

Digital Filtering: Realization



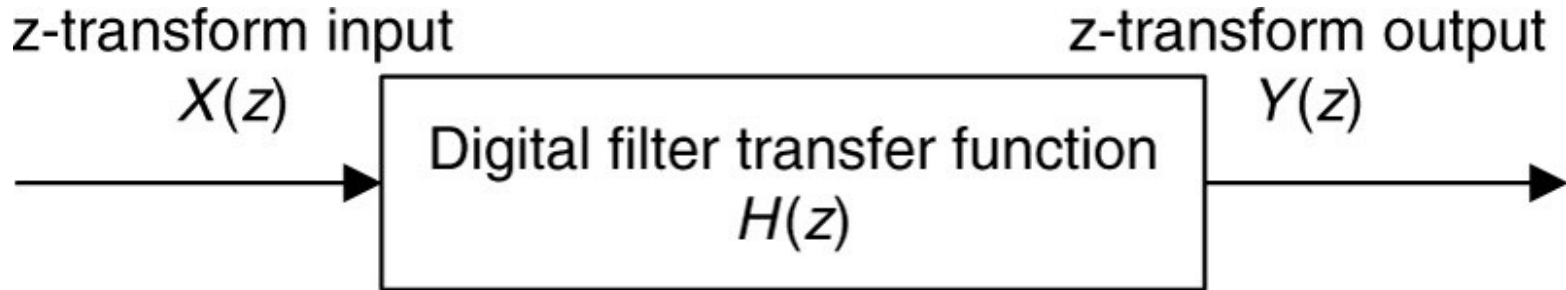
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Digital Filtering:
$$y(n) = \sum_{i=0}^M b_i x(n-i) - \sum_{j=1}^N a_j y(n-j).$$

**Matlab Implementation:
3-tap (2nd order) IIR filter**

```
>> B = [0 1]; A = [1 0 -0.5];  
>> x = [1 0.5 0.25 0.125];  
>> y = filter(B, A, x)  
  
y =  
0 1.0000 0.5000 0.7500
```

Transfer Function



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Differential
Equation:

$$y(n) = b_0x(n) + b_1x(n-1) + \cdots + b_Mx(n-M) \\ - a_1y(n-1) - \cdots - a_Ny(n-N).$$

z- Transform:

$$Y(z) = b_0X(z) + b_1X(z)z^{-1} + \cdots + b_MX(z)z^{-M} \\ - a_1Y(z)z^{-1} - \cdots - a_NY(z)z^{-N}$$

Transfer
Function:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1z^{-1} + \cdots + b_Mz^{-M}}{1 + a_1z^{-1} + \cdots + a_Nz^{-N}}$$

Example: Transfer Function

Given: $y(n) = x(n) - x(n - 2) - 1.3y(n - 1) - 0.36y(n - 2)$

z- Transform: $Y(z) = X(z) - X(z)z^{-2} - 1.3Y(z)z^{-1} - 0.36Y(z)z^{-2}$.

Rearrange: $Y(z)(1 + 1.3z^{-1} + 0.36z^{-2}) = (1 - z^{-2})X(z)$

Transfer Function: $H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-2}}{1 + 1.3z^{-1} + 0.36z^{-2}}$

Given: $H(z) = \frac{z^2 - 1}{z^2 + 1.3z + 0.36}$

Rearrange: $H(z) = \frac{(z^2 - 1)/z^2}{(z^2 + 1.3z + 0.36)/z^2} = \frac{1 - z^{-2}}{1 + 1.3z^{-1} + 0.36z^{-2}}$.

