Z - Transform

The z-transform is a very important tool in describing and analyzing digital systems.

It offers the techniques for digital filter design and frequency analysis of digital signals.

Definition of z-transform:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

Where z is a complex variable

For causal sequence, x(*n*) = 0, *n* < 0:

$$X(z) = Z(x(n)) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$= x(0)z^{-0} + x(1)z^{-1} + x(2)z^{-2} + \ldots$$

All the values of z that make the summation to exist form a ***region of convergence***.

Example 1

Problem:

Given the sequence, $x(n) = u(n)$, find the z transform of *x*(*n*).

Solution:

$$X(z) = \sum_{n=0}^{\infty} u(n) z^{-n} = \sum_{n=0}^{\infty} \left(z^{-1}\right)^n = 1 + \left(z^{-1}\right) + \left(z^{-1}\right)^2 + \ldots .$$

We know, $1 + r + r^2 + \ldots = \frac{1}{1-r}$ when $|r| < 1$.

Therefore,

$$X(z) = \frac{1}{1 - z^{-1}} = \frac{1}{1 - \frac{1}{z}} = \frac{z}{z - 1}$$

Region of convergence (ROC)

When, $|z^{-1}| < 1 \Rightarrow |z| > 1$

Example 2

Problem:

Given the sequence, $x(n) = a^n u(n)$, find the z transform of *x*(*n*).

Solution:

$$X(z) = \sum_{n=0}^{\infty} a^n u(n) z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n = 1 + (az^{-1}) + (az^{-1})^2 + \ldots.$$

Therefore, $$X(z) = \frac{1}{1 - az^{-1}} = \frac{1}{1 - \frac{a}{z}} = \frac{z}{z - a}$$

Region of convergence

When, $|az^{-1}| < 1 \Rightarrow |z| > a$

Z-Transform Table

Line No.	$x(n), n \geq 0$	z-Transform $X(z)$	Region of Convergence
1	$x(n)$	$\sum_{n=0}^{\infty} x(n)z^{-n}$	
2	$\delta(n)$	1	$\|z\| > 0$
3	$au(n)$	$\frac{az}{z-1}$	$\|z\| > 1$
4	$nu(n)$	$\frac{z}{(z-1)^2}$	$\|z\| > 1$
5	$n^2u(n)$	$\frac{z(z+1)}{(z-1)^3}$	$\|z\| > 1$
6	$a^n u(n)$	$\frac{z}{z-a}$	$\|z\| > \|a\|$
7	$e^{-na}u(n)$	$\frac{z}{(z-e^{-a})}$	$\|z\| > e^{-a}$
8	$na^n u(n)$	$\frac{az}{(z-a)^2}$	$\|z\| > \|a\|$
9	$\sin(an)u(n)$	$\frac{z\sin(a)}{z^2 - 2z\cos(a) + 1}$	$\|z\| > 1$
10	$\cos(an)u(n)$	$\frac{z[z-\cos(a)]}{z^2 - 2z\cos(a) + 1}$	$\|z\| > 1$
11	$a^n \sin(bn)u(n)$	$\frac{[a\sin(b)]z}{z^2 - [2a\cos(b)]z + a^2}$	$\|z\| > \|a\|$
12	$a^n \cos(bn)u(n)$	$\frac{z[z - a\cos(b)]}{z^2 - [2a\cos(b)]z + a^{-2}}$	$\|z\| > \|a\|$
13	$e^{-an}\sin(bn)u(n)$	$\frac{[e^{-a}\sin(b)]z}{z^2 - [2e^{-a}\cos(b)]z + e^{-2a}}$	$\|z\| > e^{-a}$
14	$e^{-an}\cos(bn)u(n)$	$\frac{z[z - e^{-a}\cos(b)]}{z^2 - [2e^{-a}\cos(b)]z + e^{-2a}}$	$\|z\| > e^{-a}$

Example 3

Problem:

