EE-301: Signals and Systems Midterm Exam 2

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Instructions: 1hours 30 mins allowed, answer all questions

Q1. (20*marks*, 20 mins short answer questions)

(a) The step response of an LTI system is given by $y(t) = e^{-0.5t}u(t)$. Calculate its impulse response $h(t)$.

(b) The impulse response of discrete-time (DT) LTI system is $h(n) = \delta(n) - 0.5\delta(n-1)$, find its step response.

(c) Find the frequency response for an LTI system described by the difference equation: $y(n) = x(n) + x(n-1)$. State whether the impulse response of the system is finite or infinite.

(d) Calculate the autocorrelation of the signal $x(t) = e^{-t}u(t)$.

Ans:

(a) $h(t) = \dfrac{dy(t)}{dt} = e^{-0.5t}\delta(t) - 0.5e^{-0.5t}u(t)$

(b) $s(n) = u(n) * h(n) = u(n) * [\delta(n) - 0.5\delta(n-1)] = u(n) - 0.5u(n-1)$

(c) Applying DTFT and the shifting property: $Y(e^{j\omega}) = X(e^{j\omega}) + e^{-j\omega}X(e^{j\omega})$, i.e,

$$Y(e^{j\omega}) = X(e^{j\omega})[1 + e^{-j\omega}] \Rightarrow H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = [1 + e^{-j\omega}],$$

Note that $h(n) = \delta(n) - \delta(n-1)$, hence it is an FIR system.

(d) $R_{XX}(t) = \int_{-\infty}^{\infty} x(\tau - t)x(\tau)d\tau$

For t > 0: $R_{XX}(t) = \int_{t}^{\infty} (e^{-(\tau-t)})(e^{-\tau})d\tau = e^{t}\int_{t}^{\infty} e^{-2\tau}d\tau = \dfrac{e^{-t}}{2}$

For t < 0: $R_{XX}(t) = \int_{0}^{\infty} (e^{-(\tau-t)})(e^{-\tau})d\tau = e^{t}\int_{0}^{\infty} e^{-2\tau}d\tau = \dfrac{e^{t}}{2}$

Thus $R_{XX}(t) = \dfrac{1}{2}e^{-|t|}$

Q2. (40 *marks*)

(a). [30] Consider a periodic CT signal x(t) shown in the figure below.

(i) Compute the general formula for Fourier series coefficients, a_k for the periodic signal x(t).

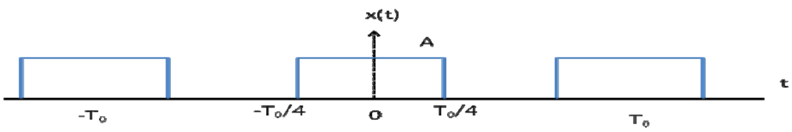
(ii) Using part (i), write down the general formula for Fourier series coefficients, a_k for signal $\frac{dx(t)}{dt}$

(iii) Using part (i), write down the general formula for Fourier series coefficients, a_k for signal x(-t)

(iv) Using part (i) , sketch the Fourier series coefficients for $k \in [-3, 3]$

(v) Using (i) and (iv) write down the Fourier series representation for $k \in [-3, 3]$, and then using this determine the continuous-time Fourier transform (CTFT) for x(t).

(b). [10] Compute the discrete-time Fourier series (DTFS) a_k for the signal $x[n] = 1 + \sin(\frac{2\pi}{6}n + \frac{\pi}{4})$, sketch the amplitude of coefficients for $k \in [-14, 14]$.



Ans:

(a)

(i) Let $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$; $\omega_0 = \frac{2\pi}{T_0}$ {F.S. expansion }

Then we have $a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-jk\omega_0 t}dt = \frac{1}{T_0}\int_{-\frac{T_0}{4}}^{\frac{T_0}{4}} Ae^{-jk\omega_0 t}dt = \frac{A}{-jk\omega_o T_0}\left(e^{\frac{-jk\omega_0 T_0}{4}} - e^{\frac{jk\omega_0 T_0}{4}}\right)$

$$= \frac{A}{jk2\pi}\left(e^{jk\frac{\pi}{2}} - e^{-jk\frac{\pi}{2}}\right) = \frac{A}{k\pi}\sin\left(\frac{k\pi}{2}\right),\ k \neq 0$$

For k=0, we have $a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)dt = \frac{1}{T_0}\int_{-\frac{T_0}{4}}^{\frac{T_0}{4}} Adt = \frac{A}{2}$

Hence we obtain $x(t) = \frac{A}{2} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{A}{k\pi}\sin\left(\frac{k\pi}{2}\right)e^{jk\omega_0 t}$

(ii) Using properties, F.S. coefficient of $\frac{dx(t)}{dt} \xleftrightarrow{F.S.} a_k \cdot jk\omega_0$, where a_k is as obtained in part (i)

