$$

$$1156.71 \leq \hat{\mu} \leq 1359.29$$

$$(1156.71, 1359.29)$$



Confidence Interval for a Population Proportion (Large Sample)

When the sample size is large, $n\pi \geq 5, n(1-\pi) \geq 5$, the sample proportion,

$$P = \frac{X}{n} = \frac{\text{Total number of successes}}{\text{Total number of trials}}$$

$$P \sim N\left(\pi, \frac{\pi(1-\pi)}{n}\right)$$

The confidence interval for a population proportion:

$$\pi = P \pm Z_{\frac{\alpha}{2}} \sqrt{\frac{P(1-P)}{n}}$$

$$\sqrt{\frac{P(1-P)}{n}}$$

The standard error of the proportion

Example (6)

The owner of the West End credit Kwick Fill Gas Station wishes to determine the proportion of customers who use a credit card or debit card to pay at the pump. He surveys 100 customers and finds that 80 paid at the pump.

- Estimate the value of the population proportion.
- Develop a 95 percent confidence interval for the population proportion.
- Interpret your findings.

Solution:

a.

$$\pi = P = \frac{X}{n} = \frac{80}{100} = 0.8$$

b.

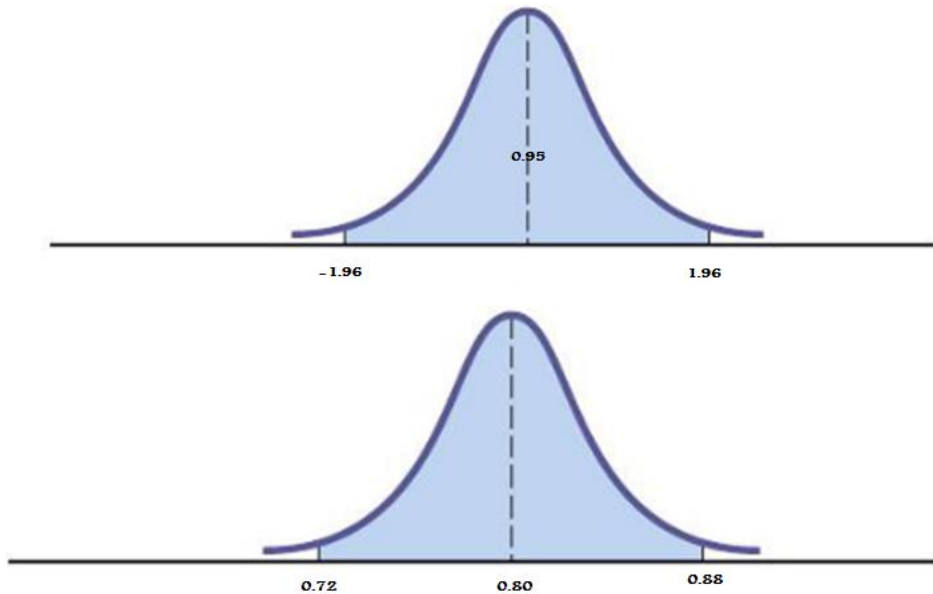
$$Z_{\frac{0.05}{2}} = Z_{0.025} = Z_{0.9750} = -1.96 \quad Z_{1-\frac{0.05}{2}} = Z_{0.9750} = 1.96$$

$$P \pm Z_{\frac{\alpha}{2}} \sqrt{\frac{P(1-P)}{n}} = 0.8 \pm 1.96 \sqrt{\frac{(0.8)(0.2)}{100}} = 0.8 \pm 1.96 \sqrt{0.0016} = 0.8 \pm 1.96(0.04) = 0.8 \pm 0.0784$$

$$0.72 \leq \hat{\pi} \leq 0.88$$

$$(0.72, 0.88)$$

- We are reasonably sure the population proportion is between 0.72 and 0.88 percent.



Example (7)

The Fox TV network is considering replacing one of its prime-time crime investigation shows with a new family-oriented comedy show. Before a final decision is made, network executives commission a sample of 400 viewers. After viewing the comedy, 0.63 percent indicated they would watch the new show and suggested it replace the crime investigation show.

- Estimate the value of the population proportion.
- Develop a 99 percent confidence interval for the population proportion.
- Interpret your findings.

Solution:

a.

$$\pi = P = 0.63$$

b.

$$Z_{\frac{0.01}{2}} = Z_{0.005} = -2.58$$

$$Z_{1-\frac{0.01}{2}} = Z_{1-0.005} = Z_{0.9950} = 2.58$$

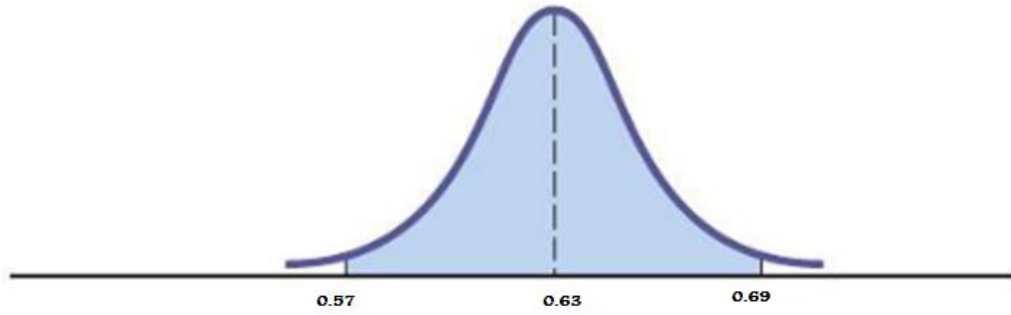
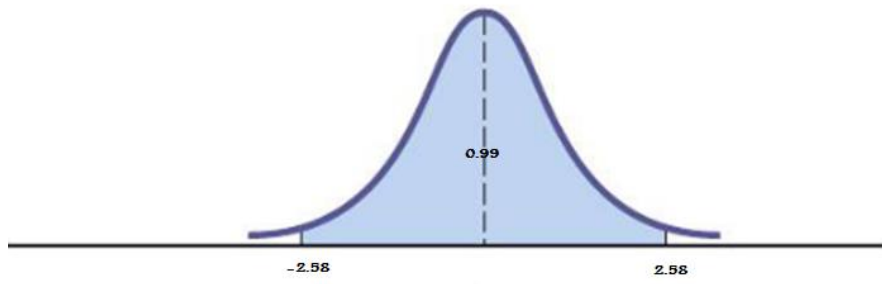
$$P \pm Z_{\frac{\alpha}{2}} \sqrt{\frac{P(1-P)}{n}} = 0.63 \pm 2.58 \sqrt{\frac{(0.63)(0.37)}{400}} = 0.63 \pm 2.58 \sqrt{0.00058275}$$

