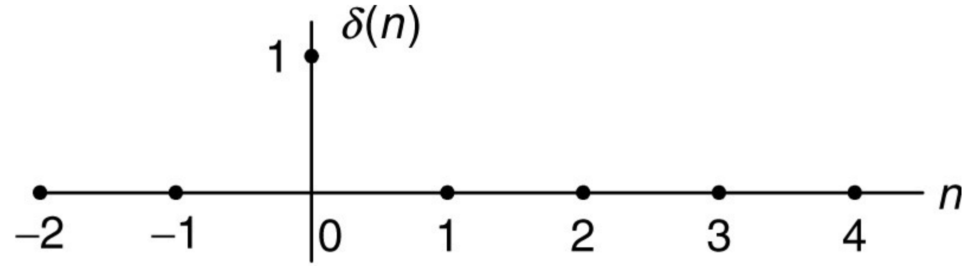
Common Digital Sequences

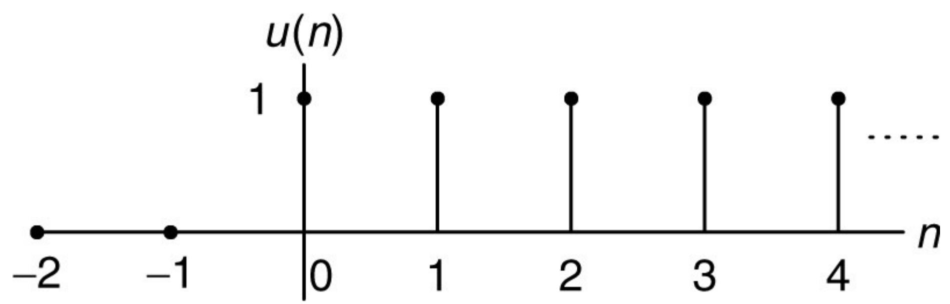
Unit-impulse sequence:

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$



Unit-step sequence:

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$



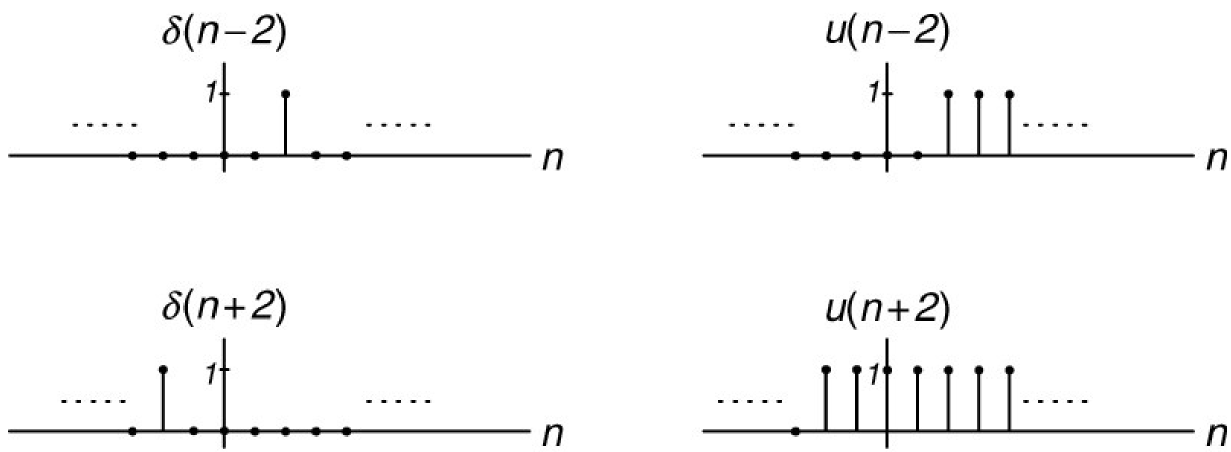
Shifted Sequences

Shifted unit-impulse

Shifted unit-step

Right shift by two samples

Left shift by two samples



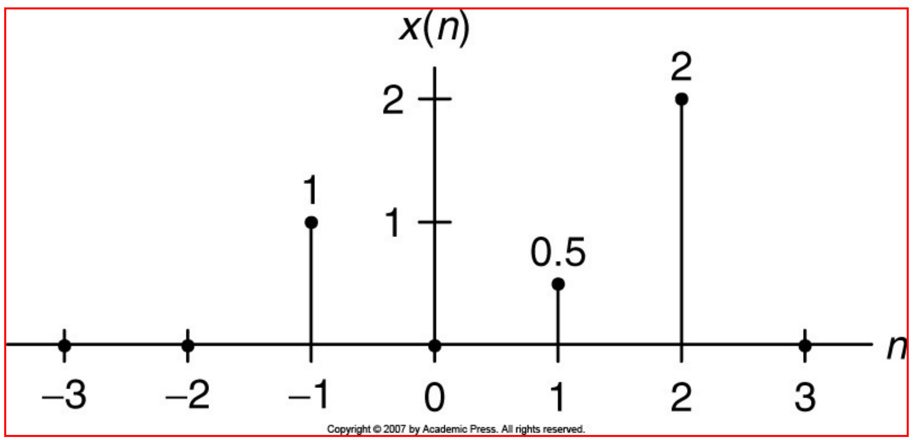
Example 1

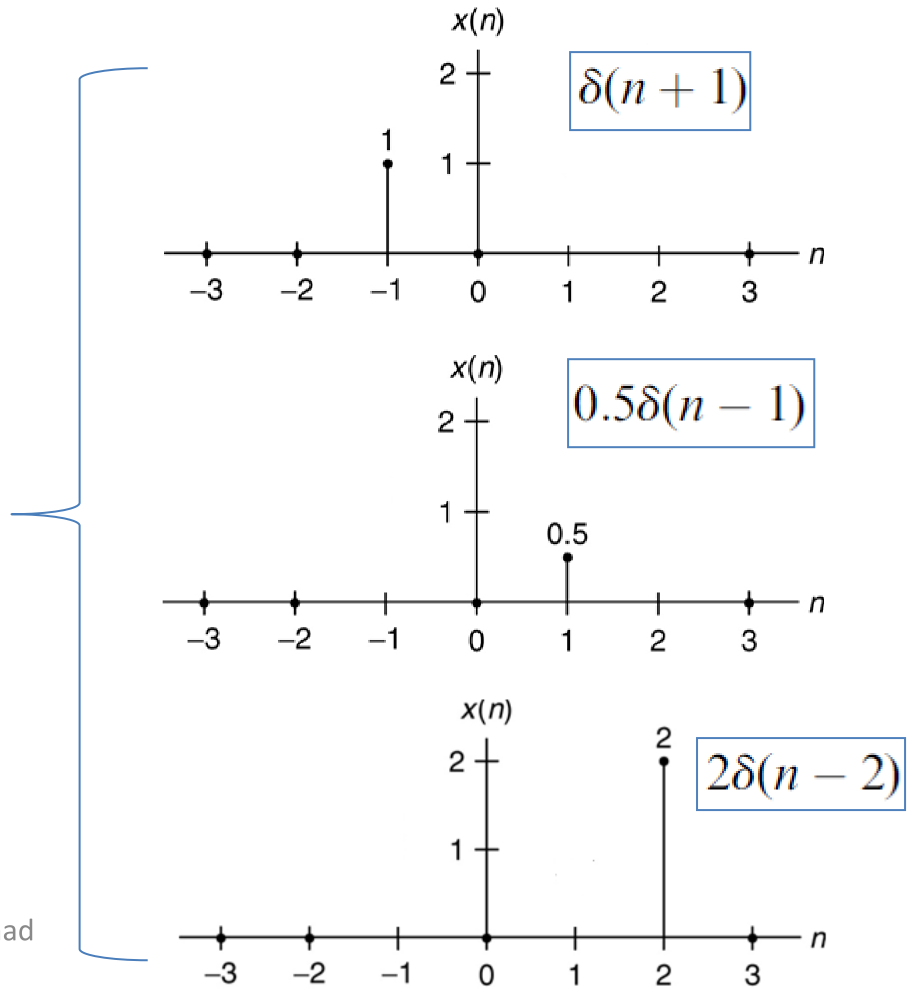
Given the following,

$$x(n) = \delta(n+1) + 0.5\delta(n-1) + 2\delta(n-2),$$

a. Sketch this sequence.

