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A new approach for Weibull modeling for reliability life data analysis

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ABSTRACT

This paper presents a proposed approach for modeling the life data for system components that have failure modes by different Weibull models. This approach is applied for censored, grouped and ungrouped samples. To support the main idea, numerical applications with exact failure times and censored data are implemented. The parameters are obtained by different computational statistical methods such as graphic method based on Weibull probability plot (WPP), maximum likelihood estimates (MLE), Bayes estimators, non-linear Benard's median rank regression. This paper also presents a parametric estimation method depends on expectation–maximization (EM) algorithm for estimation the parameters of finite Weibull mixture distributions. GOF is used to determine the best distribution for modeling life data. The performance of the proposed approach to model lifetime data is assessed. It's an efficient approach for moderate and large samples especially with a heavily censored data and few exact failure times.

1. Introduction

In survival/reliability analysis Weibull models arise in several medical and industrial applications. In medical science, Weibull models can be used to determine the progress of patients from some point in time, such as the time a surgical procedure is performed or a treatment regimen is initiated, until the occurrence of some well-defined event such as death or cessation of symptoms. In industrial applications, Weibull models have been used in life testing to determine the probability that a component manufactured will fail under a given environment. If this probability is high, changing the material, the process of manufacturing, or redesigning might be the alternatives that manufacturer might need to explore.

In the analysis of life data, life data or times-to-failure data of sample units of our product can be classified into two types: complete data (all information is available) or censored data (some of the information is missing). Complete data means that the value of each sample unit is observed or known. The most common case of censoring is what is referred to as right censored data, or suspended data. In the case of life data, these data sets are composed of units that did not fail. Grouped data analysis is used for tests in which groups of units possess the same time-to-failure or in which groups of units were suspended at the same time [1,2].

In Reliability modeling theory, the hazard function is a fundamental quantity in reliability/survival analysis. It's also called failure rate. Most population mortality data and failures of several electrical components follow a bathtub hazard model [3]. A bathtub hazard model consists of three distinct periods, the burn-in failure period or the period of infant

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mortality, the chance (random) failures period that have approximately a constant failure rate and the wear-out failure period or the old age period. Decreasing hazard model characteristic of certain types of electronic devices which has an elevated likelihood of early failures. A constant hazard model describes the devices which have no extra-ordinary of failures are expected, in survival analysis it describes individuals from population whose only risks of death are accidents or rare illness.

Finite mixture Weibull distributions arise in reliability/survival analysis which have many industrial and medical applications, notably in the analysis of failure time data (survival data), and have important mathematical properties [4,5].

Most of the distributions in the Weibull family have a characteristic shape on the Weibull probability plot (WPP). A standard Weibull distribution (2-parameter) has a straight line shape, 3-parameter Weibull is a concave curve with left vertical asymptote, twofold Weibull mixture has a single inflection point (S-shaped) with parallel asymptotes, twofold Weibull competing risk has a convex curve with a left asymptote and a right asymptote or may be considered as a straight line, curving into a second line with a steeper slope, which is called in this case Classic Bi-Weibull, and More over if the curve has multiple inflection modes, then once can suggest more than one Weibull mixture models such as competing risk with a batch problem model which is known as competing risk mixture model, a compound competing risk mixture model or may be simply twofold Weibull mixture [6,7].

The design of the manufacturing process has a significant impact on ω_i , the probability that an item is conforming when the process is in control. Ideally, one would like to have this probability equal to one so that no item produced is nonconforming. Due to quality variations, the failure distribution of an item produced is given by a mixture distribution which is defined as follows:

$$F(x) = \omega F_c(x) + (1 - \omega)F_n(x) \tag{1}$$

where ω is the probability that the item is conforming and $F_c(x)$ and $F_n(x)$ are the failure distributions of conforming and nonconforming items, respectively, with $F_c(x) < F_n(x)$ implying that the reliability of a nonconforming item is smaller than that for a conforming item [6]. Burn-in is used to improve the reliability of the item released for sale.

There are several methods can be applied for obtaining parameter estimates of the mixed Weibull distribution and 3-parameter Weibull distribution. These include graphic, moments, maximum likelihood estimation, Bayes estimators, nonlinear median rank regression and Monte Carlo simulation methods, and many others [6–10].

Kao [4] used a two-component Weibull mixture to fit the life distribution of electron valves. He bases his analysis on the observation that there are two types of failure modes, sudden catastrophic failure mode and wear-out failure mode. He analyzes this mixture using graphical approach. Fall [4] estimated the parameters of a mixture of two Weibull distributions for complete data sample by using the method of moments which was not an efficient procedure. The mixed Weibull distribution (also known as a multimodal Weibull) is used to model data that do not fall on a straight line on a Weibull probability plot (WPP). Data of this type, particularly if the data points follow an S-shape on the probability plot, may be indicative of more than one failure mode at work in the population of failure times [11].

Statisticians prefer maximum likelihood estimation (MLE) over other estimates because MLE have excellent statistical characteristics in general, particularly for large data sets. However, MLE can handle suspensions and interval data better than rank regression, particularly when dealing with a heavily censored data set with few exact failure times or when the censoring times are unevenly distributed. It can also provide estimates with one or no observed failures, which rank regression cannot do. As a rule of thumb, our recommendation is to use rank regression techniques when the sample sizes are small and without heavy censoring. When heavy or uneven censoring is present, when a high proportion of interval data is present and/or when the sample size is sufficient, MLE should be preferred. The likelihood function is a function of the data. It consists of the product of the probability density functions for each failure data point times the product of reliability function for each suspension (right censored) with the distribution parameters unknown.

EM algorithm for estimating the parameters which is called the expectation–maximization is a general method for optimizing likelihood functions and is useful in situations where data might be missing or simpler optimization methods fail. The seminal paper on this topic is by Dempster, Laird and Rubin [12], where they formulate the EM algorithm and establish its properties. It's an algorithm for computing maximum likelihood estimates of parameters when some of data are missing. It's an iterative algorithm that alternates two steps until the convergence is attained to sufficient accuracy. Given some values assumed for the unknown parameters, the E step evaluates the joint likelihood of the complete data set, suitably averaged over all values of the missing data. This is therefore an expectation of the likelihood that is conditional on the observed data. The M step maximizes this expectation over the unknown parameter values, the values providing this maximization are used for the next E step.

Elmahdy and Aboutahoun [13] suggested a procedure for parameter estimation of finite Weibull mixture distributions for modeling complete lifetime data sample by using the graphical method, Bayesian estimates and the concept of EM algorithm which was an efficient approach especially when the mixture is well mixed for moderate complete sample size.

