EE-301: Signals and Systems Midterm Exam 2

Electrical Engineering Department, College of Engineering, King Saud University (Fall 321)

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Instructions: 1hours 30 mins allowed, answer all questions

Q1. (20*marks*, 20 mins short answer questions)

(a) The step response of an LTI system is given by $y(t) = 1 + e^{-0.5t}u(t)$. Calculate its impulse response $h(t)$.

(b) Determine the frequency response $H(j\omega)$ for the system in part (a).

(c) Find the frequency response for an LTI system described by the difference equation: $y(n) = 2x(n) + (1/3)y(n-1)$.

(d) Calculate the autocorrelation of the signal $x(n) = (1/3)^n u(n-1)$.

Ans:

(a) $h(t) = \dfrac{dy(t)}{dt} = e^{-0.5t}\delta(t) - 0.5e^{-0.5t}u(t) = e^{0}\delta(t) - 0.5e^{-0.5t}u(t)$

(b) $H(j\omega) = 1 - \dfrac{0.5}{0.5 + j\omega}$

(c) Applying DTFT and the shifting property: $Y(e^{j\omega}) = 2X(e^{j\omega}) + \frac{1}{3}e^{-j\omega}Y(e^{j\omega})$, i.e,

$$Y(e^{j\omega})[1 - \frac{1}{3}e^{-j\omega}] = 2X(e^{j\omega}) \Rightarrow H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{2}{1 - \frac{1}{3}e^{-j\omega}}$$

OR get $h(n)$ in time domain by applying $x(n) = u(n)$ into the system, and then find $H(e^{j\omega})$ from it:

$h(0) = 2,\ h(1) = (1/3)\cdot 2,\ h(2) = (1/3)^2 \cdot 2,\ \cdots$, so that $h(n) = 2(1/3)^n \cdot u(n)$, and $H(e^{j\omega}) = \dfrac{2}{1 - \frac{1}{3}e^{-j\omega}}$

(d) $R_{XX}(n) = \sum_{k=0}^{\infty} x(k-n)x(k) = \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^{(k-n)} u(k-1-n) \left(\frac{1}{3}\right)^{k} u(k-1)$

For $n+1 > 1$, $R_{XX}(n) = \left(\frac{1}{3}\right)^{-n} \sum_{k=n+1}^{\infty} \left(\frac{1}{3}\right)^{2k} = \left(\frac{1}{3}\right)^{-n} \left[\sum_{m=0}^{\infty} \left(\frac{1}{9}\right)^{m} - \sum_{m=0}^{n} \left(\frac{1}{9}\right)^{m}\right] = \left(\frac{1}{3}\right)^{n} \left(\frac{1}{8}\right)$

For $n+1 \le 1$, $R_{XX}(n) = \left(\frac{1}{3}\right)^{-n} \sum_{k=1}^{\infty} \left(\frac{1}{3}\right)^{2k} = \left(\frac{1}{3}\right)^{-n} \left[\sum_{m=0}^{\infty} \left(\frac{1}{9}\right)^{m} - 1\right] = \left(\frac{1}{3}\right)^{-n} \left(\frac{1}{8}\right)$

Q2. (40 *marks*)

(a) (i) Compute the complex Fourier series (FS) representation for the periodic signal $x(t)$ shown below.

(ii) Determine the DC part and the first harmonics of the signal $x(t)$.

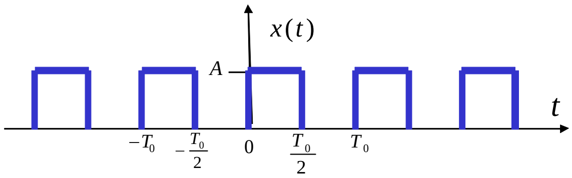
(iii) Write an expression for the FS coefficients of the signal $x(t-T_0/4)$.

(iv) Write an expression for the FS coefficients of the signal $\dfrac{dx(t)}{dt}$.

(v) Compute the Fourier transform (FT) for the signal $x(t)$.

(b) (i) Compute the FS coefficients a_k, for the discrete-time (DT) signal $x(n) = 2\cos(2\pi n)$.

(ii) Sketch the FS coefficients, a_k, obtained in (c) part (i), for the interval $k \in [-3,3]$.