Solution:





Generation of Digital Signals

Let, sampling interval, $\Delta t = T$

x(n): digital signal
x(t): analog signal

$$x(n) = x(t)|_{t=nT} = x(nT)$$

Also $$u(t)|_{t=nT} = u(nT) = u(n)$$

Example 2

Convert analog signal x(t) into digital signal x(n), when sampling period is 125 microsecond, also plot sample values.

$$\boxed{x(t) = 10e^{-5000t}u(t)}$$

Solution:

$$t = nT = n \times 0.000125 = 0.000125n$$

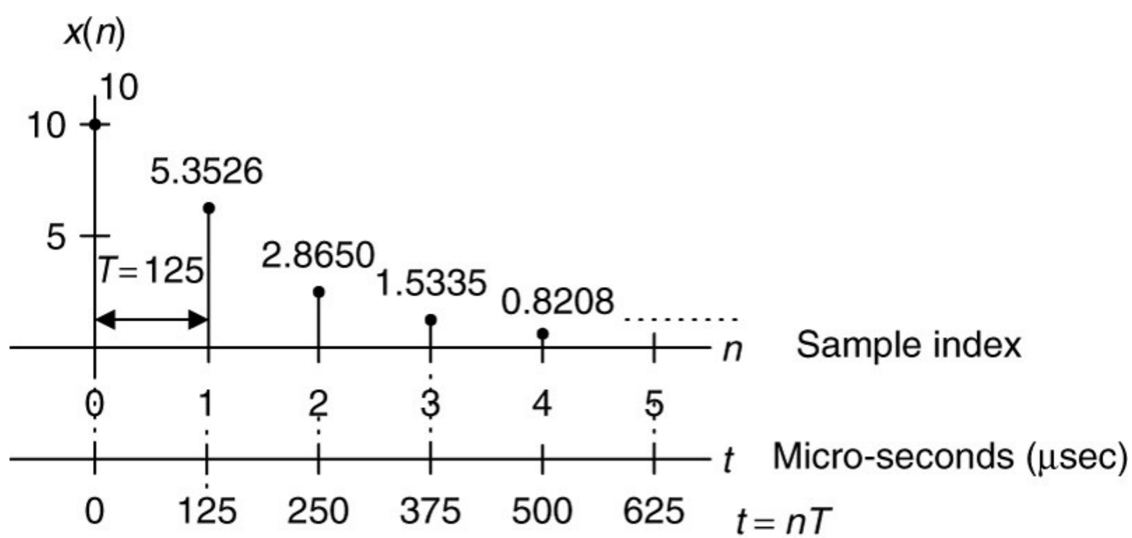
$$x(n) = x(nT) = 10e^{-5000\times 0.000125n}u(nT) = 10e^{-0.625n}u(n)$$

Example 2 (contd.)

The first five sample values:

$$x(0) = 10e^{-0.625\times 0}u(0) = 10.0$$
$$x(1) = 10e^{-0.625\times 1}u(1) = 5.3526$$
$$x(2) = 10e^{-0.625\times 2}u(2) = 2.8650$$
$$x(3) = 10e^{-0.625\times 3}u(3) = 1.5335$$
$$x(4) = 10e^{-0.625\times 4}u(4) = 0.8208$$

Plot of the digital sequence:



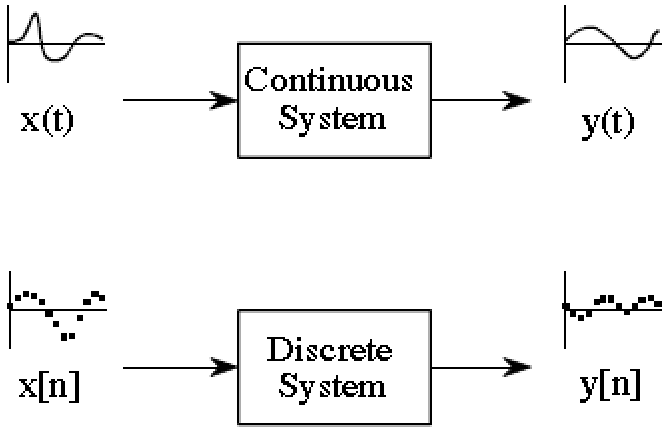
Linear System

System: A system that produces an output signal in response to an input signal.

Continuous system & discrete system.

Time, t

Sample number, n



Linear Systems: Property 1

1. **Homogeneity**
2. **Additivity**

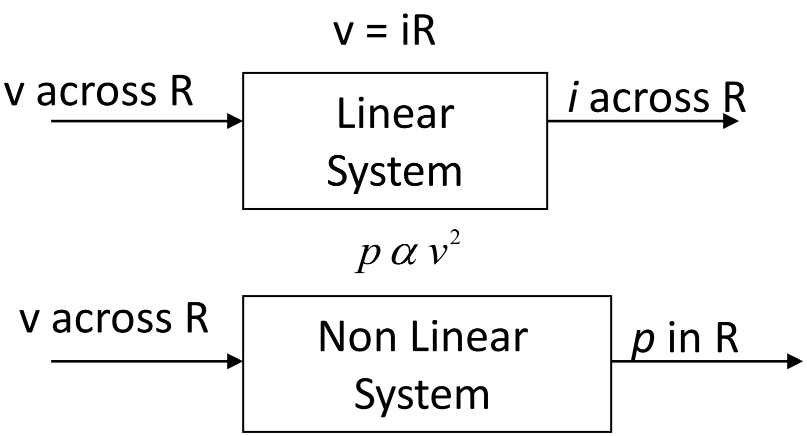
→ Must for all linear systems

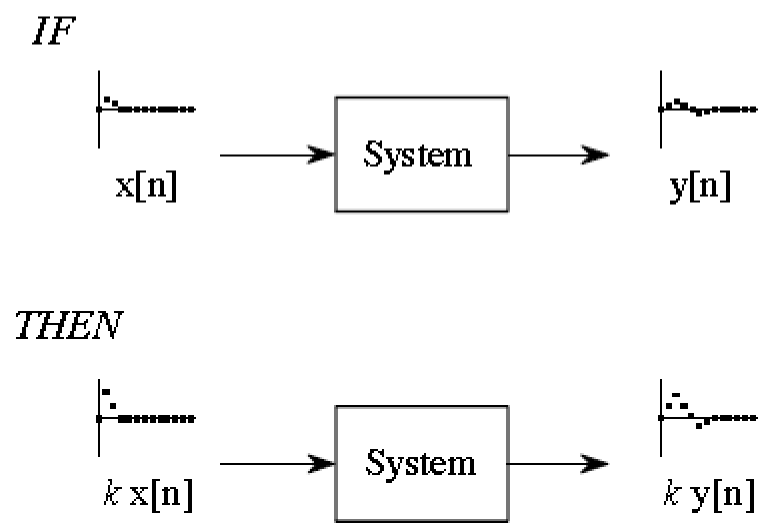
3. **Shift invariance** → Must for DSP linear systems