Life data analysis can be used to save human lives, modeling life data of crowd disasters, crime, terrorism, war and disease spreading provides a good picture of the actual system behavior [14]. life data analysis can also be used for interpretation the patterns of evolutionary dynamics of group interactions on structured populations, understanding the dynamics of scientific progress and the evolution of language as studied in languages cool as they expand [15–18].

The objective of this paper is to develop an empirical approach for modeling censored lifetime data for system components that have failure modes by different Weibull models. This paper is organized as follows. In Section 2, we survey the different weibull models such as 3-parameter Weibull, Weibull competing risk and Weibull mixture models. In Section 3, we formulate an algorithm for estimating the parameters of mixed-Weibull distribution (Weibull Mixture Model). In Section 4, we focus on Goodness of fit tests (GOF), to determine the best distribution among the suggested distributions for modeling lifetime data. In Section 5, we formulate the proposed modeling approach for life data by Weibull models. Section 6 deals with some applications to illustrate the proposed approach for modeling actual life data set and statistical inference for the selected Weibull models. Section 7 is the summary of the conclusions of this paper and some future extensions of this research.

Notation & Acronyms

F_e	number of groups of times-to-failure data points
n_j	number of units that failed in the *jth* time-to-failure data group
S	number of groups of suspension data points
n_k	number of suspension or censored units that have not failed in the *kth* group of suspension data point
r	number of units which have exact failures, $r = \sum_{j=1}^{F_e} n_j$
n'	number of suspended units or surviving units, $n' = \sum_{k=1}^{S} n_k$
n	test sample size, $n = r + n'$
t_j	ordered times-to-failure of failed units, $j = 1, 2, \ldots, F_e$
t_k	ordered operating times of suspension units, $k = 1, 2, \ldots, S$
β, α, γ	shape, scale and location parameters of 3-parameter Weibull distribution, $\beta > 0, \alpha > 0$ and $-\infty < \gamma < \infty$
m	number of subpopulations
i	index for subpopulations, $i = 1, 2, \ldots, m$
ω_i	mixing weight of *ith* subpopulation, $\sum_{i=1}^{m} \omega_i = 1$
$\boldsymbol{\theta}$	parameter vector for Weibull model of an m-mixed Weibull distribution
α_i, β_i	scale and shape parameters of subpopulation $i, \alpha_i > 0$, $\beta_i > 0$
f_i	probability density function of subpopulation i
R_i	reliability function of subpopulation i
$R(t)$	reliability function of a model
$F(t)$	cumulative distribution function (cdf) of a model
$f(t)$	probability density function (pdf) of a model
$h(t)$	hazard function (failure rate) of a model
l	log-likelihood function
N_j	the adjusted rank for the failures
WPP	Weibull probability paper
MLE	maximum likelihood estimation
EM	expectation–maximization Algorithm
GOF	goodness of fit tests

2. Weibull models

A large number of models have been derived from the two-parameter Weibull distribution and are referred to as Weibull models [6]. The various members of the Weibull family exhibit a wide variety of shapes and therefore represent different characteristics for the reliability functions. Model selection, parameter estimation and model validation are important tasks in reliability analysis. In this study we develop a simple and applicable approach to select the most appropriate Weibull model for modeling failure data.

2.1. Three-parameter Weibull model

The probability density function of 3-parameter Weibull distribution is defined mathematically as

$$f(t|\beta,\alpha,\gamma) = \frac{\beta}{\alpha}\left(\frac{t-\gamma}{\alpha}\right)^{\beta-1} e^{-\left(\frac{t-\gamma}{\alpha}\right)^{\beta}}, \qquad t > 0 \tag{2}$$

where $\beta > 0, \alpha > 0$ and $-\infty < \gamma < \infty$ are the shape, scale and location parameters of the distribution. The shape parameter is responsible for the skew of the distribution, the scale parameter is sometimes referred to as the characteristic life and the location parameter is used to shift the distribution in one direction or another to define the location of its origin and can be either positive or negative, it is sometimes called minimum life. The corresponding Reliability and hazard functions are defined respectively as

$$R(t|\beta,\alpha,\gamma) = e^{-\left(\frac{t-\gamma}{\alpha}\right)^{\beta}} \tag{3}$$

$$h(t|\beta,\alpha,\gamma) = \frac{\beta}{\alpha}\left(\frac{t-\gamma}{\alpha}\right)^{\beta-1} \tag{4}$$

If the location parameter γ is equal to zero, the three parameter model becomes two parameter model (standard Weibull model), then we have

$$f(t|\beta,\alpha) = \frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1} \exp\left[-\left(\frac{t}{\alpha}\right)^{\beta}\right], \quad t > 0 \tag{5}$$

This distribution is applied to a wide range of problems. The Weibull distribution is by far the world's most popular statistical model for life data. It is also used in many other applications, such as weather forecasting and fitting data of all kinds. It may be employed for engineering analysis with smaller sample sizes than any other statistical distribution [7].

We can estimate the parameters β, α and γ by using MLE method through a specified statistical software package.

2.2. Weibull competing risk model

Competing risk occurs when a population has two or more failure modes and the entire population is at risk from either failure mode. Even though a Weibull plot of this data will show a cusp or corner or appear curved, it is a homogenous population subject to a mixture of competing failure modes. If the first slope is shallow and the second slope steeper, this is called Classic Bi-Weibull. An example of competing risk is an automobile tire. It can fail due to puncture, or it can fail due to wear out [7].

The distribution function of a general m-fold Weibull competing risk model [6] involving m subpopulations, where m subpopulations are the standard two-parameter Weibull distributions is defined as:

$$F(t|\boldsymbol{\theta}) = 1 - \prod_{i=1}^{m}[1 - F_i(t|\beta_i, \alpha_i)] \tag{6}$$

where $\alpha_i > 0$, $\beta_i > 0$ scale and shape parameters of subpopulation i respectively and $\boldsymbol{\theta} = (\alpha_1, \alpha_2, \ldots, \alpha_m, \beta_1, \beta_2, \ldots, \beta_m)$ is the parameter vector of an m-fold Weibull competing risk model. The corresponding Reliability(survivor) function is defined as:

$$R(t|\boldsymbol{\theta}) = \prod_{i=1}^{m} R_i(t|\beta_i, \alpha_i) \tag{7}$$

which can be written as:

$$R(t|\boldsymbol{\theta}) = \prod_{i=1}^{m} \exp\left[-\left(\frac{t}{\alpha_i}\right)^{\beta_i}\right] \tag{8}$$