Find z-transform of the following sequences.

a. $x(n) = 10\sin(0.25\pi n)u(n)$ **b.** $x(n) = e^{-0.1n}\cos(0.25\pi n)u(n)$

Solution:

a. From line 9 of the Table:

$$X(z) = 10Z(\sin(0.2\pi n)u(n))$$

$$= \frac{10\sin(0.25\pi)z}{z^2 - 2z\cos(0.25\pi) + 1} = \frac{7.07z}{z^2 - 1.414z + 1}.$$

b. From line 14 of the Table:

$$X(z) = Z\big(e^{-0.1n}\cos(0.25\pi n)u(n)\big) = \frac{z(z - e^{-0.1}\cos(0.25\pi))}{z^2 - 2e^{-0.1}\cos(0.25\pi)z + e^{-0.2}}$$

$$= \frac{z(z - 0.6397)}{z^2 - 1.2794z + 0.8187}.$$

Z- Transform Properties (1)

Linearity: $Z(ax_1(n) + bx_2(n)) = aZ(x_1(n)) + bZ(x_2(n))$

a and *b* are arbitrary constants.

Example 4

Problem:

Find z- transform of $x(n) = u(n) - (0.5)^n u(n)$.

Solution:

Using z- transform table:

$$Z(u(n)) = \frac{z}{z-1}$$ Line 3

$$Z(0.5^n u(n)) = \frac{z}{z-0.5}.$$ Line 6

Therefore, we get $X(z) = \frac{z}{z-1} - \frac{z}{z-0.5}$.

Z- Transform Properties (2)

Shift Theorem:

$$Z(x(n-m)) = z^{-m}X(z)$$

Verification:

n = m

$$Z(x(n-m)) = \sum_{n=0}^{\infty} x(n-m)z^{-n}$$
$$= x(-m)z^{-0} + \ldots + x(-1)z^{-(m-1)} + x(0)z^{-m} + x(1)z^{-m-1} + \ldots.$$

Since $x(n)$ is assumed to be causal: $x(-m) = x(-m+1) = \ldots = x(-1) = 0.$

Then we achieve, $Z(x(n-m)) = x(0)z^{-m} + x(1)z^{-m-1} + x(2)z^{-m-2} + \ldots.$

$$\Rightarrow \quad Z(x(n-m)) = z^{-m}\left(x(0) + x(1)z^{-1} + x(2)z^{-2} + \ldots\right) = z^{-m}X(z).$$

Example 5

Problem:

Find z- transform of $y(n) = (0.5)^{(n-5)} \cdot u(n-5)$,

where $u(n-5) = 1$ for $n \geq 5$ and $u(n-5) = 0$ for $n < 5$.

Solution:

Using shift theorem,

$$Y(z) = Z\left[(0.5)^{n-5} u(n-5)\right] = z^{-5} Z[(0.5)^n u(n)].$$

Using z- transform table, line 6: $Y(z) = z^{-5} \cdot \frac{z}{z-0.5} = \frac{z^{-4}}{z-0.5}$.

Z- Transform Properties (3)

Convolution

In time domain, $x(n) = x_1(n) * x_2(n) = \sum_{k=0}^{\infty} x_1(n-k)x_2(k),$ Eq. (1)

In z- transform domain,

$$\boxed{X(z) = X_1(z)X_2(z).}$$

$X(z) = Z(x(n)),\ X_1(z) = Z(x_1(n)),$ and $X_2(z) = Z(x_2(n)).$

Verification:

Using z- transform in Eq. (1)

$$X(z) = \sum_{n=0}^{\infty} x(n)z^{-n} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} x_1(n-k)x_2(k)z^{-n}.$$

$$\Rightarrow X(z) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} x_2(k)z^{-k}x_1(n-k)z^{-(n-k)}.$$

$$\Rightarrow X(z) = \sum_{k=0}^{\infty} x_2(k)z^{-k}\sum_{n=0}^{\infty} x_1(n-k)z^{-(n-k)}.$$

let $m = n - k$:

$$\Rightarrow X(z) = \sum_{k=0}^{\infty} x_2(k)z^{-k}\sum_{m=0}^{\infty} x_1(m)z^{-m}$$

$$\Rightarrow X(z) = X_2(z)X_1(z) = X_1(z)X_2(z).$$

Example 6

Problem: Given the sequences,

$$x_1(n) = 3\delta(n) + 2\delta(n-1)$$
$$x_2(n) = 2\delta(n) - \delta(n-1),$$

Find the z-transform of their convolution.

Solution:

Applying z-transform on the two sequences,

$$X_1(z) = 3 + 2z^{-1}$$
$$X_2(z) = 2 - z^{-1}.$$

From the table, line 2

Therefore we get,

$$X(z) = X_1(z)X_2(z) = (3 + 2z^{-1})(2 - z^{-1})$$
$$= 6 + z^{-1} - 2z^{-2}.$$

Inverse z- Transform: Examples

Find inverse z-transform of $X(z) = 2 + \frac{4z}{z-1} - \frac{z}{z-0.5}$

Example 7

We get, $x(n) = 2Z^{-1}(1) + 4Z^{-1}\left(\frac{z}{z-1}\right) - Z^{-1}\left(\frac{z}{z-0.5}\right)$

Using table, $x(n) = 2\delta(n) + 4u(n) - (0.5)^n u(n).$

Find inverse z-transform of $X(z) = \frac{5z}{(z-1)^2} - \frac{2z}{(z-0.5)^2}$

Example 8

We get, $x(n) = Z^{-1}\left(\frac{5z}{(z-1)^2}\right) - Z^{-1}\left(\frac{2z}{(z-0.5)^2}\right) = 5Z^{-1}\left(\frac{z}{(z-1)^2}\right) - \frac{2}{0.5}Z^{-1}\left(\frac{0.5z}{(z-0.5)^2}\right)$

Using table, $x(n) = 5nu(n) - 4n(0.5)^n u(n).$

Inverse z- Transform: Examples

Example 9

Find inverse z-transform of $X(z) = \frac{10z}{z^2 - z + 1}$

Since, $X(z) = \frac{10z}{z^2 - z + 1} = \left(\frac{10}{\sin(a)}\right)\frac{\sin(a)z}{z^2 - 2z\cos(a) + 1},$

By coefficient matching, $-2\cos(a) = -1$

Hence, $\cos(a) = 0.5$, and $a = 60^\circ$ ⇒ $\sin(a) = \sin(60^\circ) = 0.866.$

Therefore, $x(n) = \frac{10}{\sin(a)} Z^{-1}\left(\frac{\sin(a)z}{z^2 - 2z\cos(a) + 1}\right) = \frac{10}{0.866}\sin(n \cdot 60^0) = 11.547\sin(n \cdot 60^0).$

Example 10

Find inverse z-transform of $X(z) = \frac{z^{-4}}{z-1} + z^{-6} + \frac{z^{-3}}{z+0.5}$

$$x(n) = Z^{-1}\left(z^{-5}\frac{z}{z-1}\right) + Z^{-1}(z^{-6} \cdot 1) + Z^{-1}\left(z^{-4}\frac{z}{z+0.5}\right)$$

⇒ $x(n) = u(n-5) + \delta(n-6) + (-0.5)^{n-4}u(n-4).$

Inverse z-Transform: Using Partial Fraction

Example 11

Problem:

Find inverse z-transform of $X(z) = \frac{1}{(1 - z^{-1})(1 - 0.5z^{-1})}$

Solution:

First eliminate the negative power of z.

$$X(z) = \frac{z^2}{z^2(1 - z^{-1})(1 - 0.5z^{-1})} = \frac{z^2}{(z - 1)(z - 0.5)}$$