Homogeneity: (deals with amplitude)

If $x[n] \rightarrow y[n]$, then $kx[n] \rightarrow ky[n]$

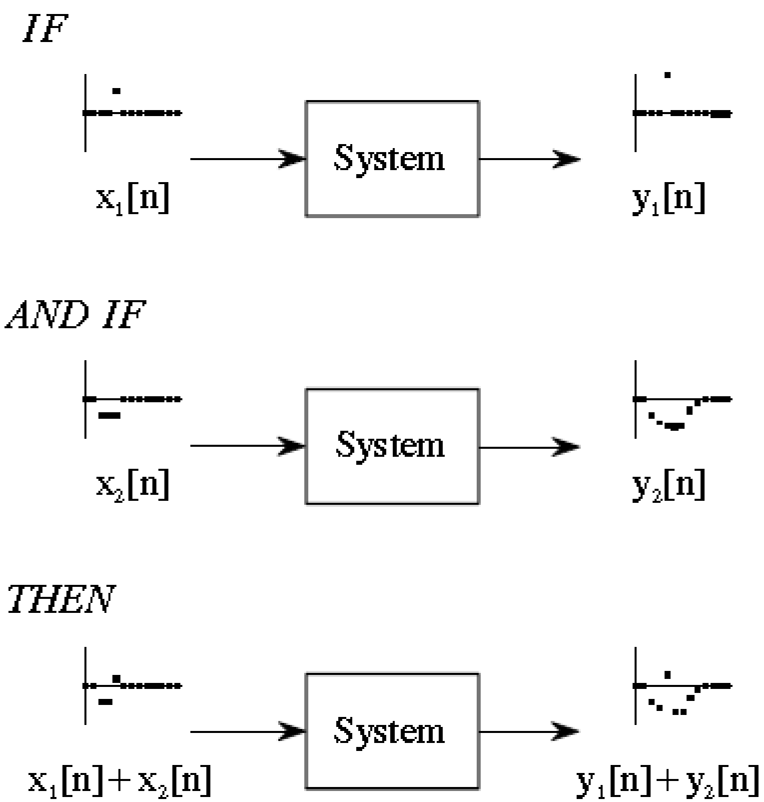
K is a constant



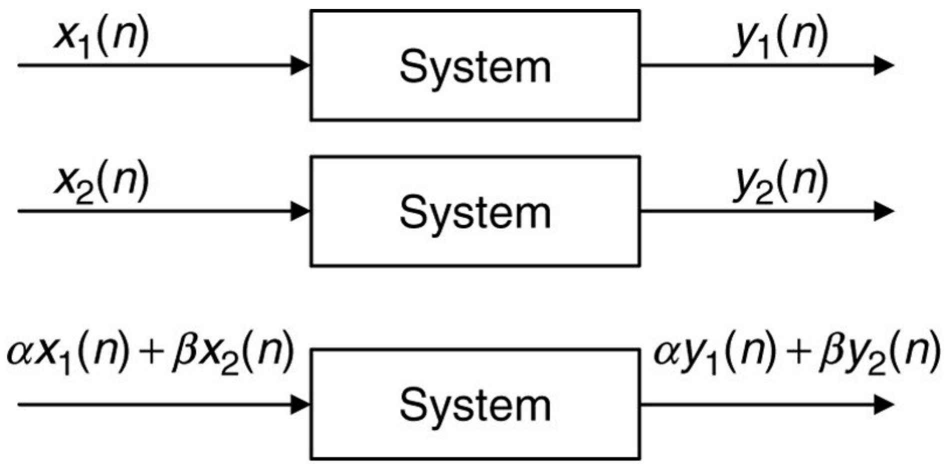


Linear Systems: Property 2

Additivity

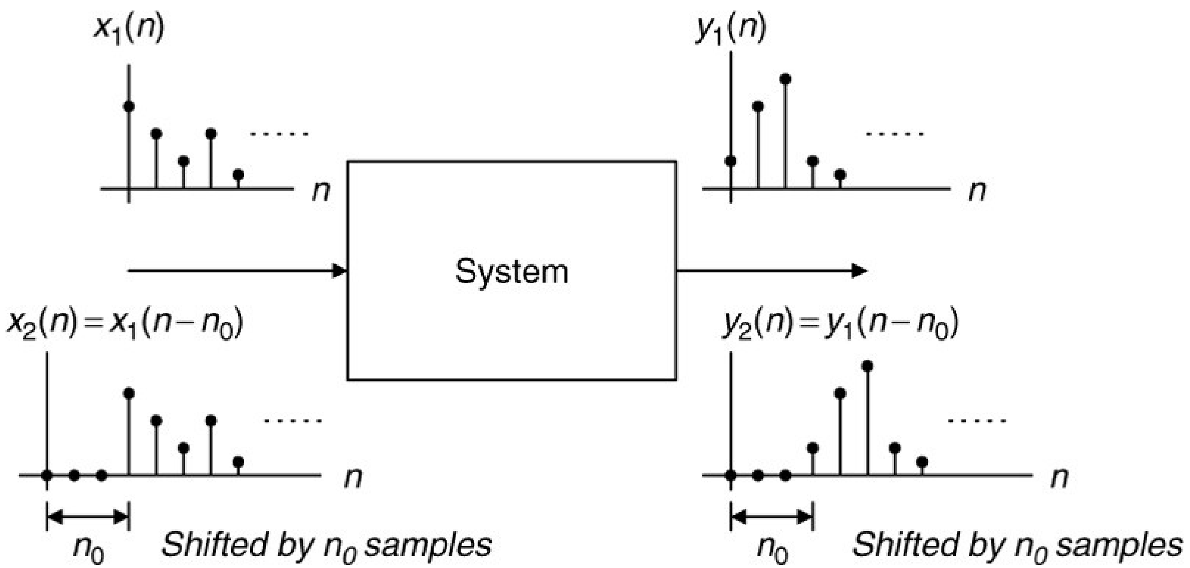


Homogeneity & Additivity

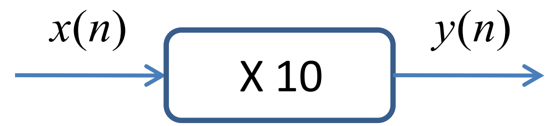


Linear Systems: Property 3

Shift (time) Invariance



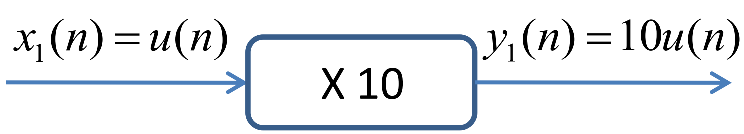
Example 3

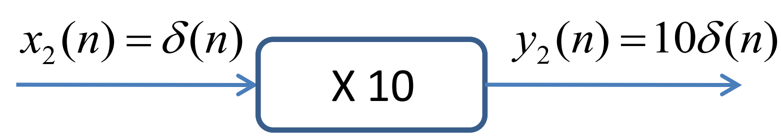


Let a digital amplifier, $y(n) = 10x(n)$

If the inputs are: $x_1(n) = u(n)$ and $x_2(n) = \delta(n)$

Outputs will be: $y_1(n) = 10u(n)$ and $y_2(n) = 10\delta(n)$, respectively.