The density function is given by

$$f(t|\boldsymbol{\theta}) = \sum_{i=1}^{m}\left\{\prod_{\substack{j=1\\ j\neq i}}^{m} R_j(t|\beta_j, \alpha_j)\right\} f_i(t|\beta_i, \alpha_i) \tag{9}$$

for $m = 2$, the density function f is given by

$$f(t) = R_1 f_2 + R_2 f_1 \tag{10}$$

The hazard functions is given by

$$h(t|\boldsymbol{\theta}) = \sum_{i=1}^{m} h_i(t|\beta_i, \alpha_i) \tag{11}$$

which can be written as:

$$h(t|\boldsymbol{\theta}) = \sum_{i=1}^{m}\left(\frac{\beta_i}{\alpha_i}\right)\left(\frac{t}{\alpha_i}\right)^{\beta_i - 1} \tag{12}$$

2.3. Weibull mixture model

When we have m-fold mixture model that involves m sub-populations, then the probability density function $f(t|\boldsymbol{\theta})$ of the mixture distribution is given as follows:

$$f(t|\boldsymbol{\theta}) = \sum_{i=1}^{m} \omega_i f_i(t|\beta_i, \alpha_i) \tag{13}$$

where $\omega_i > 0, \alpha_i > 0$, $\beta_i > 0$ are mixing weight, scale and shape parameters of subpopulation i respectively, $\sum_{i=1}^{m}\omega_i = 1$ and $\boldsymbol{\theta} = (\omega_1, \omega_2, \ldots, \omega_m, \alpha_1, \alpha_2, \ldots, \alpha_m, \beta_1, \beta_2, \ldots, \beta_m)$ is called the parameter vector of an m-mixed Weibull distribution.

$$f(t|\boldsymbol{\theta}) = \sum_{i=1}^{m} \omega_i \left(\frac{\beta_i}{\alpha_i}\right)\left(\frac{t}{\alpha_i}\right)^{\beta_i - 1} \exp\left[-(\frac{t}{\alpha_i})^{\beta_i}\right] \tag{14}$$

The reliability(survivor) function $R(t|\boldsymbol{\theta})$ of the mixture distribution is given as follows:

$$R(t|\boldsymbol{\theta}) = \sum_{i=1}^{m} \omega_i \exp\left[-\left(\frac{t}{\alpha_i}\right)^{\beta_i}\right] \tag{15}$$

Another function to describe the Reliability of system components is the hazard (failure rate) function $h(t|\boldsymbol{\theta})$ of the mixture Weibull distribution is given as follows:

$$h(t|\boldsymbol{\theta}) = \sum_{i=1}^{m} \omega_i \left(\frac{\beta_i}{\alpha_i}\right)\left(\frac{t}{\alpha_i}\right)^{\beta_i - 1} \tag{16}$$

The mixed Weibull distribution is the appropriate model when a product has two or more failure modes or causes. This occurs in many situations, such as both early failures (infant mortality) and chance failures might be involved in a burn-in test and also in the case of quality control mode with an infant mortality followed by a wear out mode.

2.3.1. The graphic method for Weibull mixture model

In graphic method we separate the observed times in such a way to failure data into two sub-populations or more, then we model each subpopulation to a single Weibull distribution. Really precise estimation of parameters in a mixture model is not possible with only moderate amounts of data, but a main concern is to determine whether some model in the given class produces a reasonable fit to the data [2]. Relatively informal methods are often helpful in examining this possibility, particularly if the data appear separable into two or more fairly distinct parts. One can estimate by plotting the sample cdf (cumulative distribution function) on Weibull plotting paper (WPP) and fit it by inspection a smooth curve which may be convex, concave or likely S-shaped and approaches to a straight line when data points become smaller. A steep slope followed by a shallow slope usually indicates a batch problem. There will usually be a knee in the curve where the slope decreases sharply [6,19]. The cumulative failure at this point (e.g., 20%) should be used for an estimate of ω. If there is no Knee, use the plotting position for the first data point [20]. Graphical approach which starts with WPP for modeling a given data set which may contains censored data is introduced in details in [21]. Probability plots of the two sets (Classic Bi-Weibull) of observations are especially useful, providing both parameter estimates and a check on the assumed form of two sub-populations distribution (simple mixture distributions), also we use these graphic parameter estimates as initial estimates in the proposed method. We can also use the facilities of Super SMITH software package to find the graphic parameter estimates of mixture Weibull distribution, a statistical separation based on rank regression, p-value or likelihood ratio tests were employed to estimate the parameters of the two Weibulls in case of Classic Bi-Weibull Model. More complex solutions for mixtures of two and three failure modes including batch effects are provided by “YBATH” software created by Carl Tarum and included with Super SMITH software package.

3. The proposed algorithm for estimating the parameters of Weibull mixture model

In this section, we introduce a proposed Algorithm for Estimating the parameters of mixed-Weibull distribution (Weibull mixture model) which is based on EM algorithm [12]. The general form of the likelihood function for a given observation of failures and right censored (suspensions) is defined as follows:

$$L(t;\theta) = \prod_{j=1}^{r} f(t_j|\boldsymbol{\theta}) \prod_{k=1}^{n'} R(t_k|\boldsymbol{\theta}) \tag{17}$$

The log-likelihood function l can be expressed as:

$$l(t;\theta) = \sum_{j=1}^{r} \ln\left[f(t_j|\theta)\right] + \sum_{k=1}^{n'} \ln\left[R(t_k|\theta)\right] \tag{18}$$

where the first sum refers to failures and the second sum refers to suspensions.

Consider a reliability life testing is applied on n units of a product which has two failure modes, a grouped ordered time-to-failure and censored data sample is obtained.

For grouped data including exact times-to-failure and censoring, the log-likelihood function l can be written as:

$$l(t;\theta) = \sum_{j=1}^{F_e} n_j \ln\left[f(t_j|\theta)\right] + \sum_{k=1}^{S} n_k \ln\left[R(t_k|\theta)\right] \tag{19}$$

For the Proposed algorithm, we augment the observed (measured) data with some unobserved (missing data). This means that we embed the observed data in a larger complete data space. Note that, missing data are not necessarily missing in the classical way. This process is called data augmentation. By Bayes formula the concept of belonging probability, $P_i(t_j, \boldsymbol{\theta}^{(h)})$, which is the posterior probability that the unit belongs to the i th subpopulation $(i = 1, 2, \ldots, m)$, knowing that it failed at time t_j is introduced as [13]:

$$P_i(t_j, \boldsymbol{\theta}^{(h)}) = \frac{\omega_i^{(h)} f_i\left(t_j|\beta_i^{(h)}, \alpha_i^{(h)}\right)}{f\left(t_j|\boldsymbol{\theta}^{(h)}\right)} \tag{20}$$

$$P_i\left(t_j, \boldsymbol{\theta}^{(h)}\right) = \frac{\omega_i^{(h)} f_i\left(t_j|\beta_i^{(h)}, \alpha_i^{(h)}\right)}{\sum_{i=1}^{m} \omega_i^{(h)} f_i\left(t_j|\beta_i^{(h)}, \alpha_i^{(h)}\right)} \tag{21}$$