Dividing both sides by z: $\frac{X(z)}{z} = \frac{z}{(z - 1)(z - 0.5)} = \frac{A}{(z - 1)} + \frac{B}{(z - 0.5)}$

Finding the constants:

$$A = (z - 1)\frac{X(z)}{z}\bigg|_{z=1} = \frac{z}{(z - 0.5)}\bigg|_{z=1} = 2,$$

$$B = (z - 0.5)\frac{X(z)}{z}\bigg|_{z=0.5} = \frac{z}{(z - 1)}\bigg|_{z=0.5} = -1$$

$$\frac{X(z)}{z} = \frac{2}{(z - 1)} + \frac{-1}{(z - 0.5)}$$

$$X(z) = \frac{2z}{(z - 1)} + \frac{-z}{(z - 0.5)}$$

Therefore, inverse z-transform is: $x(n) = 2u(n) - (0.5)^n u(n).$

Inverse z-Transform: Using Partial Fraction

Problem:

Find $y(n)$ if $Y(z) = \dfrac{z^2(z+1)}{(z-1)(z^2-z+0.5)}$.

Example 12

Solution:

Dividing both sides by z: $$\frac{Y(z)}{z} = \frac{z(z+1)}{(z-1)(z^2-z+0.5)}.$$

$$\Rightarrow \quad \frac{Y(z)}{z} = \frac{B}{z-1} + \frac{A}{(z-0.5-j0.5)} + \frac{A^*}{(z-0.5+j0.5)}$$

We first find B:

$$B = (z-1)\frac{Y(z)}{z}\bigg|_{z=1} = \frac{z(z+1)}{(z^2-z+0.5)}\bigg|_{z=1} = \frac{1\times(1+1)}{(1^2-1+0.5)} = 4.$$

Next find A:

$$A = (z-0.5-j0.5)\frac{Y(z)}{z}\bigg|_{z=0.5+j0.5} = \frac{z(z+1)}{(z-1)(z-0.5+j0.5)}\bigg|_{z=0.5+j0.5}$$

Example 12 – contd.

$$A = \frac{(0.5+j0.5)(0.5+j0.5+1)}{(0.5+j0.5-1)(0.5+j0.5-0.5+j0.5)} = \frac{(0.5+j0.5)(1.5+j0.5)}{(-0.5+j0.5)j}.$$

Using polar form

$$A = \frac{(0.707\angle 45^\circ)(1.58114\angle 18.43^\circ)}{(0.707\angle 135^\circ)(1\angle 90^\circ)} = 1.58114\angle -161.57^\circ$$

$$A^* = \bar{A} = 1.58114\angle 161.57^\circ.$$

$$P = 0.5 + 0.5j = |P|\angle\theta = 0.707\angle 45^\circ \text{ and } P^* = |P|\angle -\theta = 0.707\angle -45^\circ.$$

Now we have:

$$Y(z) = \frac{4z}{z-1} + \frac{Az}{(z-P)} + \frac{A^*z}{(z-P^*)}.$$

Therefore, the inverse z-transform is:

$$y(n) = 4Z^{-1}\left(\frac{z}{z-1}\right) + Z^{-1}\left(\frac{Az}{(z-P)} + \frac{A^*z}{(z-P^*)}\right)$$

$$\begin{aligned} y(n) &= 4u(n) + 2|A|(|P|)^n \cos(n\theta + \phi)u(n) \\ &= 4u(n) + 3.1623(0.7071)^n \cos(45^\circ n - 161.57^\circ)u(n) \end{aligned}$$

Inverse z-Transform: Using Partial Fraction

Problem:

Find $x(n)$ if $X(z) = \dfrac{z^2}{(z-1)(z-0.5)^2}$.