(iii) Using properties, F.S. coefficient of $x(-t) \xleftrightarrow{F.S.} a_{-k}$, where a_k is as obtained in part (i)

$$a_{-k} = \begin{cases} \frac{A}{-k\pi}\sin\left(\frac{-k\pi}{2}\right) = \frac{A}{k\pi}\sin\left(\frac{k\pi}{2}\right), & k \neq 0 \\ \frac{A}{2}, & k = 0 \end{cases}$$

(iv) $a_0 = \frac{A}{2}$, $a_1 = a_{-1} = \frac{A}{\pi}$, $a_2 = a_{-2} = 0$, $a_3 = a_{-3} = -\frac{A}{3\pi}$

(v) For the range $k \in [-3, 3]$, $x(t) = \sum_{k=-3}^{3} a_k e^{jk\omega_0 t}$; $\omega_0 = \frac{2\pi}{T_0}$

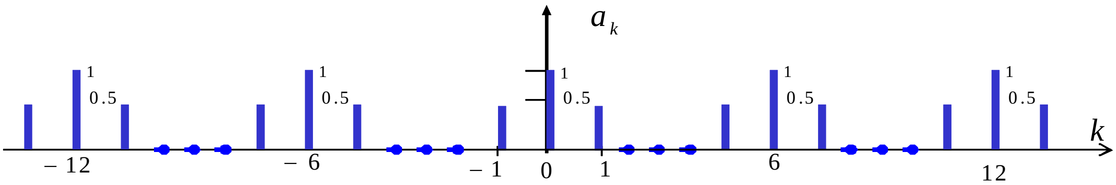
Then the F.T. is given by $X(j\omega) = \sum_{k=-3}^{3} 2\pi a_k \delta(\omega - k\omega_0)$, with a_0, $a_{\pm1}$, $a_{\pm2}$, $a_{\pm3}$ given in part (iv)

(b) $x(n) = 1 + \sin(\frac{2\pi}{6}n + \frac{\pi}{4})$. Note that $x(n)$ is periodic with fundamental period N=6, and $\omega_0 = \frac{2\pi}{6}$.

Using the Euler's formula: $x(n) = 1 + \frac{1}{2j}\left[e^{j\frac{2\pi}{6}n} e^{j\frac{\pi}{4}} - e^{-j\frac{2\pi}{6}n} e^{-j\frac{\pi}{4}} \right] = 1 + \frac{1}{2j}\left[e^{j\omega_0 n} e^{j\frac{\pi}{4}} + e^{-j\omega_0 n} e^{-j\frac{\pi}{4}} \right]$.

Hence $a_0 = 1$, $a_k = \begin{cases} \frac{1}{2j} e^{j\frac{\pi}{4}} & k = 1 \\ -\frac{1}{2j} e^{-j\frac{\pi}{4}} & k = -1 \end{cases}$. Thus $|a_1| = |a_{-1}| = \frac{1}{2}$, and $|a_{k+N}| = |a_k|$.

Sketch for $k \in [-14, 14]$:

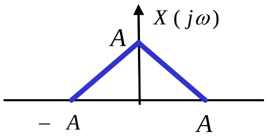


Q3. (40 *marks*)

(a) (i) Determine the frequency response $H(j\omega)$ for an LTI system with impulse response $h(t) = e^{-0.5t}u(t)$.

(ii) Sketch the magnitude and phase spectrum of the system.

(b) (i) Determine and sketch the FT $Y(j\omega)$ for the signal $y(t) = x(t)\sin 2\pi t$, given $X(j\omega)$ as shown below.



(ii) Determine the continuous-time signal $x(t)$ having the FT $X(j\omega) = \begin{cases} 2 & |\omega| < 5 \\ 0 & \text{otherwise} \end{cases}$.

(c) (i) Find the FT for the DT signal $x(n) = (1/2)^{n-1}u(n-1)$.

(ii) Determine the discrete-time signal $x(n)$ having the FT $X(e^{j\omega}) = 1 + 3e^{-j\omega} + 2e^{-j2\omega}$.

(d) Given the Fourier Transform pairs $x(t) = e^{-|t|} \xleftrightarrow{FT} X(j\omega) = \dfrac{2}{1+\omega^2}$,

(i) Use appropriate Fourier transform properties to find the Fourier transform of $y(t) = te^{-|t|}$.

(ii) Using the result in part (i), and along with the duality property, determine the Fourier transform of $y(t) = \dfrac{4t}{(1+t^2)^2}$.

[Hint: if a signal g(t) has F.T. G(jω), then for another time-domain signal G(t) same in nature to G(jω), its own F.T. can be obtained as F.T.{G(t)}=2π g(jω)].