Similarly for n' surviving units, the conditional probability of a unit is belonging to subpopulation i, given that it survived until t_k is

$$P_i(t_k, \boldsymbol{\theta}^{(h)}) = \frac{\omega_i^{(h)} R_i\left(t_k|\beta_i^{(h)}, \alpha_i^{(h)}\right)}{R\left(t_k|\boldsymbol{\theta}^{(h)}\right)} \tag{22}$$

$$P_i(t_k, \boldsymbol{\theta}^{(h)}) = \frac{\omega_i^{(h)} R_i\left(t_k|\beta_i^{(h)}, \alpha_i^{(h)}\right)}{\sum_{i=1}^{m} \omega_i^{(h)} \exp\left[-\left(\frac{t_k}{\alpha_i}\right)^{\beta_i}\right]} \tag{23}$$

Given a current estimate $\boldsymbol{\theta}^{(h)}$ define the expectation of the log-likelihood function as;

$$Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(h)}) = \sum_{j=1}^{F_e} n_j P_i(t_j, \boldsymbol{\theta}^{(h)}) \ln\left[f(t_j|\boldsymbol{\theta})\right] + \sum_{k=1}^{S} n_k P_i(t_k, \boldsymbol{\theta}^{(h)}) \ln[R(t_k|\boldsymbol{\theta})] \tag{24}$$

which can also be written as:

$$Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(h)}) = \sum_{j=1}^{F_e}\sum_{i=1}^{m} n_j P_i(t_j, \boldsymbol{\theta}^{(h)}) \ln\left[\omega_i f_i(t_j|\beta_i, \alpha_i)\right] + \sum_{k=1}^{S}\sum_{i=1}^{m} n_k P_i(t_k, \boldsymbol{\theta}^{(h)}) \ln\left[\omega_i R_i(t_k|\beta_i, \alpha_i)\right] \tag{25}$$

$$\begin{aligned} Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(h)}) = {} & \sum_{j=1}^{F_e}\sum_{i=1}^{m} n_j P_i(t_j, \boldsymbol{\theta}^{(h)}) \ln(\omega_i) + \sum_{j=1}^{F_e}\sum_{i=1}^{m} n_j P_i(t_j, \boldsymbol{\theta}^{(h)}) \ln f_i(t_j|\beta_i, \alpha_i) + \sum_{k=1}^{S}\sum_{i=1}^{m} n_k P_i(t_k, \boldsymbol{\theta}^{(h)}) \ln(\omega_i) \\ & + \sum_{k=1}^{S}\sum_{i=1}^{m} n_k P_i\left(t_k, \boldsymbol{\theta}^{(h)}\right) \ln R_i(t_k|\beta_i, \alpha_i) + \lambda\left(\sum_{i=1}^{m} \omega_i - 1\right) \end{aligned} \tag{26}$$

where λ is the lagrange multiplier with the constraint that $\sum_{i=1}^{m} \omega_i = 1$. The evaluation of this expectation is called the E step of the algorithm. In the second step, the M-step (the maximization step), we find that value $\boldsymbol{\theta}^{(h+1)}$ of $\boldsymbol{\theta}$ which maximizes $Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(h)})$.

To find $\omega_i^{(h+1)}$ of ω_i which maximize $Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(h)})$, taking the derivative of Eq. (26) with respect to ω_i equal to zero, we ge t:

$$\sum_{j=1}^{F_e} \frac{n_j P_i(t_j, \boldsymbol{\theta}^{(h)})}{\omega_i} + \sum_{k=1}^{S} \frac{n_k P_i(t_k, \boldsymbol{\theta}^{(h)})}{\omega_i} + \lambda = 0 \tag{27}$$

$$\sum_{j=1}^{F_e} n_j P_i(t_j, \boldsymbol{\theta}^{(h)}) + \sum_{k=1}^{S} n_k P_i(t_k, \boldsymbol{\theta}^{(h)}) + \lambda\omega_i = 0 \tag{28}$$

Summing both sides over i and using the fact that $\sum_{i=1}^{m} P_i(t_j, \boldsymbol{\theta}^{(h)}) = 1$, $\sum_{i=1}^{m} P_i(t_k, \boldsymbol{\theta}^{(h)}) = 1$ we get $\lambda = -n$, consequently

$$\omega_i^{(h+1)} = \frac{1}{n}\left[\sum_{j=1}^{F_e} n_j P_i(t_j, \boldsymbol{\theta}^{(h)}) + \sum_{k=1}^{S} n_k P_i(t_k, \boldsymbol{\theta}^{(h)})\right] \tag{29}$$

To find the value $\alpha_i^{(h+1)}$ of α_i which maximize $Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(h)})$, taking the derivative of Eq. (26) with respect to α_i equal to zero, we get:

$$\sum_{j=1}^{F_e} n_j P_i(t_j, \boldsymbol{\theta}^{(h)}) \frac{\partial \ln(f_i(t_j|\beta_i, \alpha_i)}{\partial \alpha_i} + \sum_{k=1}^{S} n_k P_i(t_k, \boldsymbol{\theta}^{(h)}) \frac{\partial \ln(R_i(t_k|\beta_i, \alpha_i)}{\partial \alpha_i} = 0 \tag{30}$$

$$\sum_{j=1}^{F_e} n_j P_i(t_j, \boldsymbol{\theta}^{(h)}) = \frac{1}{(\alpha_i^{(h+1)})^{\beta_i^{(h+1)}}}\left[\sum_{j=1}^{F_e} n_j P_i(t_j, \boldsymbol{\theta}^{(h)})(t_j)^{\beta_i^{(h+1)}} + \sum_{k=1}^{S} n_k P_i(t_k, \boldsymbol{\theta}^{(h)})(t_k)^{\beta_i^{(h+1)}}\right] \tag{31}$$

$$\alpha_i^{(h+1)} = \left[\frac{\sum_{j=1}^{F_e} n_j P_i(t_j, \boldsymbol{\theta}^{(h)})(t_j)^{\beta_i^{(h+1)}} + \sum_{k=1}^{S} n_k P_i(t_k, \boldsymbol{\theta}^{(h)})(t_k)^{\beta_i^{(h+1)}}}{\sum_{j=1}^{F_e} n_j P_i(t_j, \boldsymbol{\theta}^{(h)})} \right]^{1/\beta_i^{(h+1)}} \tag{32}$$