Example 13

Solution:

Dividing both sides by z:

$$\frac{X(z)}{z} = \frac{z}{(z-1)(z-0.5)^2} = \frac{A}{z-1} + \frac{B}{z-0.5} + \frac{C}{(z-0.5)^2},$$

$$\text{where } A = (z-1)\frac{X(z)}{z}\bigg|_{z=1} = \frac{z}{(z-0.5)^2}\bigg|_{z=1} = 4.$$

$$\frac{R_m}{(z-p)} + \frac{R_{m-1}}{(z-p)^2} + \cdots + \frac{R_1}{(z-p)^m} \qquad R_k = \frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\left((z-p)^m \frac{X(z)}{z}\right)\bigg|_{z=p}$$

m = 2, p = 0.5

$$B = R_2 = \frac{1}{(2-1)!}\frac{d}{dz}\left\{(z-0.5)^2\frac{X(z)}{z}\right\}_{z=0.5}$$

$$= \frac{d}{dz}\left(\frac{z}{z-1}\right)\bigg|_{z=0.5} = \frac{-1}{(z-1)^2}\bigg|_{z=0.5} = -4$$

Example 13 – contd.

$$C = R_1 = \frac{1}{(1-1)!} \frac{d^0}{dz^0} \left\{ (z-0.5)^2 \frac{X(z)}{z} \right\}_{z=0.5}$$

$$= \left. \frac{z}{z-1} \right|_{z=0.5} = -1.$$

Then $X(z) = \frac{4z}{z-1} + \frac{-4z}{z-0.5} + \frac{-1z}{(z-0.5)^2}.$

From Table:

$$Z^{-1}\left\{\frac{z}{z-1}\right\} = u(n),$$

$$Z^{-1}\left\{\frac{z}{z-0.5}\right\} = (0.5)^n u(n),$$

$$Z^{-1}\left\{\frac{z}{(z-0.5)^2}\right\} = 2n(0.5)^n u(n).$$

Finally we get,

$$x(n) = 4u(n) - 4(0.5)^n u(n) - 2n(0.5)^n u(n).$$

Partial Function Expansion Using MATLAB

Problem:

$$X(z) = \frac{1}{(1 - z^{-1})(1 - 0.5z^{-1})}$$

Example 14

Solution:

The denominator polynomial can be found using MATLAB:

```
>> conv([1 -1], [1 -0.5])
D =
1.0000 -1.5000 0.5000
```

Therefore,

$$X(z) = \frac{1}{(1 - z^{-1})(1 - 0.5z^{-1})} = \frac{1}{1 - 1.5z^{-1} + 0.5^{-2}} = \frac{z^2}{z^2 - 1.5z + 0.5}$$

and $\frac{X(z)}{z} = \frac{z}{z^2 - 1.5z + 0.5}$.

```
>> [R,P,K] = residue([1 0], [1 -1.5 0.5])
R =        P =          K =
  2          1.0000       []
 -1          0.5000
```

The solution is:

$$X(z) = \frac{2z}{z - 1} - \frac{z}{z - 0.5}.$$

Partial Function Expansion Using MATLAB

Problem:

$$Y(z)=\frac{z^2(z+1)}{(z-1)(z^2-z+0.5)}$$

Example 15

Solution:

```
>> N = conv([1 0 0], [1 1])
N =
   1 1 0 0
>> D = conv([1 -1], [1 -1 0.5])
D =
   1.0000  -2.0000 1.5000 -0.5000
```

$$Y(z)=\frac{z^2(z+1)}{(z-1)(z^2-z+0.5)}=\frac{z^3+z^2}{z^3-2z^2+1.5z-0.5}$$

and $\frac{Y(z)}{z}=\frac{z^2+z}{z^3-2z^2+1.5z-0.5}$.

```
>> [R,P,K] = residue([1 1 0], [1 -2 1.5 -0.5])
R =
4.0000
   -1.5000 - 0.5000i
   -1.5000 + 0.5000i
P =
   1.0000
   0.5000 + 0.5000i
   0.5000 - 0.5000i
K =
   []
```

$$X(z)=\frac{Bz}{z-p_1}+\frac{Az}{z-p}+\frac{A^*z}{z-p^*},$$

where $B=4$,

$p_1=1$,

$A=-1.5-0.5j$,

$p=0.5+0.5j$,

$A^*=-1.5+0.5j$, and

$p=0.5-0.5j$.

