(Ex) Let $T: \mathbb{R}^2 \to M_2(\mathbb{R})$, $B_{\mathbb{R}^2} = \{(2,0), (1,1)\}$ and $B_{M_2(\mathbb{R})} = \left\{ \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \right\}$. If the matrix of $T$ is $M_T = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 3 & 0 & 0 & 2 \end{bmatrix}$. find $T$?

Solution The goal is To find $T(a,b)$.

Now, since $B_{\mathbb{R}^2}$ is a basis, $(a,b) = x(2,0) + y(1,1)$

$$\Rightarrow \begin{array}{l} 2x + y = a \\ y = b \end{array}$$

$$\Rightarrow x = \frac{a-b}{2}$$

(STEP1) So, $[(a,b)]_{B_{\mathbb{R}^2}} = \left[ \frac{a-b}{2} \quad b \right]$

(STEP2) So, $T(a,b) = \left[ \frac{a-b}{2} \quad b \right] \begin{bmatrix} 1 & 1 & -1 & 0 \\ 3 & 0 & 0 & 2 \end{bmatrix}$

$$= \left[ \frac{a+5b}{2} \quad \frac{a-b}{2} \quad \frac{b-a}{2} \quad 2b \right]$$

(STEP3) Now $T(a,b) = \frac{a+5b}{2} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} +$

$$\frac{a-b}{2} \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix} +$$

$$\frac{b-a}{2} \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} +$$

$$2b \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} a+5b & \frac{3(a-b)}{2} \\ \frac{a-b}{2} & b \end{bmatrix}$$

Hence, $T(a,b) = \begin{bmatrix} a+5b & \frac{3(a-b)}{2} \\ \frac{a-b}{2} & b \end{bmatrix}$.

## ** Rank and Nullity:

Let $T: V_1 \to V_2$ be a linear transformation.

① Nullity $(T)$ = Dim $(\text{Ker } T)$
② Rank of $T$ = Dim $(\text{Im } T)$

Notice that: $\text{Dim}(T) = \text{Nullity}(T) + \text{Rank of } T$
$= \text{Dim}(\text{Im}(T)) + \text{Dim}(\text{Ker}(T))$

Example: Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be linear transformation where $T(x,y,z) = (x+2y, 0, 3z)$.

① Find Ker $T$ and Im $T$?
② Find Nullity of $T$ and rank of $T$?

Solution: (1) Let $v = (x,y,z) \in \text{Ker } T$

$\Rightarrow T(x,y,z) = (0,0,0)$
$\Rightarrow (x+2y, 0, 3z) = (0,0,0)$
$\Rightarrow 3z = 0$ and $x+2y = 0$
$\Rightarrow z = 0$ and $y = t \wedge x = -2t$

So, $\text{Ker } T = \{(-2t, t, 0) : \text{where } t \in \mathbb{R}\}$

Now, To find image of $T$, suppose that $v \in \mathbb{R}^3$ and there exists $v' \in \mathbb{R}^3$ (domain) s.t.

$T(v') = v \Rightarrow T(a,b,c) = (x,y,z)$
$\Rightarrow (a+2b, 0, 3c) = (x,y,z)$
$\Rightarrow x = a+2b \Rightarrow$ (there is no constraints)
$y = 0$
$z = 3c \Rightarrow$ (there is no costraints)

$\Rightarrow \text{Im}(T) = \{(x,y,z) : y = 0\}$
$= \{(x,0,z)\}$

(For Justifying: $(3,0,5) = T(1,1,\frac{5}{3})$).

(2) $B_{\text{Ker} T} = \{(-2,1,0)\} \Rightarrow \text{Dim}(\text{Ker } T) = 1 = \text{Nullity}(T)$

$B_{\text{Im } T} = \{(1,0,0), (0,0,1)\} \Rightarrow \text{Dim}(\text{Im}(T)) = 2$
$= \text{Rank}(T)$

## ** Eigen Values and Eigen Vectors :

**Definition :** Let $T: V \to V$ be a linear transformation. $0 \neq A \in V$ is called Eigen vector of $T$ if

$$\boxed{T(A) = \lambda A} \quad \text{for some } \lambda \in \mathbb{R}.$$

and

$\lambda \in \mathbb{R}$ is called Eigen value of $T$ if

There exists $0 \neq A \in V$ where $\boxed{T(A) = \lambda A}$

**Remark :** ① The eigenvector always $\neq 0$ , but the eigen value may $= 0$

② If $T: V \to V$ is linear transformation, it is not necessary that there exists eigen vectors or eigen values.