$$= 0.63 \pm 2.58(0.02414) = 0.63 \pm 0.0623$$

$$0.57 \leq \hat{\pi} \leq 0.69$$

(0.57 , 0.69)

c .We are reasonably sure the population proportion is between 0.57 and 0.69 percent .



Note:

If the value of estimated proportion(p) not mentioned we substitute it by 0.5(as studies and reachers recommended)

Choosing an appropriate sample size for the population mean

$$e = \pm Z \frac{\sigma}{\sqrt{n}} \quad \text{Or} \quad e = \frac{UCL-LCL}{2}$$
$$e = Z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

The sample size for estimating the population mean:

$$n = \left(\frac{Z_{\alpha/2} \sigma}{e} \right)^2$$

Example (8)

A student in public administration wants to determine the mean amount members of city councils in large cities earn per month as remuneration for being a council member. The error in estimating the mean is to be less than \$100 with a 95 percent level of confidence. The student found a report by the Department of Labor that estimated the standard deviation to be \$1,000. What is the required sample size?

Solution:

Given in the problem:

- E, the maximum allowable error, is \$100
- The value of z for a 95 percent level of confidence is 1.96,
- The estimate of the standard deviation is \$1,000.

$$n = \left(\frac{\left(Z_{\frac{\alpha}{2}} \right) \sigma}{e} \right)^2 = \left(\frac{(1.96)(1000)}{100} \right)^2 = 384.16 \approx 385$$

Example (9)

A population is estimated to have a standard deviation of 10. if a 95 percent confidence interval is used and an interval of ± 2 is desired. How large a sample is required?

Solution: Given in the problem:

- E, the maximum allowable error, is 2
- The value of z for a 95 percent level of confidence is 1.96,
- The estimate of the standard deviation is 10.

$$n = \left(\frac{\left(Z_{\frac{\alpha}{2}} \right) \sigma}{e} \right)^2 = \left(\frac{(1.96)10}{2} \right)^2 = 96.04 \approx 97$$

Example (10)

If a simple random sample of 326 people was used to make a 95% confidence interval of (0.57,0.67), what is the margin of error (e)?

Solution:

$$e = \frac{\text{upper} - \text{lower}}{2} = \frac{0.67 - 0.57}{2} = \frac{0.1}{2} = 0.05$$

Example (11)

If $n=34$, the standard deviation 4.2 (σ), $1 - \alpha = 95\%$.What is the maximum allowable error (E) ?

Solution:

$$e = \pm Z_{\frac{\alpha}{2}} \left(\frac{\sigma}{\sqrt{n}} \right)$$

$$e = \pm 1.96 \left(\frac{4.2}{\sqrt{34}} \right) = \pm 1.96(0.7203) = \pm 1.41$$

The maximum allowable error (e) = 1.41

Choosing an appropriate sample size for the population proportion

The margin error for the confidence interval for a population proportion:

$$e = Z_{\frac{\alpha}{2}} \sqrt{\frac{\pi(1 - \pi)}{n}}$$

Solving "e" equation for "n" yields the following result:

$$n = \left(\frac{Z_{\frac{\alpha}{2}} \sqrt{\pi(1 - \pi)}}{e} \right)^2$$

Or

$$n = \pi(1 - \pi) \left(\frac{Z_{\frac{\alpha}{2}}}{e} \right)^2$$

$$n = \frac{\left(Z_{\frac{\alpha}{2}} \right)^2 \pi(1 - \pi)}{e^2}$$

Example (12)

The estimate of the population proportion is to be within plus or minus 0.05, with a 95 percent level of confidence. The best estimation of the population proportion is 0.15. How large a sample is required?

Solution:

$$\begin{aligned}n &= \frac{\left(Z_{\frac{\alpha}{2}}\right)^2 \pi(1-\pi)}{e^2} = \frac{(1.96)^2 0.15(1-0.15)}{(0.05)^2} = \frac{3.8416(0.15 \times 0.85)}{0.0025} \\ &= \frac{3.8416 \times 0.1275}{0.0025} = \frac{0.4898}{0.0025} = 195.92 \approx 196\end{aligned}$$

Example (13)

The estimate of the population proportion is to be within plus or minus 0.10, with a 99 percent level of confidence. How large a sample is required?

Solution:

$$\begin{aligned}n &= \frac{\left(Z_{\frac{\alpha}{2}}\right)^2 \pi(1-\pi)}{e^2} = \frac{(2.58)^2 0.5(1-0.5)}{(0.10)^2} = \frac{6.6564(0.5 \times 0.5)}{0.01} \\ &= \frac{6.6564 \times 0.25}{0.01} = \frac{1.6641}{0.01} = 166.41 \approx 167\end{aligned}$$