Similarly, we can find the value $\beta_i^{(h+1)}$ of β_i which maximize $Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(h)})$, taking the derivative of Eq. (26) with respect to β_i equal to zero and by using Eq. (32), we can get:

$$\begin{aligned} &\frac{1}{\beta_i^{(h+1)}} \sum_{j=1}^{F_e} n_j P_i(t_j, \boldsymbol{\theta}^{(h)}) + \sum_{j=1}^{F_e} n_j P_i(t_j, \boldsymbol{\theta}^{(h)}) \ln(t_j) \\ &- \frac{\left[\sum_{j=1}^{F_e} n_j P_i(t_j, \boldsymbol{\theta}^{(h)})(t_j)^{\beta_i^{(h+1)}} \ln(t_j) + \sum_{k=1}^{S} n_k P_i(t_k, \boldsymbol{\theta}^{(h)})(t_k)^{\beta_i^{(h+1)}} \ln(t_k)\right] \sum_{j=1}^{F_e} n_j P_i(t_j, \boldsymbol{\theta}^{(h)})}{\sum_{j=1}^{F_e} n_j P_i(t_j, \boldsymbol{\theta}^{(h)})(t_j)^{\beta_i^{(h+1)}} + \sum_{k=1}^{S} n_k P_i(t_k, \boldsymbol{\theta}^{(h)})(t_k)^{\beta_i^{(h+1)}}} = 0 \end{aligned} \tag{33}$$

$$g(\beta_i^{(h+1)}) = \frac{1}{\beta_i^{(h+1)}} + \frac{\sum_{j=1}^{F_e} n_j P_i(t_j, \boldsymbol{\theta}^{(h)}) \ln(t_j)}{\sum_{j=1}^{F_e} n_j P_i(t_j, \boldsymbol{\theta}^{(h)})} - \frac{\sum_{j=1}^{F_e} n_j P_i(t_j, \boldsymbol{\theta}^{(h)})(t_j)^{\beta_i^{(h+1)}} \ln(t_j) + \sum_{k=1}^{S} n_k P_i(t_k, \boldsymbol{\theta}^{(h)})(t_k)^{\beta_i^{(h+1)}} \ln(t_k)}{\sum_{j=1}^{F_e} n_j P_i(t_j, \boldsymbol{\theta}^{(h)})(t_j)^{\beta_i^{(h+1)}} + \sum_{k=1}^{S} n_k P_i(t_k, \boldsymbol{\theta}^{(h)})(t_k)^{\beta_i^{(h+1)}}} = 0 \tag{34}$$

Taking a good initial guess of $\boldsymbol{\theta}^{(h)}$, consequently knowing $P_i(t_j, \boldsymbol{\theta}^{(h)}), P_i(t_k, \boldsymbol{\theta}^{(h)})$ and by solving Eq. (34) using a numerical method such as Newton–Raphson, updating Eqs. (29), (32) and (34) we can find MLE estimates of $\omega_i^{(h+1)}$, $\beta_i^{(h+1)}$ and $\alpha_i^{(h+1)}$ of subpopulation i.

Algorithm 1. The Proposed Algorithm for Estimating the Parameters for Weibull Mixture Model

The proposed algorithm for estimating the parameters for Weibull mixture model can be summarized as follows:

step 1: input given data t_j, t_k, n_j, n_k.
step 2: initialize parameters $\boldsymbol{\theta}_i^{(0)} = (\omega_i^{(0)}, \alpha_i^{(0)}, \beta_i^{(0)})$, define a convergence tolerance $\epsilon > 0$.
step 3: let h = 0.
step 4: compute $p_i(t_j, \boldsymbol{\theta}^{(h)})$ and $p_i(t_k, \boldsymbol{\theta}^{(h)})$.
step 5: let h = h + 1.
step 6: compute $\boldsymbol{\theta}_i^{(h+1)} = (\omega_i^{(h+1)}, \alpha_i^{(h+1)}, \beta_i^{(h+1)})$.
step 7: if $\left|\frac{\boldsymbol{\theta}_i^{(h+1)} - \boldsymbol{\theta}_i^{(h)}}{\boldsymbol{\theta}_i^{(h+1)}}\right| < \epsilon$, stop, $\hat{\boldsymbol{\theta}}_i = \boldsymbol{\theta}_i^{(h+1)}$ where $\hat{\boldsymbol{\theta}}_i = (\hat{\omega}_i, \hat{\alpha}_i, \hat{\beta}_i)$.
step 8: if $\left|\frac{\boldsymbol{\theta}_i^{(h+1)} - \boldsymbol{\theta}_i^{(h)}}{\boldsymbol{\theta}_i^{(h+1)}}\right| > \epsilon$, goto step 4.
step 9: compute the log-likelihood function l.
step10: display the estimated parameters $\hat{\boldsymbol{\theta}}_i = (\hat{\omega}_i, \hat{\alpha}_i, \hat{\beta}_i)$ and l.

4. Goodness of fit tests (GOF)

GOF is used to determine the best distribution among the suggested distributions for modeling lifetime data. After using graphical method, we should examine the goodness-of-fit statistics for parametric models. The goodness-of-fit statistics presented in this paper are r-squared value denoted by r^2 and $-2\ln\left(L\left(\widehat{\theta}\right)\right)$ value, where $\ln\left(L\left(\widehat{\theta}\right)\right)$ is the natural logarithm of the maximum likelihood for the proposed model which are discussed in details in the following section, also confidence and prediction bounds are estimated to assess goodness-of-fit statistics.

5. The proposed Weibull modeling approach for life data

The proposed approach for modeling life data by Weibull models depends on five main steps:

Step 1: Collecting a sample of life data.
Step 2: Plotting the data and interpreting the plot.
Step 3: Preliminary model selection.
Step 4: Parameter estimation.
Step 5: Goodness of fit tests and final model selection.

In step 1. Standard lifetime data consists of the exact " age" of the units that have failed (failure data) and those that have not failed (suspended data). The failure data may be found in the laboratory or in the field. Laboratory data is often obtained under controlled conditions which can differ substantially from the operational environment of the components and based on a properly planned experiment. Field data may be collected from different operating conditions-different temperatures or different humidity. The times-to-failure (or cycles-to-failure or mileage-to-failure) are needed plus the current age of units that have not failed (censored).