Partial Function Expansion Using MATLAB

Problem:

$$X(z) = \frac{z^2}{(z-1)(z-0.5)^2}$$

Example 16

Solution:

```
>> D = conv(conv([1  -1], [1  -0.5]), [1  -0.5])
D =
   1.0000  -2.0000  1.2500  -0.2500
```

$$X(z) = \frac{z^2}{(z-1)(z-0.5)^2} = \frac{z^2}{z^3 - 2z^2 + 1.25z - 0.25} \Rightarrow \frac{X(z)}{z} = \frac{z}{z^3 - 2z^2 + 1.25z - 0.25}.$$

```
>> [R,P,K] = residue([1 0], [1  -2  1.25  -0.25])
R =
     4.0000
    -4.0000
    -1.0000
P =
    1.0000
    0.5000
    0.5000
K =
    []
```

$$\Rightarrow X(z) = \frac{4z}{z-1} - \frac{4z}{z-0.5} - \frac{z}{(z-0.5)^2}$$

Difference Equation Using Z-Transform

The procedure to solve difference equation using z-transform:

1. Apply z-transform to the difference equation.
2. Substitute the initial conditions.
3. Solve for the difference equation in z-transform domain.
4. Find the solution in time domain by applying the inverse z-transform.

Example 17

Problem:

Solve the difference equation when the initial condition is $y(-1) = 1$.

$$y(n) - 0.5y(n-1) = 5(0.2)^n u(n).$$

Solution:

Taking z-transform on both sides:

$$Y(z) - 0.5\left(y(-1) + z^{-1}Y(z)\right) = 5Z\left(0.2^n u(n)\right)$$

Substituting the initial condition and z-transform on right hand side using Table:

$$Y(z) - 0.5\left(1 + z^{-1}Y(z)\right) = 5z/(z-0.2).$$

Arranging Y(z) on left hand side:

$$Y(z) - 0.5z^{-1}Y(z) = 0.5 + 5z/(z-0.2).$$

$$\Rightarrow \quad Y(z)(1 - 0.5z^{-1}) = (5.5z - 0.1)/(z-0.2)$$

$$\Rightarrow \quad Y(z) = \frac{(5.5z - 0.1)}{(1 - 0.5z^{-1})(z-0.2)} = \frac{z(5.5z-0.1)}{(z-0.5)(z-0.2)}$$

Example 17 - contd.

$$\Rightarrow \quad \frac{Y(z)}{z} = \frac{5.5z - 0.1}{(z-0.5)(z-0.2)} = \frac{A}{z-0.5} + \frac{B}{z-0.2}$$

Solving for A and B:

$$A = (z-0.5)\frac{Y(z)}{z}\bigg|_{z=0.5} = \frac{5.5z-0.1}{z-0.2}\bigg|_{z=0.5} = \frac{5.5\times 0.5 - 0.1}{0.5-0.2} = 8.8333,$$

$$B = (z-0.2)\frac{Y(z)}{z}\bigg|_{z=0.2} = \frac{5.5z-0.1}{z-0.5}\bigg|_{z=0.2} = \frac{5.5\times 0.2 - 0.1}{0.2-0.5} = -3.3333.$$

Therefore, $$Y(z) = \frac{8.8333z}{(z-0.5)} + \frac{-3.3333z}{(z-0.2)}$$

Taking inverse z-transform, we get the solution:

$$\boxed{y(n) = 8.3333(0.5)^n u(n) - 3.3333(0.2)^n u(n)}$$

Example 18

Problem:

A DSP system is described by the following differential equation with zero initial condition:

$$y(n) + 0.1y(n-1) - 0.2y(n-2) = x(n) + x(n-1)$$

a. Determine the impulse response $y(n)$ due to the impulse sequence $x(n) = \delta(n)$.

b. Determine system response $y(n)$ due to the unit step function excitation, where $u(n) = 1$ for $n \geq 0$.