Ans:

(a)

(i) Let $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$; $\omega_0 = \dfrac{2\pi}{T_0}$ {F.S. expansion }

Then we have

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-jk\omega_0 t}dt = \frac{1}{T_0}\int_0^{\frac{T_0}{2}} Ae^{-jk\omega_0 t}dt = \left[\frac{A}{-jk\omega_0 T_0}e^{-jk\omega_0 t}\right]_0^{\frac{T_0}{2}}$$

$$= \frac{A}{-jk\omega_o T_0}\left(e^{\frac{-jk\omega_0 T_0}{2}} - 1\right) = \frac{A}{jk2\pi}\left(1 - e^{-jk\pi}\right) = \frac{A}{jk2\pi}\left[1-(-1)^k\right]$$

since $\omega_o T_0 = 2\pi$ and $e^{-jk\pi} = (-1)^k$. Thus,

$$a_k = 0 \qquad k = 2m \neq 0 \text{ (even coefficients)}$$

$$a_k = \frac{A}{jk\pi} \qquad k = 2m+1 \text{ (odd coefficients)}$$

$$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)dt = \frac{1}{T_0}\int_0^{\frac{T_0}{2}} Adt = \frac{A}{2}$$

Hence we obtain

$$x(t) = \frac{A}{2} + \frac{A}{j\pi}\sum_{m=-\infty}^{\infty}\frac{1}{2m+1}e^{j(2m+1)\omega_0 t}$$

(ii) D. C. part, $a_0 = \dfrac{A}{2}$, first harmonics $a_{\pm 1} = \pm\dfrac{A}{j\pi}$

(iii) F.S. coefficient of $x(t - \frac{T_0}{4}) \xleftrightarrow{F.S.} a_k \cdot e^{-jk\omega_0 \frac{T_0}{4}}$

(iv) F.S. coefficient of $\frac{dx(t)}{dt} \xleftrightarrow{F.S.} a_k \cdot jk\omega_0$

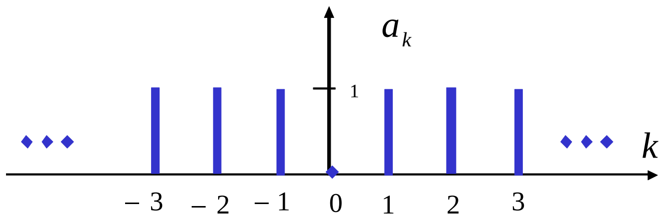
(v) F.T. of periodic signal can be obtained from the FS coefficients as: $x(t) = \sum_{k=-\infty}^{\infty} 2\pi [a_k] \delta(\omega - k\omega_0)$

(b)

(i) $x(n) = 2\frac{1}{2}\left[e^{j2\pi n} + e^{-j2\pi n}\right] = \left[e^{j\omega_0 n} + e^{-j\omega_0 n}\right]$; $\omega_0 = \frac{2\pi}{N} = 2\pi$, and $N = 1$

$a_k = 1$, $k = \pm 1$, and $a_{k+N} = a_k$. For real signal, $a_{-k} = a^*{}_k$.

(ii) Sketch:



Q3. (40 *marks*)

(a) Let $x_1(t) \xleftrightarrow{FT} X_1(j\omega)$ and $x_2(t) \xleftrightarrow{FT} X_2(j\omega)$; be two different FT pairs.

Show that $x_1(t)\cdot x_2(t) \xleftrightarrow{FT} \frac{1}{2\pi}[X_1(j\omega) * X_2(j\omega)]$, where $*$ denotes the convolution operation.

(b) (i) Determine the frequency response $H(j\omega)$ for an LTI system with impulse response $h(t) = e^{-2|t|}$.

(ii) Sketch the magnitude and phase spectrum of the system.

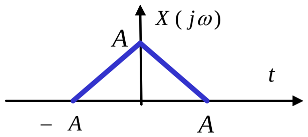
(iii) Calculate the amplitude distortion caused on the DC part of the input signal $x(t) = 1 + \cos 2\pi t$ when processed by the system.

(c) (i) Find the FT for the DT signal $x(n) = (1/2)^n u(n)$.

(ii) Determine the discrete-time signal $x(n)$ having the FT $X(e^{j\omega}) = \cos^2 \omega$.

(d) (i) Determine and sketch the FT $Y(j\omega)$ for the signal $y(t) = x(t)\cos\omega_0 t$, given $X(j\omega)$ as shown below.

(ii) Determine the continuous-time signal $x(t)$ having the FT $X(j\omega) = 0.5P_a(\omega-\omega_0) + 0.5P_a(\omega+\omega_0)$; where $P_a(\omega)$ denotes a rectangular pulse in frequency domain.