In step 2. The WPP is based on the Weibull transformation which is defined as:

$$y = \ln\ln[1/(1 - F(t))] \text{ and } x = \ln(t) \tag{35}$$

where t represents lifetime of the unit and $F(t)$ is the cdf (cumulative distribution function). A plot of y versus x is called the Weibull probability plot (WPP). The data are ranked (i.e., rearranged so the earliest failure or suspension is listed first and the oldest failure or suspension is last), plotted on Weibull probability paper (WPP). One of the main tasks when plotting WPP is calculating the y-values that correspond $F(t)$ values. There are a number of methods such as Kaplan–Meier, Median Rank and Benard's Median Rank can be used to estimate $F(t)$. We can define Benard's Median Rank, $\widehat{F}(t_j)$ for each failure data (uncensored observation) t_j as follows [7,16]:

$$\widehat{F}(t_j) = (N_j - 0.3)/(n + 0.4) \tag{36}$$

where n is the sample size (the sum of failures and suspensions) and N_j is the adjusted rank for the failures only which is defined as:

$$N_j = \frac{(\text{Reverse Rank}) \times (\text{Previous Adjusted Rank}) + (n+1)}{(\text{Reverse Rank}) + 1} \tag{37}$$

we consider, previous adjusted rank equals 0 for the first uncensored data (observed failure).

The plot is interpreted as follows: Most of the distributions in the Weibull family have a characteristic shape on the Weibull probability plot (WPP). A standard Weibull distribution (2-parameter) has a straight line shape, 3-parameter Weibull is a concave curve with left vertical asymptote, twofold Weibull mixture has a single inflection point (S-shaped) with parallel asymptotes, twofold Weibull competing risk has a convex curve with a left asymptote and a right asymptote or may be considered as a straight line, curving into a second line with a steeper slope, which is called in this case Classic Bi-Weibull, and More over if the curve has multiple inflection points which probably caused by a mixture of failure modes, then one can suggest more than one Weibull mixture models such as competing risk with a batch problem model which is known as competing risk mixture model, a compound competing risk mixture model or may be simply twofold Weibull mixture.

In step 3. For preliminary model selection, one can use the least squares fit criterion as follows:

1. let x_i and y_i , $1 \leqslant i \leqslant n$, denote the Weibull transformed values in the WPP plotting of the data set. let $y(x_i;\theta)$, $1 \leqslant i \leqslant n$ denote the Weibull transformed values for the model with parameter vector θ which can be determined at first graphically.
2. define the objective function $J(\theta)$ as:

$$J(\theta) = \sum_{i=1}^{n} [y(x_i;\theta) - y_i]^2 \tag{38}$$

which is the sum of squares of the residuals.

3. We can use the $J(\theta)$ value to compare the quality of the curve fit for two or more Weibull models used to describe the same data. The Weibull model that gives the smallest $J(\theta)$ value gives the best fit. We denote the sum of the squares of the deviation of the y values from their mean $\bar{y}$ by S, which can be computed from

$$S = \sum_{i=1}^{n} [y_i - \bar{y}]^2 \tag{39}$$

This formula can be used to compute another measure of the quality and the goodness of the curve fit, the coefficient of determination, also known as the r-squared value. It's defined as:

$$r^2 = 1 - \frac{J}{S} \tag{40}$$

For perfect fit, $J = 0$ and thus $r^2 = 1$. Thus the closer r^2 is to 1, the better the fit. The largest r^2 can be is 1 where r is known as the correlation coefficient.

In step 4 There are several methods can be applied for obtaining efficient parameter estimates of Weibull models. Here, we use maximum likelihood estimation method (MLE) or non-linear median rank regression method for different Weibull models and the proposed algorithm for estimating the parameters of Weibull mixture model for censored lifetime data which is discussed in Section 3. These methods depend on the initial graphical parameter estimates.

In step 5 Here, GOF tests and some statistical inferences are investigated, GOF is used to determine the best distribution for modeling lifetime data. After estimation of parameters of candidates Weibull models, we can repeat both **step 3** and **step 4** until we reach, the largest r^2 to select the best model for modeling the data set. Besides estimation of the coefficient of determination r^2, $-2\ln\left(L\left(\widehat{\theta}\right)\right)$ value, where $\ln\left(L\left(\widehat{\theta}\right)\right)$ is the natural logarithm of the maximum likelihood for the proposed model is used to determine the best distribution for modeling lifetime data and the confidence intervals of parameter estimates are calculated by using Fisher's matrix which measure their statistical precision. $-2\ln\left(L\left(\widehat{\theta}\right)\right)$ value is asymptotically effective and unbiased since it depends on the maximum likelihood function, The best model for the data, is the model with the lowest $-2\ln\left(L\left(\hat{\theta}\right)\right)$ value. The priority is for the Weibull model which has largest r^2 and the lowest $-2\ln\left(L\left(\hat{\theta}\right)\right)$ value and consequently fit the data perfectly. The proposed modeling approach is executed using Matlab Program. We also calculate the confidence intervals of estimated parameters for mixed Weibull distribution using Weibull++ software package which performs statistical analysis depends on Fisher's matrix, with this specified confidence intervals we are interested in obtaining an interval of real numbers that we expect contains the true value of the estimated parameter.

6. Applications

In this section, we introduce two applications to illustrate the proposed approach for modeling actual data set. In Application 1, we have a moderate sample size of 50 data points (Failures 25 and suspensions 25), In Application 2, we have a large sample size with a heavily censored data set with few exact failure times (Failures 66 and suspensions 238) for 304 units. By applying The proposed approach for modeling life data by Weibull models. The obtained Models and the estimated parameters will be reasonable and accurate. This means that the performance of the proposed approach is efficient.

6.1. Application 1

The data in Table 1 represent distance traveled in thousands of kilometers before throttle (a component of a vehicle) failure, or being suspended before failure [6]. The sample size is 50, Failures (25) and suspensions (25). Note that – in Table 4 means suspension (negative age indicates suspension).

6.2. Statistical inference for different Weibull models of application 1

Applying the proposed modeling approach for data sample of application 1. First, We notice that the shape of the data points on Weibull probability plot (WPP) in Fig. 1 is a concave upward curve has a cusp or may be considered as a steep slope followed by a shallow slope which indicates a mixture of failure modes (a batch problem), so we can suggest the Weibull mixture model as a reasonable fit of the failure data. Based on the estimated parameter vector $\hat{\theta}$, coefficient of determination, r^2 and $-2\ln\left(L\left(\hat{\theta}\right)\right)$ value in Table 2 for 3-parameter Weibull model, Weibull mixture model and Weibull competing risk

Table 1
Data Set.