Differential Equation: $y(n) = x(n) - x(n - 2) - 1.3y(n - 1) - 0.36y(n - 2)$

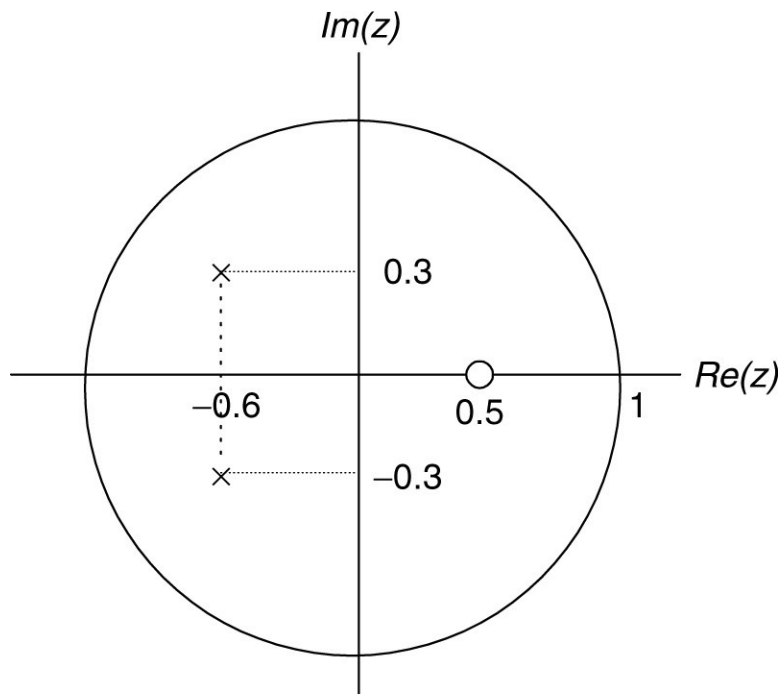
Pole - Zero from Transfer Function

$$H(z) = \frac{z^{-1} - 0.5z^{-2}}{1 + 1.2z^{-1} + 0.45z^{-2}}$$

$$H(z) = \frac{(z^{-1} - 0.5z^{-2})z^2}{(1 + 1.2z^{-1} + 0.45z^{-2})z^2} = \frac{z - 0.5}{z^2 + 1.2z + 0.45} = \frac{(z - 0.5)}{(z + 0.6 - j0.3)(z + 0.6 + j0.3)}$$

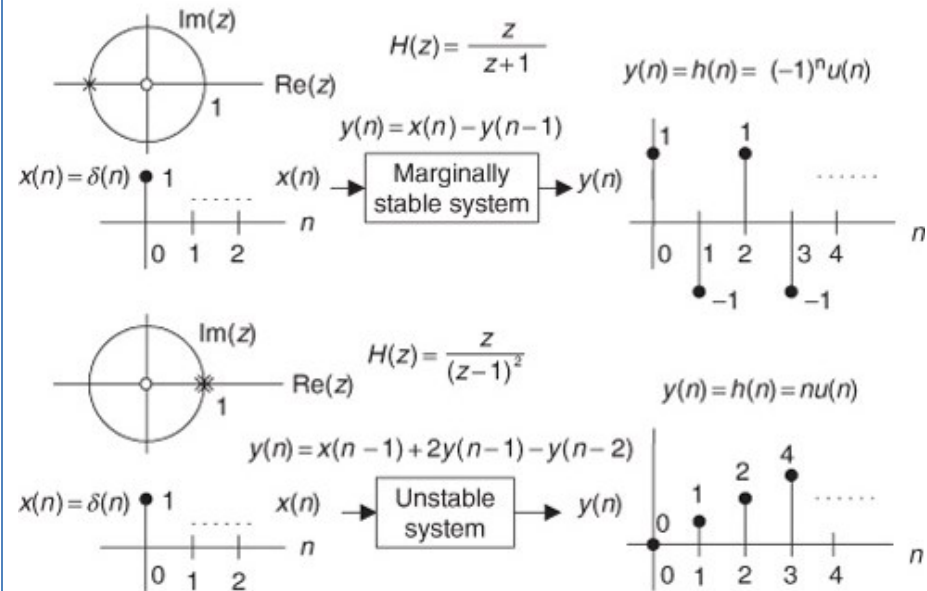
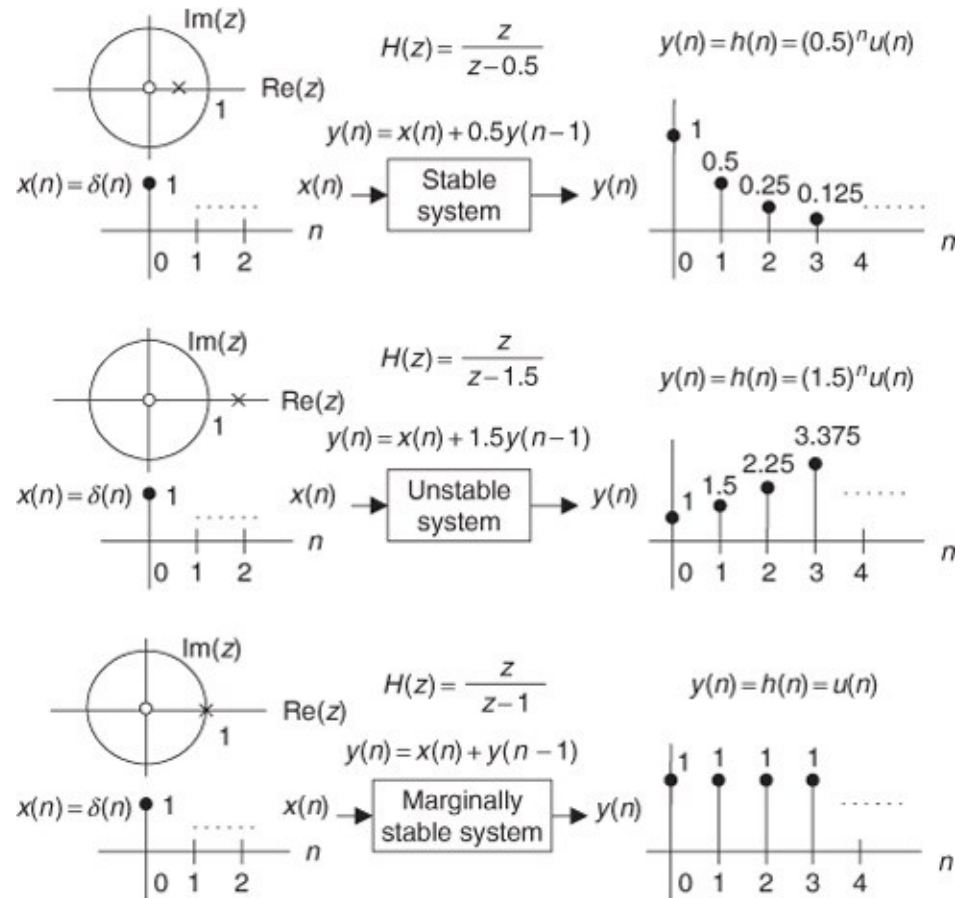
The system is stable.