**Example :** Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ where $T(x,y) = (-y,x)$

suppose that $T(x,y) = \lambda(x,y)$ for some $\lambda \in \mathbb{R}$

$$\Rightarrow (-y,x) = (\lambda x, \lambda y)$$

$$\Rightarrow -y = \lambda x \quad \text{and} \quad x = \lambda y$$

$$\Rightarrow -y = \lambda(\lambda y)$$

$$\Rightarrow -y = \lambda^2 y$$

$$\Rightarrow (\lambda^2+1)\, y = 0$$

as $\lambda^2+1 \neq 0 \Rightarrow y = 0$

$$\Rightarrow x = 0$$

$\Rightarrow v = (0,0)$ (is not eigen vector by remark 1)

**Example :** Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ where $T(x,y) = (x+2y, 3x+2y)$

Suppose that $T(x,y) = \lambda(x,y)$ for some $\lambda \in \mathbb{R}$

$$\Rightarrow (x+2y, 3x+2y) = (\lambda x, \lambda y)$$

$$\Rightarrow x+2y = \lambda x \quad \text{and} \quad 3x+2y = \lambda y$$

$$\Rightarrow \left.\begin{aligned} (1-\lambda)x + 2y &= 0 \\ 3x + (2-\lambda)y &= 0 \end{aligned}\right\} \text{Homogenouse}$$

we are looking for non-zero solutions

which means that $\begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} = 0$

$$\Leftrightarrow (1-\lambda)(2-\lambda) - 6 = 0$$
$$\Leftrightarrow \lambda^2 - 3\lambda - 4 = 0$$
$$\Rightarrow \boxed{\lambda = 4} \text{ or } \boxed{\lambda = -1}$$

So, the Eigen values : $\boxed{\lambda = 4}$ and $\boxed{\lambda = -1}$

<u>Now</u> Let $\lambda = -1$, we get:

$$\begin{cases} 2x + 2y = 0 \\ 3x + 3y = 0 \end{cases}$$

$$\Leftrightarrow x + y = 0$$

$S = \left\{ \begin{bmatrix} t \\ -t \end{bmatrix}, t \in \mathbb{R} \right\}$ the set of all eigen vectors Coordinates to $\boxed{\lambda = -1}$

Let $\lambda = 4$, we get :

$$\begin{cases} -3x + 2y = 0 \\ 3x - 2y = 0 \end{cases}$$

$$\Leftrightarrow x = \frac{2}{3} y$$

$S = \left\{ \begin{bmatrix} 2/3\, t \\ t \end{bmatrix}, t \in \mathbb{R} \right\}$ the set of all eigen vectors Coordinates to $\boxed{\lambda = 4}$

** Next method will ease the method to find eigen values and eigenvectors for any linear transformation.

we will start with the following definition:

Def : (characterstic Polynomial) of $T$.

Let $T : V \longrightarrow V$ be a linear transformation where $M$ is the matrix of $T$. The charchterstic Polynomial $= \Delta = \det(M - \lambda I)$ where $\lambda \in \mathbb{R}$.

for example: $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ ; $T(x,y) = (2x - y, 4x)$

$T(1,0) = (2,4)$ $\quad T(0,1) = (-1,0)$

$$\Rightarrow M = \begin{bmatrix} 2 & -1 \\ 4 & 0 \end{bmatrix} \Rightarrow \Delta = |M - \lambda I|$$
$$= \left| \begin{bmatrix} 2 & -1 \\ 4 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right|$$

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$$= \left| \begin{bmatrix} 2-\lambda & -1 \\ 4 & -\lambda \end{bmatrix} \right| = (2-\lambda)(-\lambda) + 4$$
$$= \lambda^2 - 2\lambda + 4 .$$

Rule : The ~~WNRN~~ eigen values of $T$ is the solution of $\Delta = 0$ (i.e. $|M - \lambda I| = 0$)

for example : In Previous example, let

$$\Delta = 0$$
$$\Leftrightarrow \lambda^2 - 2\lambda + 4 = 0$$
$$\Leftrightarrow \lambda = \frac{2 \pm \sqrt{4-16}}{2} = \frac{2 \pm \sqrt{-12}}{2} \notin \mathbb{R}$$
$\Rightarrow$ There is no eigen values of $T$.