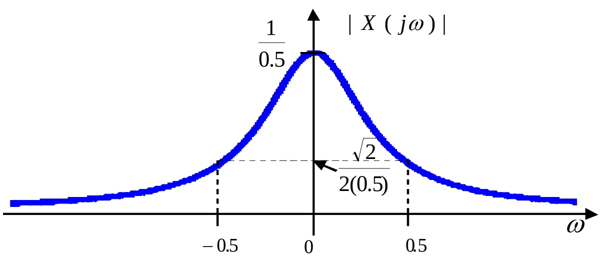
Ans:

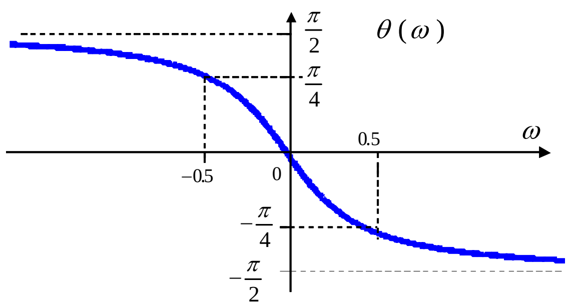
(a)

(i) Note that $h(t) = e^{-0.5t} \xleftrightarrow{FT} H(j\omega) = \dfrac{1}{0.5 + j\omega}$

Thus $|H(j\omega)| = \dfrac{0.5 - j\omega}{0.5^2 + \omega^2} = \dfrac{\sqrt{0.5^2 + \omega^2}}{(0.5^2 + \omega^2)}$, and $\theta(\omega) = -\tan^{-1}(2\omega)$,

(ii) Magnitude and Phase spectra sketches:

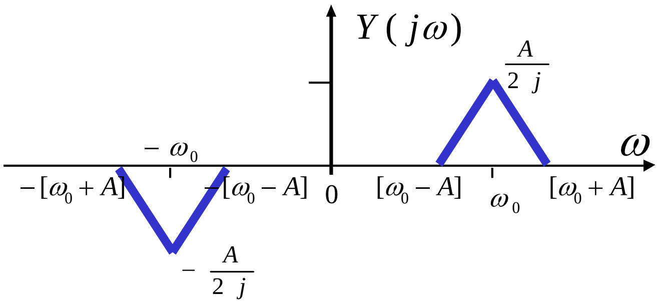




(b)

(i) Using the multiplication (in time) property,

$$Y(j\omega) = \frac{1}{2\pi} X(\omega) * [\frac{1}{2j} 2\pi\delta(\omega - \omega_0) - \frac{1}{2j} 2\pi\delta(\omega + \omega_0)] = \frac{1}{2j} X(\omega - \omega_0) - \frac{1}{2j} X(\omega + \omega_0);\ \omega_0 = 2\pi$$



(ii) Using inverse F.T., $x(t) = \frac{1}{2\pi}\int_{-5}^{5} 2e^{j\omega t} d\omega = \frac{2}{2\pi(jt)}\left[e^{j5t} - e^{-j5t}\right] = \frac{2}{\pi t}\sin(5t)$

(c)

(i) Note that $x(n) = (1/2)^n u(n) \xleftrightarrow{FT} X(e^{j\omega}) = \frac{1}{1-0.5e^{-j\omega}}$

Using DTFT property, $x(n) = (1/2)^{n-1} u(n-1) \xleftrightarrow{FT} X(e^{j\omega}) = \frac{e^{-j\omega}}{1-0.5e^{-j\omega}}$

(ii) Note that $\delta(n) \xleftrightarrow{FT} 1$, $\delta(n-1) \xleftrightarrow{FT} e^{-j\omega}$, $\delta(n-2) \xleftrightarrow{FT} e^{-j2\omega}$. Thus using the linearity property of F.T., $x(n) = \delta(n) + 3\delta(n-1) + 2\delta(n-2)$

(d)

(i) Note that $x(t) = e^{-|t|} \xleftrightarrow{FT} X(j\omega) = \frac{2}{1+\omega^2}$

Using the freq differentiation property, $tx(t) \xleftrightarrow{FT} j\frac{d}{d\omega}[X(j\omega)]$

Hence $y(t) = te^{-|t|} \xleftrightarrow{FT} Y(j\omega) = j\frac{d}{d\omega}[\frac{2}{1+\omega^2}] = \frac{-4j\omega}{(1+\omega^2)^2}$

(ii) Using duality property,

$$g(t) \xleftrightarrow{FT} G(j\omega) \;, \quad G(t) \xleftrightarrow{FT} 2\pi\, g(j\omega)$$

Hence,

$$te^{-|t|} \xleftrightarrow{FT} \frac{-4j\omega}{(1+\omega^2)^2},$$

$$\frac{-4jt}{(1+t^2)^2} \xleftrightarrow{FT} 2\pi\omega\, e^{-|\omega|}$$

$$\frac{4t}{(1+t^2)^2} \xleftrightarrow{FT} j2\pi\omega\, e^{-|\omega|}$$