The zeros do not affect system stability.



System Stability

Depends on poles' location



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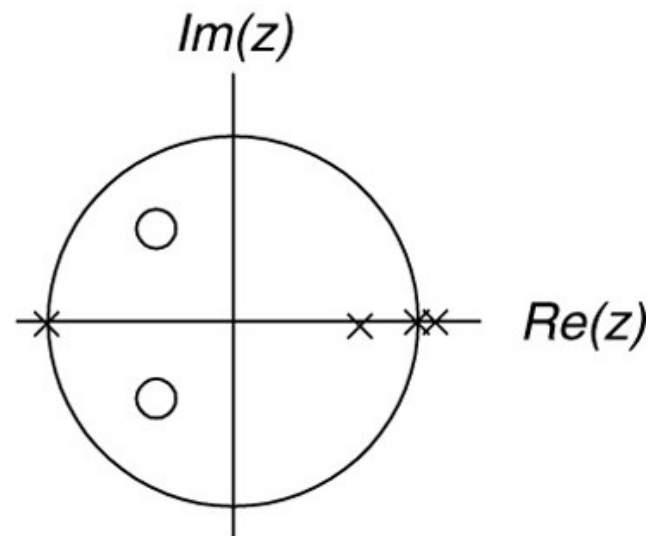
Example: System Stability

$$H(z) = \frac{z^2 + z + 0.5}{(z - 1)^2(z + 1)(z - 0.6)}$$

Zeros are $z = -0.5 \pm j0.5$.

Poles: $z = 1, |z| = 1$; $z = 1, |z| = 1$; $z = -1, |z| = 1$; $z = 0.6, |z| = 0.6 < 1$.

Since the outermost pole is multiple order (2nd order) at $z = 1$ and is on the unit circle, the system is unstable.



