

Midterm Exam 2, Semester I, 1446

Department of Mathematics, College of Science, KSU

Course: Math 481 — Maximum Marks: 25 — Duration: 1.5 Hours

immediate

Question 1

[3+3 points]

Prove the following statements related to the Lebesgue outer measure m^* :

1. $m^*({0}) = 0$,
2. $m^*((0, \infty)) = \infty$.

Question 2

[1+3 points]

Let the function $f(x)$ be defined as:

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \cap [0, 1], \\ 0, & x \in \mathbb{Q}^c \cap [0, 1], \end{cases}$$

where \mathbb{Q} denotes the set of rational numbers, and \mathbb{Q}^c denotes the set of irrational numbers. Prove that:

1. $f(x)$ is not Riemann integrable.
2. $f(x)$ is Lebesgue integrable.

Question 3

[5+5 points]

Evaluate the following limits:

(a)

$$\lim_{n \rightarrow \infty} \int_0^{\infty} \left(1 + \frac{x}{n}\right)^n e^{-2x} dx,$$

(b)

$$\lim_{n \rightarrow \infty} \int_0^1 \frac{nx}{1 + n^2 x^2} dx,$$

Question 4

[5 points]

Prove the following integral equality:

$$\int_0^1 \sin(x) \log(x) dx = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n(2n)!}.$$

End of Exam