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Summary

* Linear system $AX = B$

If A is rectengular
1. Gauss
2. Gauss - Jordan

If A is square
1. Gauss
2. Gauss - Jordan
3. Inverse - method
4. Cramer

* suppose $AX = B$ is Linear system where <u>A is square matrix</u>

If $B = 0$
system is Homogenous

$|A| \neq 0$
A^{-1} is existed
The system has unique soluton (zero - solution)

$|A| = 0$
A^{-1} is not existed
The system has ∞ many solutions
There are non-trivial solution
(we can find them through Gauss)

If $B \neq 0$
system is not - Homogenous

$|A| \neq 0$
A^{-1} is existed
The system has unique solution
$\boxed{A^{-1} \cdot B}$

$|A| = 0$
A^{-1} is not existed
we use Gauss method to decide

No Solution

∞ many Solutions

(1) Let $A \in M_3(\mathbb{R})$ where A is invertible. If $B \in M_3(\mathbb{R})$ is singular, find $|2A^{-1}A^T + 3B\,\text{adj}(B)|$?

Solution As B is singular then $|B| = 0$

Now $\text{adj}(B) = |B|\,B^{-1} = 0$

So, $3B\,\text{adj}(B) = 0$

There fore, $2A^{-1}A^T + 3B\,\text{adj}(B) = 2A^{-1}A^T$

Now

$$|2A^{-1}A^T + 3B\,\text{adj}(B)| = |2A^{-1}A^T| = 2^3\frac{|A|}{|A|} = 2^3 = 8$$

(2) Let $(x, y, z) = (1, -1, 1)$ be a solution of the following system:

$$2x - y + z = r$$
$$x + 2y - z = s$$
$$3x + 4y + rz = t$$

find r, s and t.

Solution Since $(1, -1, 1)$ is a solution, by substitute in equations

(From E1) $2 + 1 + 1 = r \Rightarrow r = 4$

(From E2) $1 - 2 - 1 = s \Rightarrow s = -2$

$3 - 4 + 4 = t \Rightarrow t = 3$

(3) write a relation of α, β and γ to make the following system is consistent:

$$x + 2y + 3z = \alpha$$
$$2x + 5y + 9z = \beta$$
$$x + 3y + 6z = \gamma$$

Solution we will write the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & \alpha \\ 2 & 5 & 9 & \beta \\ 1 & 3 & 6 & \gamma \end{array}\right] \xrightarrow[-R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & \alpha \\ 0 & 1 & 3 & -2\alpha+\beta \\ 0 & 1 & 3 & -\alpha+\gamma \end{array}\right] \xrightarrow{-R_2+R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & \alpha \\ 0 & 1 & 3 & -2\alpha+\beta \\ 0 & 0 & 0 & \alpha-\beta+\gamma \end{array}\right]$$

the system is consistent iff $\alpha - \beta + \gamma = 0$

③

(4) Find the value of m where $\begin{bmatrix} 1 & 1 & 1 \\ 1 & m & 2 \\ 1 & 10 & m \end{bmatrix}$ is invertible?

<u>solution</u> $\begin{vmatrix} 1 & 1 & 1 \\ 1 & m & 2 \\ 1 & 10 & m \end{vmatrix} \xrightarrow[-R_1+R_3]{-R_1+R_2} \begin{vmatrix} 1 & 1 & 1 \\ 0 & m-1 & 1 \\ 0 & 9 & m-1 \end{vmatrix}$

$= (m-1)^2 - 9$

<u>Now</u> A is invertible iff $(m-1)^2 - 9 \neq 0$

$\Leftrightarrow (m-1)^2 \neq 9$

$\Leftrightarrow (m-1) \neq \pm 3$

$m \neq -2$ or $m \neq 4$

So, $m \in \mathbb{R} - \{-2, 4\}$

(5) Let $\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 2 & 3 & \alpha & 3 \\ 1 & \alpha & 3 & 2 \end{array}\right]$ be the augmented matrix of a linear system. Find the value of α where the system has unique solution. Find the solution of the system then.

<u>Solution</u> clearly, the system is not homogenous. So, it has a unique solution iff $\begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & \alpha \\ 1 & \alpha & 3 \end{bmatrix}$ is invertible iff $\begin{vmatrix} 1 & 1 & -1 \\ 2 & 3 & \alpha \\ 1 & \alpha & 3 \end{vmatrix} \neq 0$.