Digital Filter: Frequency Response

$$H(z)|_{z=e^{j\omega T}} = H(e^{j\omega T}) = |H(e^{j\omega T})| \angle H(e^{j\omega T})$$

Magnitude frequency response

Phase response

Putting $\Omega = \omega T$

$$H(e^{j\Omega}) = H(z)|_{z=e^{j\Omega}} = |H(e^{j\Omega})| \angle H(e^{j\Omega})$$

Example: Given $y(n) = 0.5x(n) + 0.5x(n-1)$ Sampling rate = 8k Hz

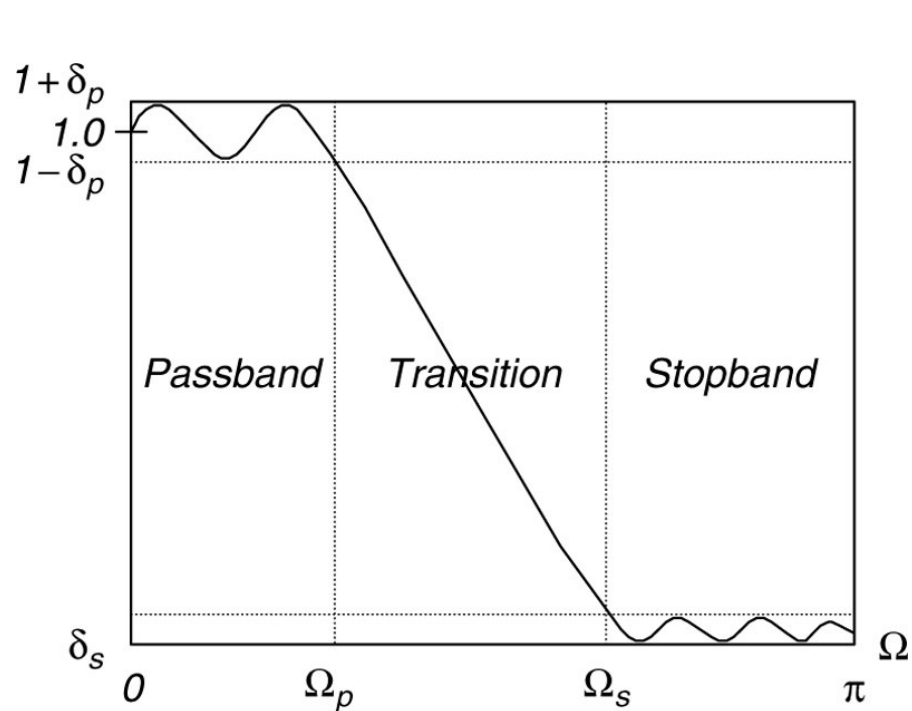
Transfer function: $H(z) = \frac{Y(z)}{X(z)} = 0.5 + 0.5z^{-1}$.

Frequency response: $H(e^{j\Omega}) = 0.5 + 0.5e^{-j\Omega}$
 $= 0.5 + 0.5 \cos(\Omega) - j0.5 \sin(\Omega)$

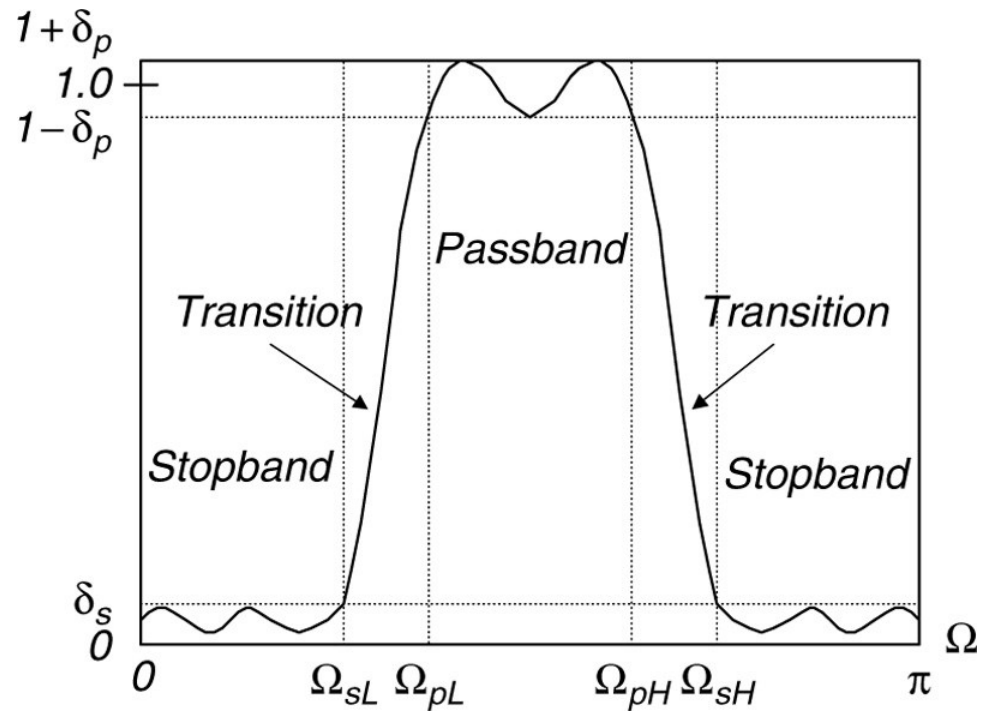
Complete Plot!