0.478	0.834	2.4	3.393	6.122	−0.484	−1.472	−1.847	−5.9	−7.878
0.583	0.944	2.639	3.904	6.331	−0.626	−1.579	−2.55	−6.226	−7.884
0.753	0.959	2.944	4.829	6.531	−0.85	−1.61	−2.568	−6.711	−10.263
0.753	1.377	2.981	5.328	11.019	−1.071	−1.729	−3.791	−6.835	−13.103
0.801	1.534	3.392	5.562	12.986	−1.318	−1.792	−4.443	−6.947	−23.245

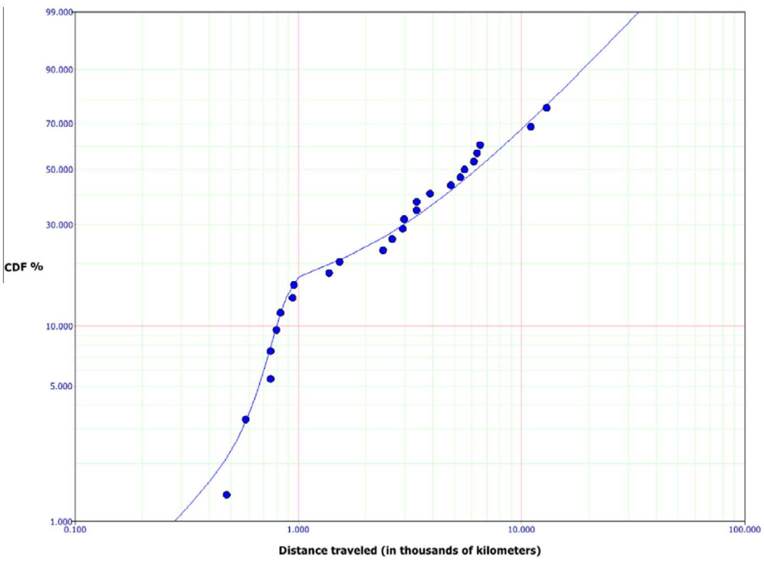

Fig. 1. Cumulative distribution function of mixed-Weibull distribution for data given in Table 1.

Table 2
Estimated parameter vector $\widehat{\theta}$, coefficient of determination, r^2 and $-2\ln\left(L\left(\widehat{\theta}\right)\right)$ value obtained for different Weibull models of application 1.

Model	$\widehat{\theta}$	r^2	$-2\ln\left(L\left(\widehat{\theta}\right)\right)$
3-parameter Weibull model	$(8.1338, 0.7911, 0.4479)$	0.9585	148.9508
Weibull mixture model	$(0.1287, 7.3257, 0.8433, 1.2448, 10.0705)$	0.9730	147.823
Weibull competing risk model	$(0.5561, 94.9132, 1.435, 10.5855)$	0.8837	163.1774

Table 3
Confidence intervals of estimated parameters of Weibull mixture and 3-parameter Weibull models of application 1.

Weibull model	Estimated parameter	Approximated 95% confidence intervals	
		Lower	Upper
Weibull mixture model	$\widehat{\omega}_1 = 0.1287$	0.0498	0.2938
	$\widehat{\beta}_1 = 7.3257$	2.9724	18.0549
	$\widehat{\alpha}_1 = 0.8433$	0.7286	0.976
	$\widehat{\omega}_2 = 0.8713$	0.0877	0.9979
	$\widehat{\beta}_2 = 1.2448$	0.8506	1.8217
	$\widehat{\alpha}_2 = 10.0705$	6.8792	14.7422
3-parameter Weibull model	$\widehat{\alpha} = 8.1338$	4.8709	13.5825
	$\widehat{\beta} = 0.7911$	0.5769	1.0848
	$\widehat{\gamma} = 0.4479$		

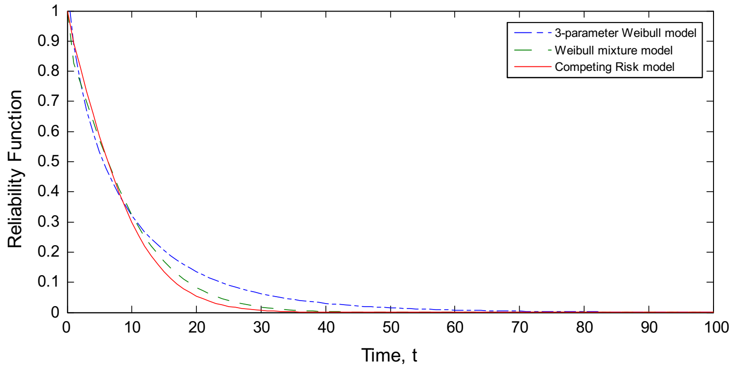

Fig. 2. A comparison of fitted reliability functions of failure times obtained by different models – application 1.

Table 4
Data Set.

1×1	3×2	4×1	9×2	11×1	17×2
22×1	23×3	31×1	35×1	36×1	37×4
39×3	40×2	44×2	45×1	46×3	47×8
48×2	49×3	50×6	51×2	52×14	-9.5×15
-12.5×16	-52.5×76	-53.5×131			

model respectively, we consider both Weibull mixture model and 3-parameter Weibull model potential candidates to fit the data while competing risk model is rejected. We found that Weibull mixture model has the smallest $-2\ln\left(L\left(\widehat{\theta}\right)\right)$ value and the largest r^2, so we can analyze these data as a Weibull mixture model. The confidence intervals of parameter estimates for potential candidates which fit life data are calculated by using Fisher's matrix, see Table 3.

Based on the estimated parameters in Table 2 and using Eq. (3), Eq. (8) and Eq. (15), we obtain reliability (survivor) functions of the three different models as illustrated in Fig. 2. Furthermore, we obtain 95% confidence intervals for the estimated parameters obtained by using the proposed approach, see Table 3. We can deduce that the suggested model is the best fit.

Table 5
Estimated parameter vector $\widehat{\theta}$, coefficient of determination, r^2 and $-2\ln\left(L\left(\widehat{\theta}\right)\right)$ value obtained for different Weibull models of application 2.