<u>Now</u> $\begin{vmatrix} 1 & 1 & -1 \\ 2 & 3 & \alpha \\ 1 & \alpha & 3 \end{vmatrix} \xrightarrow[-R_1+R_3]{-2R_1+R_2} \begin{vmatrix} 1 & 1 & -1 \\ 0 & 1 & 2+\alpha \\ 0 & \alpha-1 & 4 \end{vmatrix} =$

$4 - (\alpha-1)(2+\alpha) = 4 - (2\alpha + \alpha^2 - 2 - \alpha)$

$= -\alpha^2 - \alpha + 6$

If $|A| \neq 0 \Rightarrow -\alpha^2 - \alpha + 6 \neq 0$

$\Rightarrow \alpha^2 + \alpha - 6 \neq 0$

$\Rightarrow (\alpha+3)(\alpha-2) \neq 0$

$\Rightarrow \alpha \neq -3$ or $\alpha \neq 2$

$\Rightarrow \alpha \in \mathbb{R} - \{-3, 2\}$.

In this case, the unique solution is $A^{-1} \cdot \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$

⑥ Let $A \in M_n(\mathbb{R})$ such that $A^2 - 3A + I_n = 0$. Find A^{-1}?

Solution

$\because A^2 - 3A + I_n = 0 \Rightarrow 3A - A^2 = I_n$

$\Rightarrow A(3I - A) = I_n$

Notice that

$(3I - A)A = 3A - A^2 = I$

So,

$A^{-1} = 3I - A$

⑦ Let $A, B \in M_3(\mathbb{R})$ where $|A| = -3$ $|B| = 3$. Find $|A^T B^3 \operatorname{adj}(A^2) B^{-1}|$.

Solution : Notice that $|A| = 3$, $|B| = -1$

also, $\operatorname{adj}(A^2) = |A^2| \cdot (A^2)^{-1}$

$= |A| \cdot |A| \cdot (A^{-1})^2 = |A|^2 \cdot (A^{-1})^2$

so, $|\operatorname{adj}(A^2)| = \left| |A|^2 \cdot (A^{-1})^2 \right|$

$= (|A|^2)^3 \cdot |A^{-1}| \cdot |A^{-1}| = \dfrac{|A|^6}{|A|^2} = |A|^4$ (*

Now

$|A^T B^3 \operatorname{adj}(A^2) B^{-1}| = |A^T| \cdot |B^3| \cdot |\operatorname{adj}(A^2)| \cdot |B^{-1}|$

by (*)

$= |A| \cdot |B|^3 \cdot |A|^4 \cdot \dfrac{1}{|B|}$

$= (3)(-1)^3 (3)^4 (-1) = 3^5$

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⑤ Let $A = \begin{bmatrix} 1 & 0 & 5 \\ 1 & 1 & 1 \\ 0 & 1 & -4 \end{bmatrix}$. ① find $|-4I_3 - A|$, and decide whether the system $(-4I_3 - A)X = 0$ has unique solution or not? Find all solution.

$$-4I_3 - A = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 5 \\ 1 & 1 & 1 \\ 0 & 1 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 0 & -5 \\ -1 & -5 & -1 \\ 0 & -1 & 0 \end{bmatrix} \quad ; \; |-4I_3 - A| = 0 \Rightarrow (-4I_3 - A)X = 0 \text{ has } \infty \text{ many solutions.}$$

So, the Linear system $(-4I_3 - A)X = 0$ is can be Presented by the following augmented matrix:

$$\left[\begin{array}{ccc|c} -5 & 0 & -5 & 0 \\ -1 & -5 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{-1/5 R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ -1 & -5 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{-1/5 R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 + R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad (\infty \text{ many solutions})$$

$\left.\begin{array}{l} \text{The number of Equations} = 2 \\ \text{The number of Veriables} = 3 \end{array}\right\} \Rightarrow$ The solution will be written by Parameter t

$$\Rightarrow \left.\begin{array}{l} x + z = 0 \\ y = 0 \end{array}\right\} \quad \text{Put } x = t \Rightarrow z = -t$$

$$\text{So, } S = \left\{ \begin{bmatrix} t \\ 0 \\ -t \end{bmatrix}, t \in \mathbb{R} \right\}$$