$$|H(e^{j\Omega})| = \sqrt{(0.5 + 0.5 \cos(\Omega))^2 + (0.5 \sin(\Omega))^2} \quad \text{and} \quad \angle H(e^{j\Omega}) = \tan^{-1} \left(\frac{-0.5 \sin(\Omega)}{0.5 + 0.5 \cos(\Omega)} \right)$$

Digital Filter: Frequency Response - contd.



Low Pass Filter (LPF)

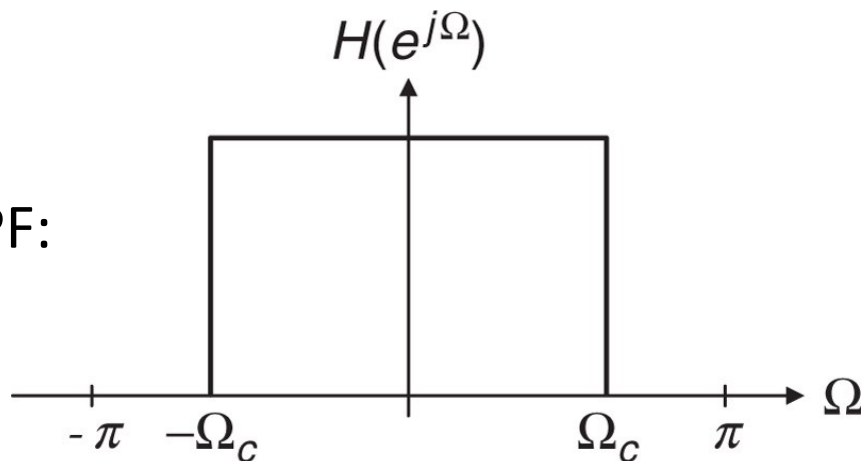


Band Pass Filter (BPF)

Matlab: Frequency Response $[h, w] = \text{freqz}(B, A, N)$

Impulse Response of FIR Filters

Frequency response of ideal LPF:



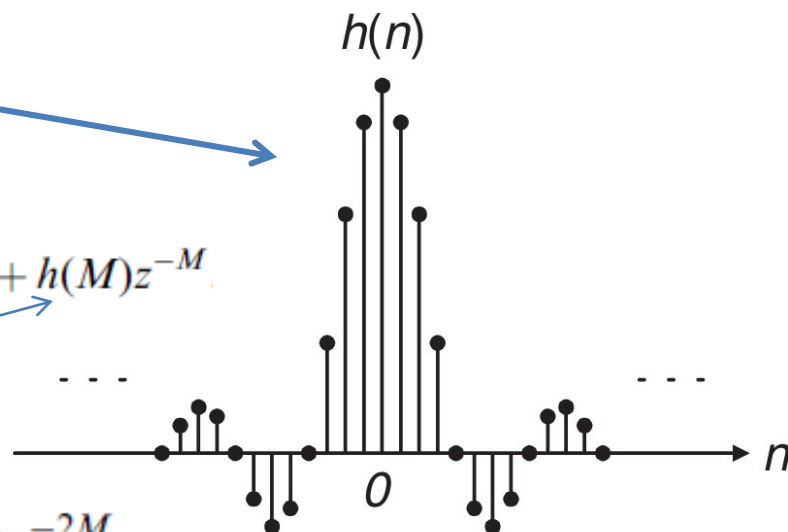
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Impulse response of ideal LPF:

After truncating $2M+1$ major components:

$$H(z) = h(M)z^M + \dots + h(1)z^1 + h(0) + h(1)z^{-1} + \dots + h(M)z^{-M}$$

symmetric



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Making causal,

$$H(z) = b_0 + b_1z^{-1} + \dots + b_{2M}(2M)z^{-2M}$$

Where, $b_n = h(n - M)$ for $n = 0, 1, \dots, 2M$.