Solution:

Taking z-transform on both sides:

$$Y(z) + 0.1Y(z)z^{-1} - 0.2Y(z)z^{-2} = X(z) + X(z)z^{-1}$$

a.

Applying $X(z) = Z(\delta(n)) = 1$ on right side

$$Y(z)(1 + 0.1z^{-1} - 0.2z^{-2}) = 1(1 + z^{-1})$$

$$\Rightarrow \quad Y(z) = \frac{1 + z^{-1}}{1 + 0.1z^{-1} - 0.2z^{-2}}$$

Example 18 – contd.

We multiply the numerator and denominator by z^2

$$Y(z) = \frac{z^2 + z}{z^2 + 0.1z - 0.2} = \frac{z(z+1)}{(z-0.4)(z+0.5)}$$

$$\Rightarrow \quad \frac{Y(z)}{z} = \frac{z+1}{(z-0.4)(z+0.5)} = \frac{A}{z-0.4} + \frac{B}{z+0.5}$$

Solving for A and B:

$$A = (z-0.4)\frac{Y(z)}{z}\bigg|_{z=0.4} = \frac{z+1}{z+0.5}\bigg|_{z=0.4} = \frac{0.4+1}{0.4+0.5} = 1.5556$$

$$B = (z+0.5)\frac{Y(z)}{z}\bigg|_{z=-0.5} = \frac{z+1}{z-0.4}\bigg|_{z=-0.5} = \frac{-0.5+1}{-0.5-0.4} = -0.5556.$$

Therefore, $$Y(z) = \frac{1.5556z}{(z-0.4)} + \frac{-0.5556z}{(z+0.5)}$$

Hnece the impulse response:

$$y(n) = 1.5556(0.4)^n u(n) - 0.5556(-0.5)^n u(n).$$

Example 18 - contd.

b.

The input is step unit function: $x(n) = u(n)$

Corresponding z-transform: $X(z) = \frac{z}{z-1}$

$$Y(z) + 0.1\,Y(z)z^{-1} - 0.2\,Y(z)z^{-2} = X(z) + X(z)z^{-1}$$ [Slide 24]

$$Y(z) = \left(\frac{z}{z-1}\right)\left(\frac{1+z^{-1}}{1+0.1z^{-1}-0.2z^{-2}}\right) = \frac{z^2(z+1)}{(z-1)(z-0.4)(z+0.5)}$$

$$Y(z) = \frac{2.2222z}{z-1} + \frac{-1.0370z}{z-0.4} + \frac{-0.1852z}{z+0.5}$$

Do the middle steps by yourself!

$$y(n) = 2.2222u(n) - 1.0370(0.4)^n u(n) - 0.1852(-0.5)^n u(n).$$

Example 19

Problem:

Determine the z-transform and the ROC of the signal: $x(n) = \left[3(2^n) - 4(3^n)\right]u(n)$

Solution:

Let $x_1(n) = 2^n u(n)$ and $x_2(n) = 3^n u(n)$

Therefore $x(n) = 3x_1(n) - 4x_2(n)$

Applying z-transform: $X(z) = 3X_1(z) - 4X_2(z)$

But

$$X_1(z) = \frac{1}{1-2z^{-1}}, \quad \text{ROC}: |z| > 2$$

$$X_2(z) = \frac{1}{1-3z^{-1}}, \quad \text{ROC}: |z| > 3$$

Intersection

Finally, $$X(z) = \frac{3}{1-2z^{-1}} - \frac{4}{1-3z^{-1}}, \quad \text{ROC}: |z| > 3$$

Example 20

Problem:

Determine the z-transform and the ROC of the signal: $x(n) = \alpha^n u(n) + b^n u(-n-1)$

Solution:

$$X(z) = \sum_{n=0}^{\infty} \alpha^n z^{-n} + \sum_{n=-\infty}^{-1} b^n z^{-n} = \sum_{n=0}^{\infty} (\alpha z^{-1})^n + \sum_{l=1}^{\infty} (b^{-1} z)^l$$

ROC_1: $|z| > |\alpha|$
ROC_2: $|z| < |b|$

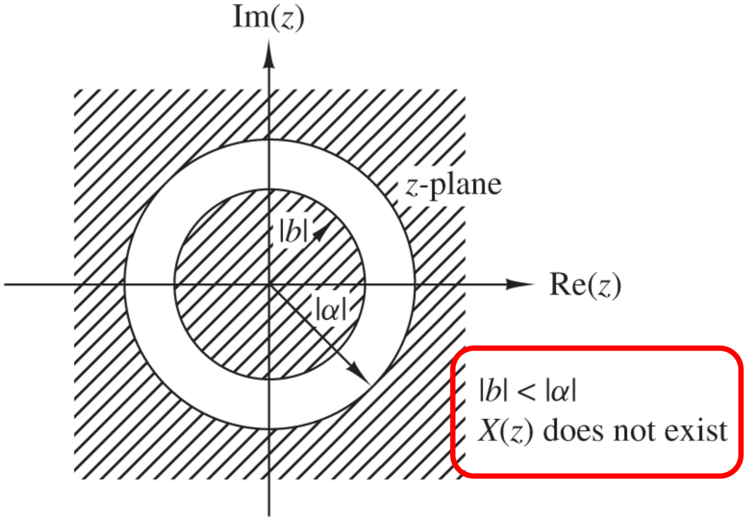
Case (a). $|b| < |\alpha|$

ROCs do not overlap, so X(z) does not exist.

Case (b). $|b| > |\alpha|$

ROC of X(z) is $|\alpha| < z < |b|$

Example 20 - contd.

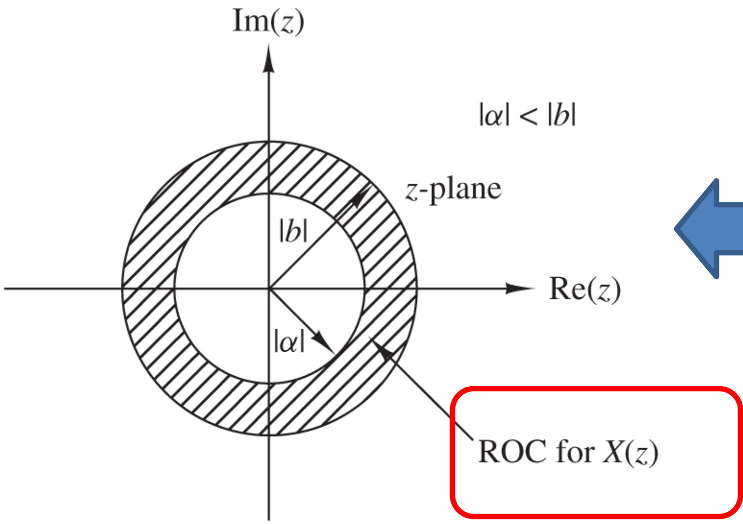


(a)

$$\sum_{l=1}^{\infty} (b^{-1}z)^l$$

$$A + A^2 + A^3 + ... = A(1 + A + A^2 + ...) = \frac{A}{1-A}$$

$$\frac{b^{-1}z}{1-b^{-1}z} = \frac{1}{1-bz^{-1}}$$



(b)

$$X(z) = \frac{1}{1-\alpha z^{-1}} - \frac{1}{1-bz^{-1}} = \frac{b-\alpha}{\alpha + b - z - \alpha b z^{-1}}$$