If we apply combined input to the system: $x(n) = 2x_1(n) + 4x_2(n) = 2u(n) + 4\delta(n)$

The output will be: $y(n) = 10x(n) = 10(2u(n) + 4\delta(n)) = \boxed{20u(n) + 40\delta(n)}$

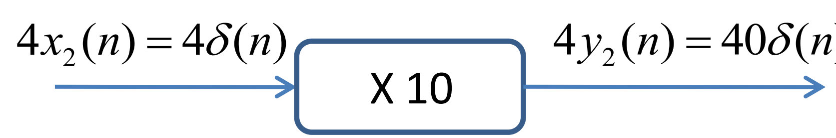
Individual outputs:

$$2y_1(n) = 2 \times 10x_1(n) = 20u(n)$$

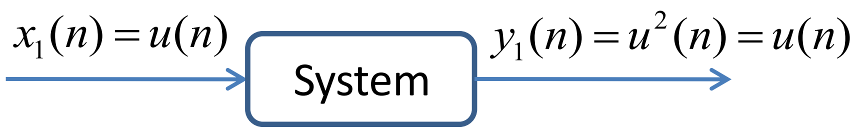
$$4y_2(n) = 4 \times 10x_2(n) = 40\delta(n)$$

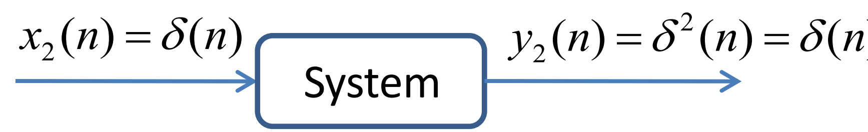
+

Linear System



Example 4

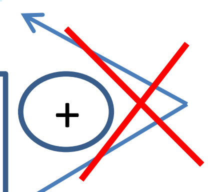




If the input is: $4x_1(n) + 2x_2(n)$

Then the output is:
$$
\begin{aligned}
y(n) &= x^2(n) = (4x_1(n) + 2x_2(n))^2 \\
&= (4u(n) + 2\delta(n))^2 = 16u^2(n) + 16u(n)\delta(n) + 4\delta^2(n) \\
&= 16u(n) + 20\delta(n).
\end{aligned}
$$

Individual outputs:
$$4y_1(n) = 4 \times x_1^{\,2}(n) = 4u(n)$$
$$2y_2(n) = 2 \times x_2^{\,2}(n) = 2\delta(n)$$

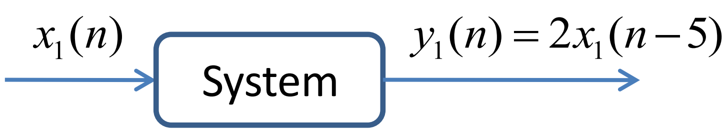


Non Linear System

Example 5 (a)

Given the linear system $y(n) = 2x(n-5)$, find whether the system is time invariant or not.

Solution:



Let the shifted input be: $x_2(n) = x_1(n - n_0)$

Therefore system output: $y_2(n) = 2x_2(n-5) = 2x_1(n - n_0 - 5)$ ← Equal

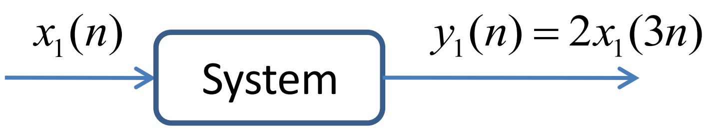
Shifting $y_1(n) = 2x_1(n-5)$ by n_0 samples leads to $y_1(n - n_0) = 2x_1(n - 5 - n_0)$.

Time Invariant

Example 5 (b)

Given the linear system $y(n) = 2x(3n)$, find whether the system is time invariant or not.

Solution:



Let the shifted input be: $x_2(n) = x_1(n - n_0)$

Therefore system output: $y_2(n) = 2x_2(3n) = 2x_1(3n - n_0)$

NOT Equal

Shifting $y_1(n) = 2x_1(3n)$ by n_0 samples leads to

$$y_1(n - n_0) = 2x_1(3(n - n_0)) = 2x_1(3n - 3n_0)$$

NOT Time Invariant

Difference Equation

A causal, linear, and time invariant system can be represented by a difference equation as follows:

$$\underbrace{y(n) + a_1 y(n-1) + \ldots + a_N y(n-N)}_{\text{Outputs}} = \underbrace{b_0 x(n) + b_1 x(n-1) + \ldots + b_M x(n-M)}_{\text{Inputs}}$$

After rearranging:

$$y(n) = -a_1 y(n-1) - \ldots - a_N y(n-N) + b_0 x(n) + b_1 x(n-1) + \ldots + b_M x(n-M)$$

Finally:

$$\boxed{y(n) = -\sum_{i=1}^{N} a_i y(n-i) + \sum_{j=0}^{M} b_j x(n-j)}$$

Example 6

Identify non zero system coefficients of the following difference equations.

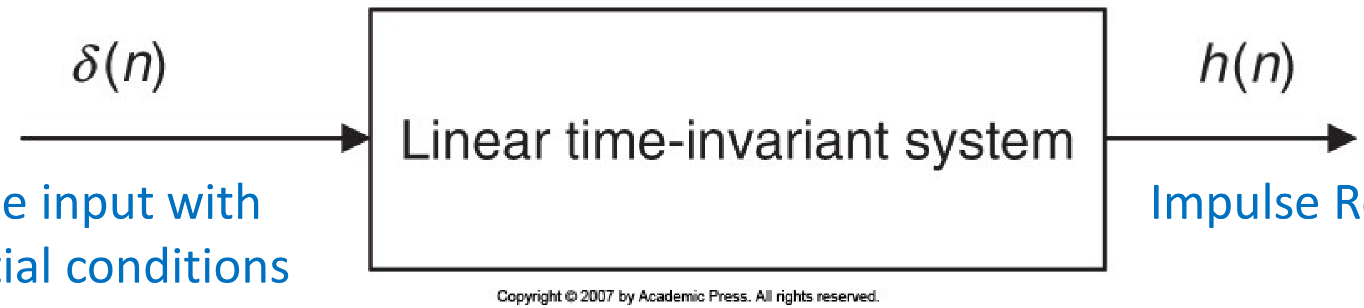
Solution:

$$y(n) = 0.25y(n-1) + x(n) \longrightarrow b_0 = 1, \quad a_1 = -0.25$$

Solution:

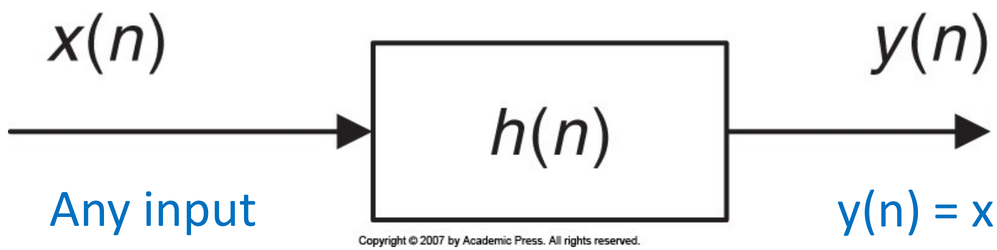
$$y(n) = x(n) + 0.5x(n-1) \longrightarrow b_0 = 1, \quad b_1 = 0.5$$

System Representation Using Impulse Response



Impulse input with zero initial conditions

Impulse Response



Any input

$y(n) = x(n) \otimes h(n)$

Convolution

$$y(n) = \ldots + h(-1)x(n+1) + h(0)x(n) + h(1)x(n-1) + h(2)x(n-2) + \ldots .$$

Example 7 (a)

Given the linear time-invariant system:

$$y(n) = 0.5x(n) + 0.25x(n-1) \text{ with an initial condition } x(-1) = 0,$$

a. Determine the unit-impulse response $h(n)$.

b. Draw the system block diagram.

c. Write the output using the obtained impulse response.

Solution:

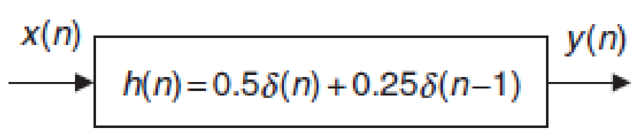
a. let $x(n) = \delta(n)$

$$h(n) = y(n) = 0.5x(n) + 0.25x(n-1) = 0.5\delta(n) + 0.25\delta(n-1)$$

Therefore,

$$h(n) = \begin{cases} 0.5 & n = 0 \\ 0.25 & n = 1 \\ 0 & elsewhere \end{cases}$$

b.



c. $y(n) = h(0)x(n) + h(1)x(n-1)$

Example 7 (b)

Given the difference equation

$$y(n) = 0.25y(n-1) + x(n) \text{ for } n \geq 0 \text{ and } y(-1) = 0,$$

a. Determine the unit-impulse response $h(n)$.

b. Draw the system block diagram.

c. Write the output using the obtained impulse response.

Solution:

a. let $x(n) = \delta(n)$ Then $h(n) = 0.25h(n-1) + \delta(n)$

$$h(0) = 0.25h(-1) + \delta(0) = 0.25 \times 0 + 1 = 1$$
$$h(1) = 0.25h(0) + \delta(1) = 0.25 \times 1 + 0 = 0.25$$
$$h(2) = 0.25h(1) + \delta(2) = 0.25 \times 0.5 + 0 = 0.0625$$
$$\ldots$$

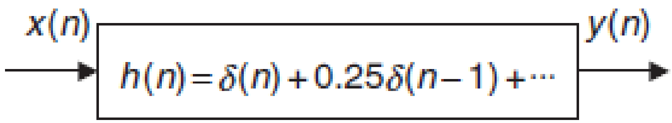
With the calculated results, we can predict the impulse response as

$$h(n) = (0.25)^n u(n) = \delta(n) + 0.25\delta(n-1) + 0.0625\delta(n-2) + \ldots.$$

Infinite!

Example 7 (b) - contd.

b.



c.

$$
\begin{aligned}
y(n) &= h(0)x(n) + h(1)x(n-1) + h(2)x(n-2) + \ldots \\
&= x(n) + 0.25x(n-1) + 0.0625x(n-2) + \ldots
\end{aligned}
$$

Finite Impulse Response (FIR) system:

When the difference equation contains no previous outputs, i.e. *‘a’* coefficients are zero. < See example 7 (a) >

Infinite Impulse Response (IIR) system:

When the difference equation contains previous outputs, i.e. *‘a’* coefficients are not all zero. < See example 7 (b) >

BIBO Stability

BIBO: Bounded In and Bounded Out

A stable system is one for which every bounded input produces a bounded output.

$$y(n) = \ldots + h(-1)x(n+1) + h(0)x(n) + h(1)x(n-1) + h(2)x(n-2) + \ldots.$$

Let, in the worst case, every input value reaches to maximum value *M.*

$$y(n) = M(\ldots + h(-1) + h(0) + h(1) + h(2) + \ldots).$$

Using absolute values of the impulse responses,

$$y(n) < M(\ldots + |h(-1)| + |h(0)| + |h(1)| + |h(2)| + \ldots).$$

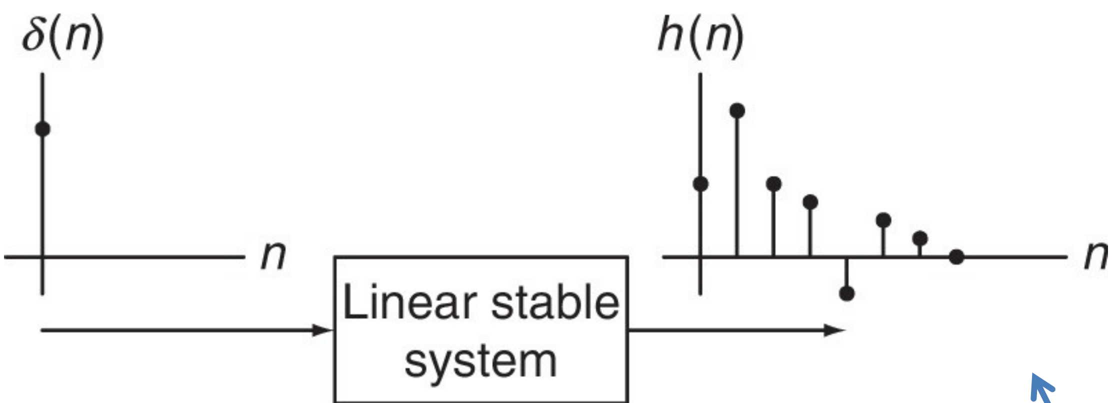
If the impulse responses are finite number, then output is also finite.

Stable system.

BIBO Stability – contd.

To determine whether a system is stable, we apply the following equation:

$$S = \sum_{k=-\infty}^{\infty} |h(k)| = \ldots + |h(-1)| + |h(0)| + |h(1)| + \ldots < \infty.$$



Impulse response is decreasing to zero.

Example 8

Given a linear system given by: $y(n) = 0.25y(n-1) + x(n)$ for $n \geq 0$ and $y(-1) = 0$

Which is described by the unit-impulse response: $h(n) = (0.25)^n u(n)$

Determine whether the system is stable or not.

Solution:

$$S = \sum_{k=-\infty}^{\infty} |h(k)| = \sum_{k=-\infty}^{\infty} \left|(0.25)^k u(k)\right|$$

Using definition of step function:

$u(k) = 1$ for $k \geq 0$

$$\Rightarrow \quad S = \sum_{k=0}^{\infty} (0.25)^k = 1 + 0.25 + 0.25^2 + \ldots$$

For a < 1, we know $\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}$ where $a = 0.25 < 1$

Therefore $S = 1 + 0.25 + 0.25^2 + \ldots = \frac{1}{1-0.25} = \frac{4}{3} < \infty$

The summation is finite, so the system is stable.