Ideal Low Pass Filter

Impulse
Response:

$$h(n) = \begin{cases} \frac{\Omega_c}{\pi} & n = 0 \\ \frac{\sin(\Omega_c n)}{n\pi} & \text{for } n \neq 0 \end{cases} \quad -M \leq n \leq M$$

Example:

3-tap FIR LPF with cutoff freq. = 800 Hz and sampling rate = 8k Hz.

$$\Omega_c = 2\pi f_c T_s = 2\pi \times 800/8000 = 0.2\pi \text{ radians}$$

$$2M + 1 = 3$$

$$h(0) = \frac{\Omega_c}{\pi} \quad \text{for } n = 0$$

$$h(n) = \frac{\sin(\Omega_c n)}{n\pi} = \frac{\sin(0.2\pi n)}{n\pi}, \quad \text{for } n \neq 0$$



$$h(0) = \frac{0.2\pi}{\pi} = 0.2$$

$$h(1) = \frac{\sin[0.2\pi \times 1]}{1 \times \pi} = 0.1871$$

Using symmetry: $h(-1) = h(1) = 0.1871$

Ideal Low Pass Filter - contd.

Delaying $h(n)$ by
 $M = 1$ sample,

$$b_0 = h(0 - 1) = h(-1) = 0.1871$$

$$b_1 = h(1 - 1) = h(0) = 0.2$$

$$b_2 = h(2 - 1) = h(1) = 0.1871$$

**Filter
 coefficients**

Transfer function $H(z) = 0.1871 + 0.2z^{-1} + 0.1871z^{-2}$

Differential Eq: $y(n) = 0.1871x(n) + 0.2x(n - 1) + 0.1871x(n - 2)$

Frequency response $H(e^{j\Omega}) = 0.1871 + 0.2e^{-j\Omega} + 0.1871e^{-j2\Omega}$

$$= e^{-j\Omega} (0.1871e^{j\Omega} + 0.2 + 0.1871e^{-j\Omega})$$

$$= e^{-j\Omega} (0.2 + 0.3742 \cos(\Omega))$$

$$e^{jx} + e^{-jx} = 2 \cos(x)$$

Magnitude: $|H(e^{j\Omega})| = |0.2 + 0.3472 \cos \Omega|$

Complete Plot!