Ans:

(a) $$FT\{x_1(t)\cdot x_2(t)\} = \int_{-\infty}^{\infty} x_1(t)\cdot x_2(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} x_1(t)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} X_2(\nu) e^{j\nu} d\nu\right) e^{-j\omega t} dt \quad [\text{let } \lambda = \omega - \nu]$$

$$= \frac{1}{2\pi}\int_{t=-\infty}^{\infty}\int_{\lambda=-\infty}^{\infty} x_1(t) X_2(\omega-\lambda)\, e^{-j\lambda t}\, d\lambda\, dt \quad [\text{integrating w.r.t. } t \text{ first, we get}]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\lambda) X_2(\omega-\lambda) d\lambda = \frac{1}{2\pi} X_1(j\omega) * X_2(j\omega)$$

(b)

(i) Signal $x(t) = e^{-2|t|}$ can be written as

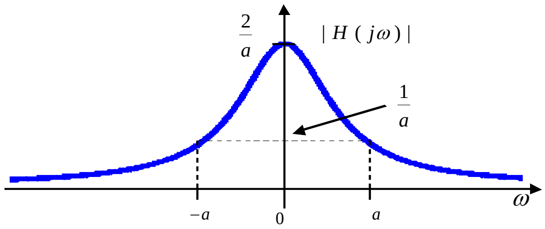
$$x(t) = e^{-a|t|} = \begin{cases} e^{-at} & t > 0 \\ e^{at} & t < 0 \end{cases} \text{ ; where } a=2.$$

Thus $$X(\omega) = \int_{-\infty}^{0} e^{(a-j\omega)t} dt + \int_{0}^{\infty} e^{-(a+j\omega)t} dt = \frac{1}{a-j\omega} + \frac{1}{a+j\omega} = \frac{2a}{a^2+\omega^2}$$

Hence we get

$$e^{-a|t|} \leftrightarrow \frac{2a}{a^2+\omega^2}$$

(ii) Magnitude spectrum sketch (note that $\theta(\omega) = 0$):



(iii) DC part is the component of $x(t)$ with the frequency $\omega = 0$. From $|H(j\omega)|$ in part (ii), this component has amplitude distortion of 2/*a* =1.

(c) (i) $X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} (1/2)^n u(n) e^{-j\omega n} = \dfrac{1}{1-0.5e^{-j\omega}}$

(ii) Recall that the DTFT of a delayed unit impulse is a complex exponential: $\delta(n-n_0) \overset{DTFT}{\Longleftrightarrow} e^{-jn_0\omega}$

Therefore, the inverse DTFT of $X\left(e^{j\omega}\right) = \cos^2\omega$ may easily be found if we expand it in terms of complex exponentials: $X\left(e^{j\omega}\right) = \left(\frac{1}{2}e^{j\omega} + \frac{1}{2}e^{-j\omega}\right)^2 = \frac{1}{2} + \frac{1}{4}e^{j2\omega} + \frac{1}{4}e^{-j2\omega}$

Thus, it follows that $x(n)$ is

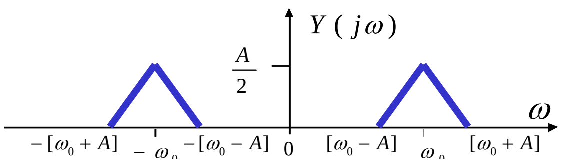
$$x(n) = \frac{1}{2}\delta(n) + \frac{1}{4}\delta(n+2) + \frac{1}{4}\delta(n-2)$$ [note: directly calculating DTFT^{-1}\{ $\cos^2\omega$ \} gives same answer]

(d)

(i) Using the multiplication (in time) property,

$$Y(j\omega) = \frac{1}{2\pi} X(\omega) * [\pi\delta(\omega+\omega_0) + \pi\delta(\omega-\omega_0)] = \frac{1}{2} X(\omega+\omega_0) + X(\omega-\omega_0)$$

<u>Sketch</u>:



(ii) Note that $X(j\omega) = \dfrac{1}{2\pi} P_a(\omega) * [\pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)]$

Therefore using the convolution (in frequency) property, $x(t) = \dfrac{\sin at}{\pi t}\cos\omega_0 t$

<u>Sketch</u>: $x(t)$ is a cosine wave but with the amplitude of a sinc function.