Digital Convolution

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

$$= \ldots + h(-1)x(n+1) + h(0)x(n) + h(1)x(n-1) + h(2)x(n-2) + \ldots$$

The sequences are interchangeable.

Commutative

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

$$x[n] * h[n] = h[n] * x[n]$$

Convolution sum requires h(n) to be reversed and shifted.

If h(n) is the given sequence, h(-n) is the reversed sequence.

Reversed Sequence

Given a sequence,

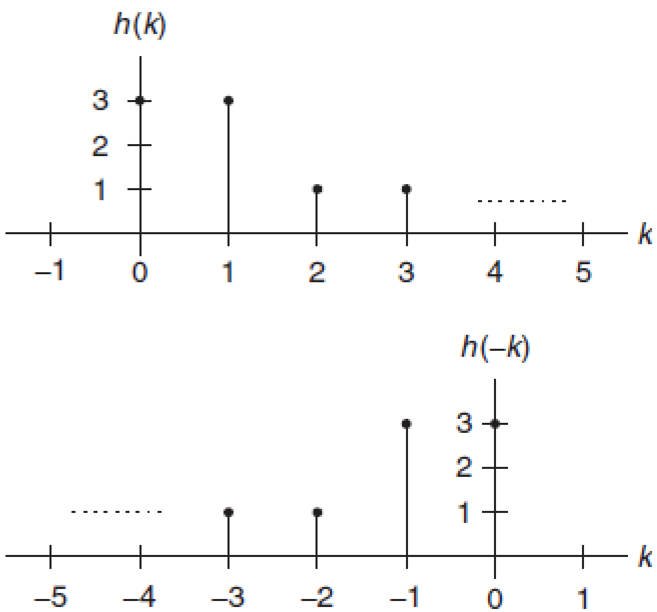
$$h(k) = \begin{cases} 3, & k = 0,1 \\ 1, & k = 2,3 \\ 0 & elsewhere \end{cases}$$

where k is the time index or sample number,

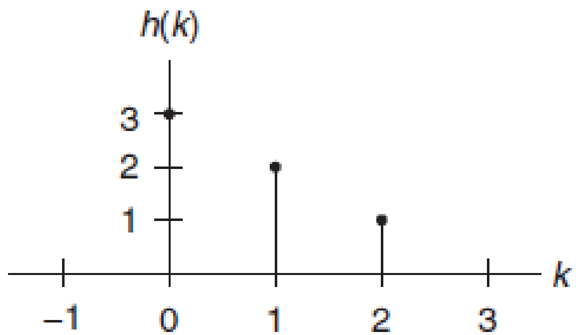
a. Sketch the sequence $h(k)$ and reversed sequence $h(-k)$.

Solution:

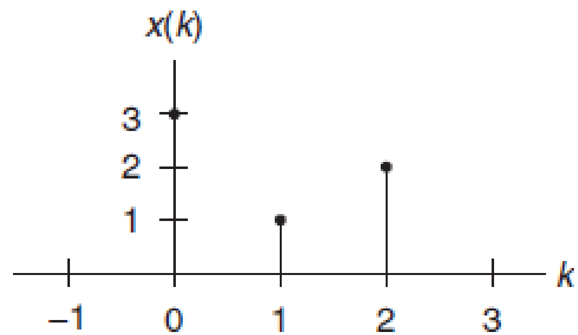
a.



Convolution Using Table Method
Example 9



Length = 3



Length = 3

Solution:

Convolution sum using the table method.

k:	-2	-1	0	1	2	3	4	5	
$x(k)$:			**3**	**1**	**2**				
$h(-k)$:	1	2	3						$y(0) = 3 \times 3 = 9$
$h(1-k)$		1	2	3					$y(1) = 3 \times 2 + 1 \times 3 = 9$
$h(2-k)$			1	2	3				$y(2) = 3 \times 1 + 1 \times 2 + 2 \times 3 = 11$
$h(3-k)$				1	2	3			$y(3) = 1 \times 1 + 2 \times 2 = 5$
$h(4-k)$					1	2	3		$y(4) = 2 \times 1 = 2$
$h(5-k)$						1	2	3	$y(5) = 0$ (no overlap)

Convolution length = 3 +3 – 1 = 5

Convolution Using Table Method Example 10

$$x(n) = \begin{cases} 1 & n = 0,1,2 \\ 0 & otherwise \end{cases} \text{ and } h(n) = \begin{cases} 0 & n = 0 \\ 1 & n = 1,2 \\ 0 & otherwise \end{cases}$$

Length = 3

Length = 2

Solution:

k:	-2	-1	0	1	2	3	4	5	$\ldots$	
$x(k)$:			**1**	**1**	**1**				$\ldots$	
$h(-k)$:	1	1	0							$y(0) = 0$ (no overlap)
$h(1-k)$		1	1	0						$y(1) = 1 \times 1 = 1$
$h(2-k)$			1	1	0					$y(2) = 1 \times 1 + 1 \times 1 = 2$
$h(3-k)$				1	1	0				$y(3) = 1 \times 1 + 1 \times 1 = 2$
$h(4-k)$					1	1	0			$y(4) = 1 \times 1 = 1$
$h(n-k)$						1	1	0		$y(n) = 0,\ n \geq 5$ (no overlap) Stop