Model	$\widehat{\theta}$	r^2	$-2\ln\left(L\left(\widehat{\theta}\right)\right)$
3-parameter Weibull model	$(91.9004, 2.6003, -2.135)$	0.1777	796.0627
Weibull mixture model	$(0.3955, 0.8379, 406.9116, 8.7576, 60.1097)$	0.9739	759.2794
Competing Risk model	$(0.8602, 928.8263, 10.0298, 61.7253)$	0.9826	754.7394

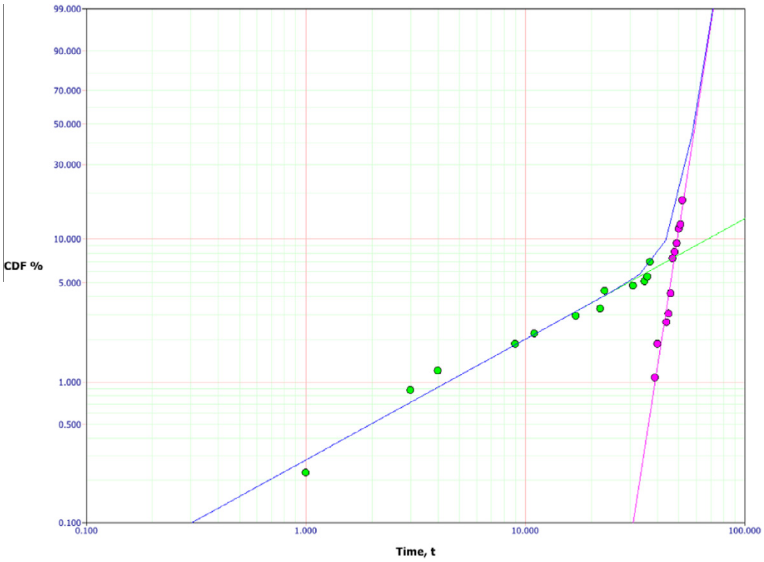

Fig. 3. Cumulative distribution function of Weibull competing risk model for data given in Table 4.

Table 6
Confidence intervals of estimated parameters of Weibull competing risk and Weibull mixture models of application 2.

Weibull model	Estimated parameter	Approximated 95% confidence intervals	
		Lower	Upper
Weibull competing risk model			
	$\widehat{\beta}_1 = 0.8602$	0.6539	1.1316
	$\widehat{\alpha}_1 = 928.8263$	412.3583	2092.1571
	$\widehat{\beta}_2 = 10.0298$	7.8762	12.7722
	$\widehat{\alpha}_2 = 61.7253$	59.0631	64.5075
Weibull mixture model			
	$\widehat{\omega}_1 = 0.3955$	0.0096	0.9778
	$\widehat{\beta}_1 = 0.8379$	0.462	1.5198
	$\widehat{\alpha}_1 = 406.9116$	36.0037	4598.8888
	$\widehat{\omega}_2 = 0.6045$	0.0043	0.9982
	$\widehat{\beta}_2 = 8.7576$	3.7538	20.4314
	$\widehat{\alpha}_2 = 60.1097$	42.1046	85.8144

6.3. Application 2

The data in Table 4 [7] show the Locomotive Power units Overhaul Life, Southern Pacific's high horsepower diesel electric locomotives had 16 power assemblies (units) each. Here we consider the case of failure of a power unit results in a power loss and unscheduled replacement. Failures (66) and suspensions (238) for 304 units are listed by age in months and units failed. Negative age indicates suspension, note that – in Table 4 means suspension.

6.4. Statistical inference for different Weibull models of application 2

Applying the proposed approach for modeling data sample of application 2. Based on the estimated parameter vector $\widehat{\theta}$, coefficient of determination, r^2 and $-2\ln\left(L\left(\widehat{\theta}\right)\right)$ value in Table 5 which are requirements of proposed approach for

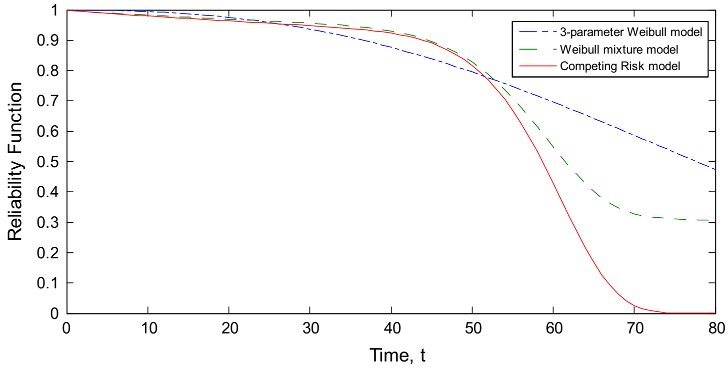

Fig. 4. A comparison of fitted reliability functions of failure times obtained by different models – application 2.

3-parameter Weibull model, Weibull mixture model and competing risk model respectively, we consider Weibull competing risk model and Weibull mixture model potential candidates to fit the data while 3-parameter Weibull model is rejected. We can analyze these data as a Weibull competing risk for two reasons, the first reason is due to the Weibull probability plot (WPP) for the data which is a convex curve with a left asymptote and a right asymptote or may be considered as a straight line, curving into a second line with a steeper slope, which is called in this case Classic Bi-Weibull as shown in Fig. 3. and the second reason, we found that Weibull competing risk model has the smallest $-2\ln\left(L\left(\hat{\theta}\right)\right)$ value and the largest r^2. The confidence intervals of parameter estimates for potential candidates which fit lifetime data are calculated by using Fisher's matrix, see Table 6.

Based on the estimated parameters in Table 5 and using Eq. (3), Eq. (8) and Eq. (15), we obtain reliability (survivor) functions of the three different models as illustrated in Fig. 4. Furthermore, we obtain 95% confidence intervals for the estimated parameters obtained by using the proposed approach, see Table 6. We can deduce that the suggested model is the best fit.

7. Conclusion

In concluding this paper, the proposed approach characterizes the distribution of life data or lifetimes to failure/suspension for the system components by different Weibull models. It's an efficient approach especially when the mixture is well mixed for moderate and large samples with a heavily censored data set and few exact failure times. It can be applied to the complete, censored, grouped and ungrouped samples. This paper also presents a comparison of the fitted reliability functions of the 3-parameter Weibull, competing risk and Weibull mixture models. Numerical application with censored, grouped and ungrouped life data used through the proposed approach. GOF based on r^2 and $-2\ln\left(L\left(\hat{\theta}\right)\right)$ value is used to determine the best distribution for modeling life data, the priority is for the Weibull model which has the smallest $-2\ln\left(L\left(\hat{\theta}\right)\right)$ value and the largest r^2. We can use the proposed method for other finite Weibull models, it can be applied on a simple Weibull mixture and Weibull competing risk mixture due to infant mortality and chance failure modes or Weibull competing risk mixture due to chance and wear-out failure modes.

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Appendix A. Supplementary data

Supplementary data associated with this article can be found, in the online version, at http://dx.doi.org/10.1016/j.amc.2014.10.036.

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