Example : find the eigen values of $T : P_1(\mathbb{R}) \longrightarrow P_1(\mathbb{R})$ ; $T(a+bx) = -b + ax$

Solution

(STEP 1) we will find $M$ respect to the basis $\{1, x\}$

$$T(1) = T(1+0x) = x$$
$$= \boxed{0} + \boxed{1}\, x$$
$$T(x) = T(0+1x) = -1$$
$$= \boxed{-1} \cdot 1 + \boxed{0}\, x$$

So, $M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

[STEP 2] $\Delta = |M - \lambda I| = \left| \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right|$

$$= \left| \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} \right| = \lambda^2 + 1$$

[STEP 3] let $\lambda^2 + 1 = 0 \Rightarrow \lambda^2 = -1 \Rightarrow \lambda \notin \mathbb{R}$

So, there is eigen values !!

## Rule 2

Let $\lambda$ be an eigen values of $T$. So, there is an eigen vector $\underline{\underline{X}}$ related to $\lambda$ where $(M-\lambda I)X=0$.

**Example** find the eigen values of $A=\begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$ ?

Solution: $|A-\lambda I|=0$

$$\left| \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\Rightarrow \quad \lambda^2 + 3\lambda + 2 = 0$$

$$\Rightarrow \quad \lambda = -1 \quad \text{or} \quad \lambda = -2$$

To find the eigen vector $X$ of $\boxed{\lambda=-1}$

Put $(A-\lambda I)X=0$

$$\left( \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 3 & -12 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left. \begin{array}{l} 3x_1 - 12x_2 = 0 \\ x_1 - 4x_2 = 0 \end{array} \right\}$$

$$\Rightarrow \quad x_1 - 4x_2 = 0$$

$$\Rightarrow \quad x_1 = t \quad \text{and} \quad x_2 = \frac{1}{4}t$$

So, $x = \begin{bmatrix} 1 \\ 1/4 \end{bmatrix} t$ where $t \neq 0$ (Because eigen vector $\neq 0$)

By the same manner we calcular the eigen vector $x$ of $\boxed{\lambda=-2}$

Example : Find the eigen vectors of $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Solution Firstly, we will find the eigen values $\lambda$ of $A$. For that

Put $|A - \lambda I| = 0$

$$\Rightarrow \left| \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda)^3 = 0$$

$$\Rightarrow \boxed{\lambda = 2}$$

Now, suppose the eigen vector of $\lambda = 2$ is $X$.

Hence $(A - \lambda I) X = 0$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow x_2 = 0$$

$$\text{So, } X = \begin{bmatrix} s \\ 0 \\ r \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + r \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

where 's' and r is <u>not zeros</u>!!

★★ Eigen space :

Let $T: V \longrightarrow V$ be a linear transformation. If $\lambda$ is the Eigen value of $T$ then <u>the set</u> of all eigen vectors of $\lambda$ is called Eigen space.

example : Let $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ ; $T(x, y, z) = (x, y, -2z)$

$$\left.\begin{aligned} T(1,0,0) &= (1,0,0) \\ T(0,1,0) &= (0,1,0) \\ T(0,0,1) &= (0,0,-2) \end{aligned}\right\} \Rightarrow M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

To find eigen values, Put

$\det(M-\lambda I)=0 \Rightarrow \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & -2-\lambda \end{vmatrix} = 0$

$\Rightarrow \boxed{(1-\lambda)^2(-2-\lambda)=0} \rightarrow (*)$

$\Rightarrow \boxed{\lambda=1}$ and $\boxed{\lambda=-2}$ are eigen values.

To find the eigen vectors of $\boxed{\lambda=1}$.

Let $(M-\lambda I)X=0$

$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\Rightarrow -3x_3=0 \Rightarrow x_3=0$.

Hence the eigen vectors

$X = \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} ;\ x \text{ and } y \neq 0 \text{ at same time} \right\}$

$x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

So, $\text{Dim}(X) = 2$ ( Look at (*) it is the Power of $(1-\lambda)$ ) — Rule

Dimension of the eigen space!

** Diagnolization :

Let $T: V \longrightarrow V$ be a Linear transformation. Let $M$ be the matrix of Linear transformation respects to the basis $B$. If $M$ is diognal then $T$ is called diognalizable.

Rule : Let $T: V \rightarrow V$ be Linear transformation. If $\lambda_1, \dots, \lambda_m$ are distinct eigen values and $X_1, \dots, X_m$ are related eigen vactors. Then $\{X_1, \dots, X_m\}$ is Linear independent.