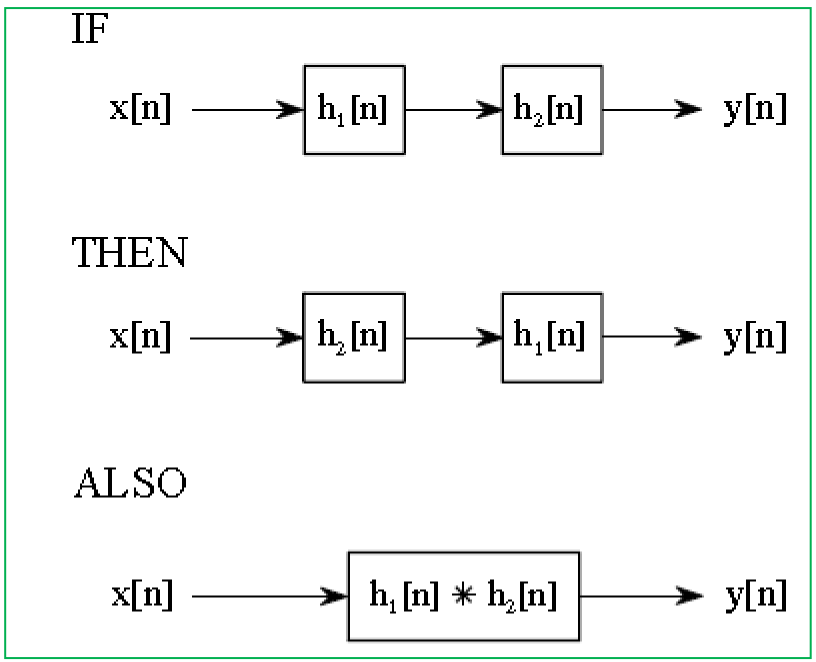
Convolution length = 3 + 2 – 1 = 4

Convolution Properties

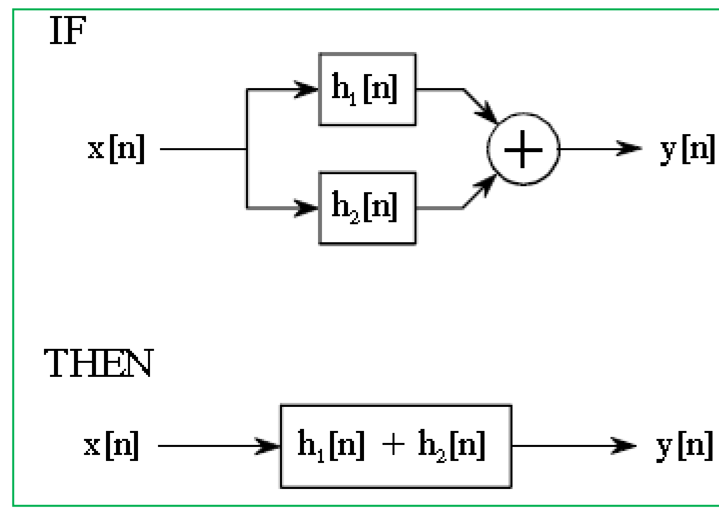
Commutative: $a[n]*b[n]=b[n]*a[n]$

Associative: $(a[n]*b[n])*c[n]=a[n]*(b[n]*c[n])$

Distributive: $a[n]*b[n]+a[n]*c[n]=a[n]*(b[n]+c[n])$



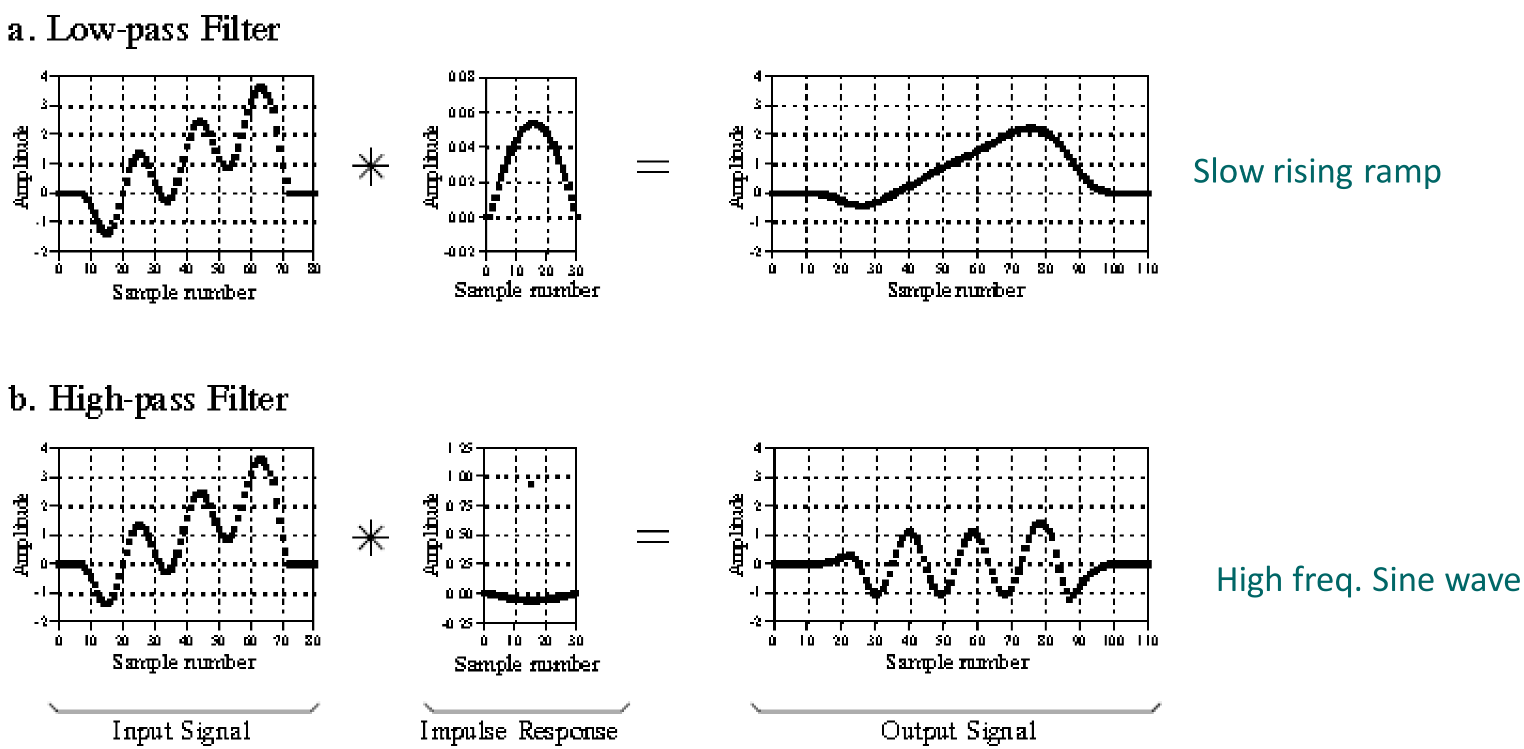
Associative



Distributive

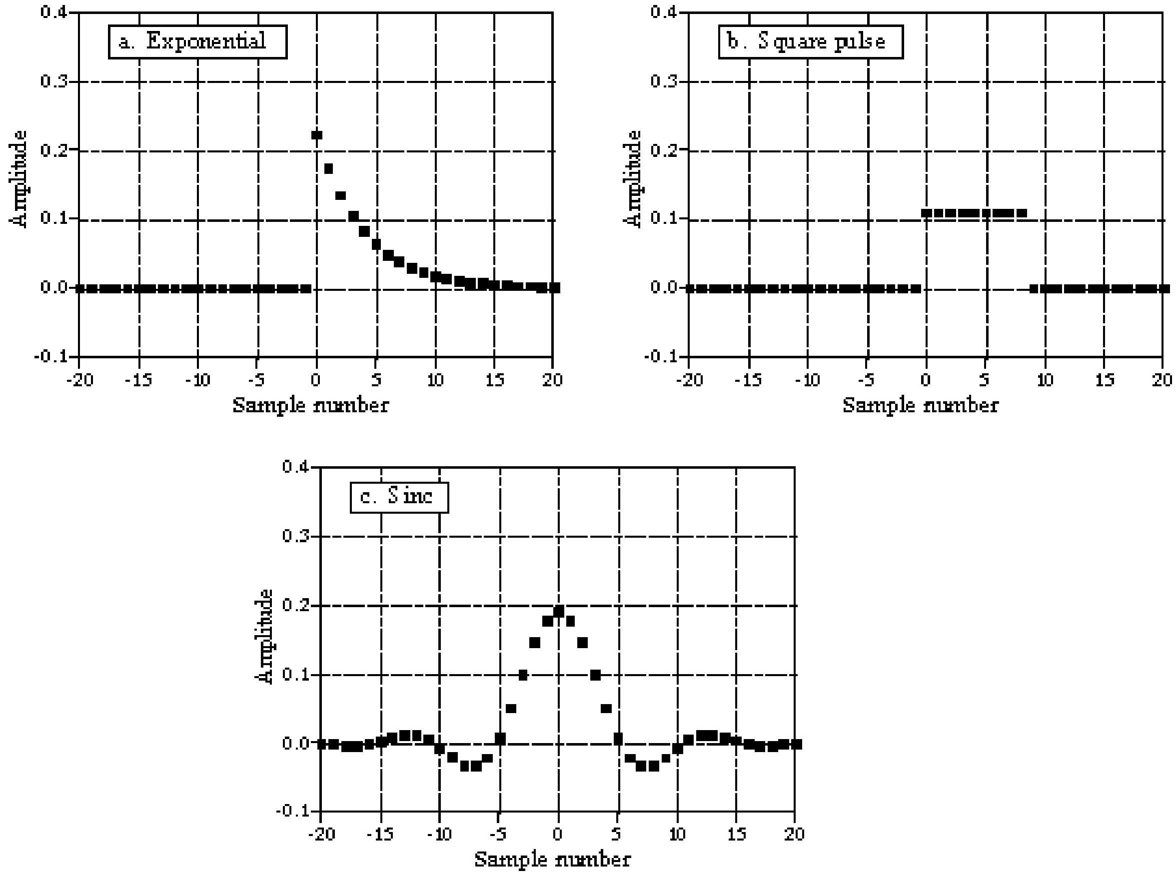
Examples of Convolution

Kernel



Low Pass Filters

Kernel: formed by a group of positive adjacent points that provide smoothing.

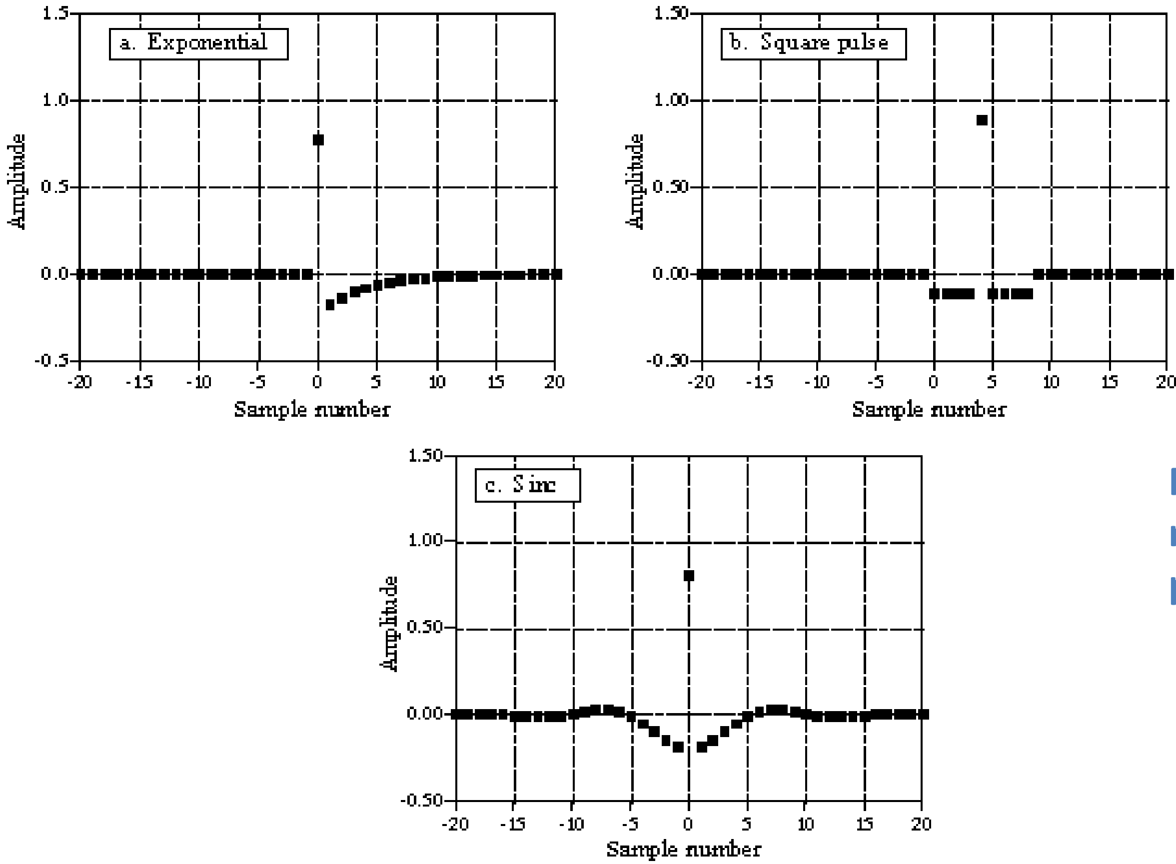


For reducing noise

Sum of the points must be one. Gain of *one* in DC.

High Pass Filters

Kernel: delta function – corresponding low-pass filter.



Peak is surrounded by many adjacent negative points.

Sum of the points must be zero. Zero gain at DC (zero frequency).

Signal-to-Noise Ratio (SNR)

Bel or decibel (dB):

A bel: The power is changed by a *factor of ten*.

3 bels → Power of 10×10 × 10 = 1000 times.

Decibel (dB): One-tenth of a bel.

30 dB → Power of 10×10 × 10 = 1000 times.

$$dB = 10\log_{10}\frac{P_2}{P_1}$$

$$dB = 20\log_{10}\frac{A_2}{A_1}$$

0 dB → 10^0 times = 1 time = equal power.

Clean signal, s(n), with variance = 0.5

Noise signal, v(n), with variance = 1

Noisy signal, x(n) = s(n) + Kv(n)

Find K so that SNR = 20 dB.

$$\sigma^2_x = \sigma^2_s + K^2\sigma^2_v$$

$$SNR = 20dB = 10\log_{10}\left(\frac{\text{var}(s(n))}{\text{var}(v(n))}\right) = 10\log_{10}\left(\frac{0.5}{K^2}\right)$$

$$\log_{10}\frac{0.5}{K^2} = 2 \Rightarrow \frac{0.5}{K^2} = 10^2 \Rightarrow K = 0.07071$$

Periodicity

Example 11

Consider the following continuous signal for the current $i(t) = \cos(20\pi t)$

which is sampled at 12.5 ms. Will the resulting discrete signal be periodic?

The continuous radian frequency is ω = 20π radians. Since the sampling rate interval T_s = 12.5 msec = 0.0125 sec, then

$$x(n) = \cos(2\pi(10)(0.0175)n) = \cos\left(\frac{2\pi}{8}n\right) = \cos\left(\frac{\pi}{4}n\right)$$

Since for periodicity we must have: $\frac{2\pi}{\theta_o} = \frac{N}{k}$

We get, $\frac{2\pi}{2\pi/8} = \frac{N}{k} = \frac{16\pi}{2\pi} = \frac{8}{1}$

For k = 1 we have N = 8, which is the fundamental period.

If N/k is a rational number (ratio of two integers) then x(n) is periodic and the period is $N = k\left(\frac{2\pi}{\theta_0}\right)$

The smallest value of N that satisfies the above equation is called the fundamental period. If $2\pi/\theta_0$ is not a rational number, then $x(n)$ is not periodic.

Figure Acknowledgement

Most of the figures are taken from the following books:

Li Tan, *Digital Signal Processing, Fundamentals and Applications*, Elsevier, 2008.

Steven W. Smith, *Digital Signal Processing: A Practical Guide for Engineers and Scientists*, Newnes, Elsevier, 2003.