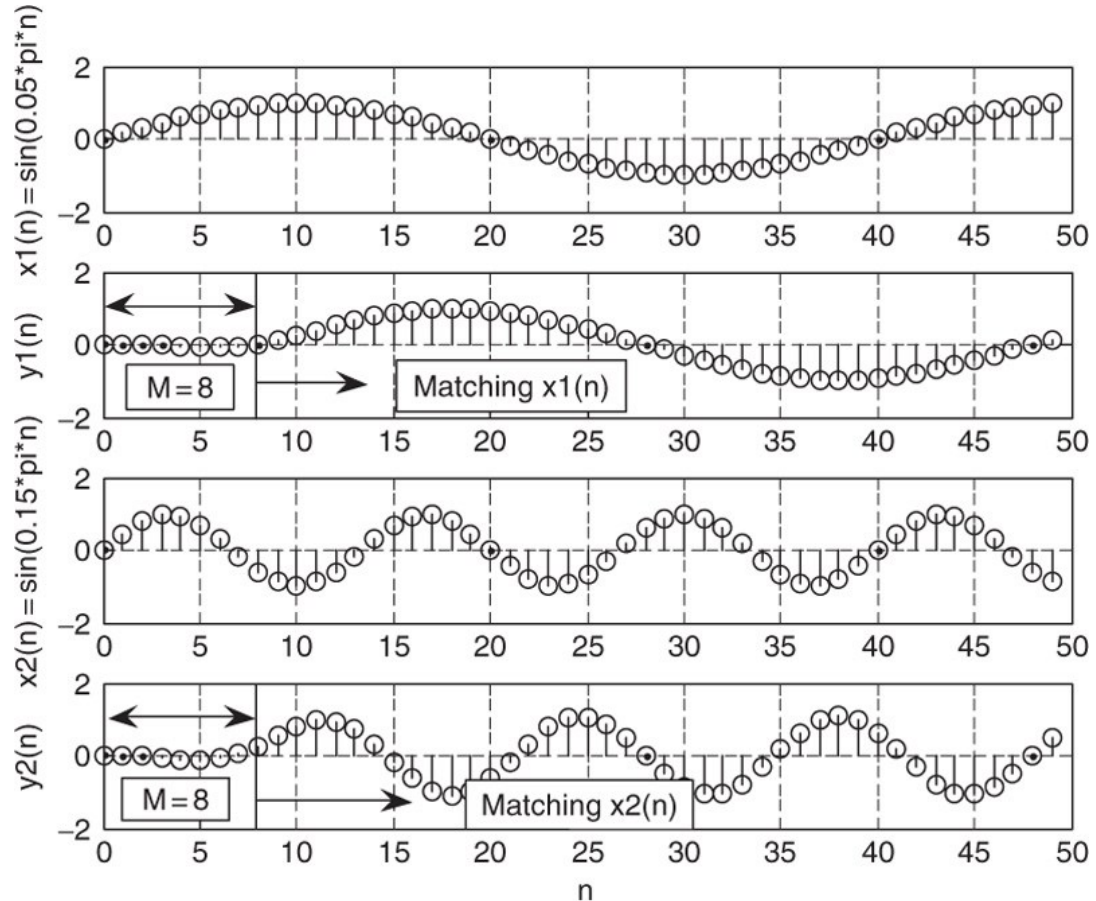
Phase: and $\angle H(e^{j\Omega}) = \begin{cases} -\Omega & \text{if } 0.2 + 0.3472 \cos \Omega > 0 \\ -\Omega + \pi & \text{if } 0.2 + 0.3472 \cos \Omega < 0 \end{cases}$

Linear Phase

If filter has linear phase property, the output will simply be a delayed version of input.

Let, 17-tap FIR filter with linear phase property.

8 samples delay



8 samples delay

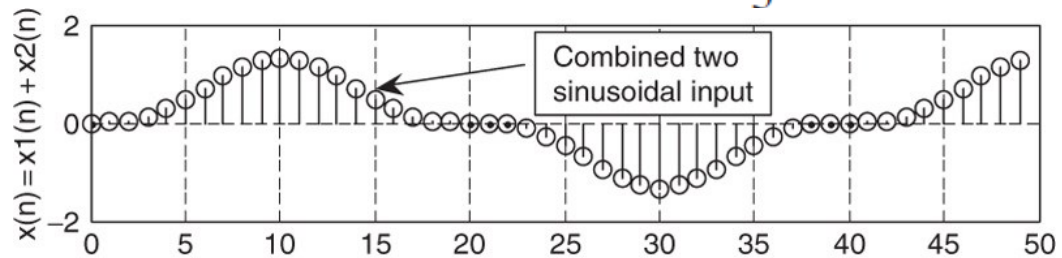
Nonlinear Phase

Input: $x(n) = x_1(n) + x_2(n) = \sin(0.05\pi n)u(n) - \frac{1}{3} \sin(0.15\pi n)u(n)$

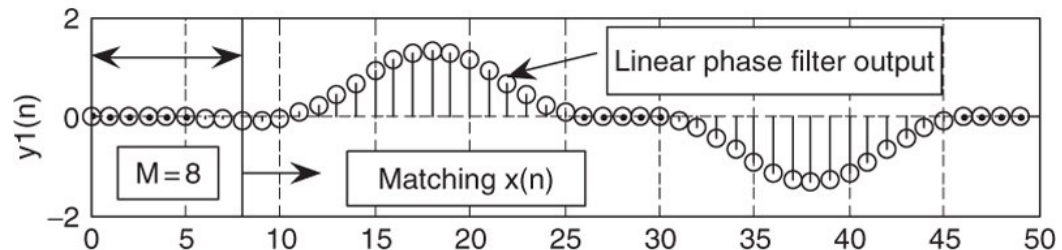
Linear phase filter output: $y_1(n) = \sin[0.05\pi(n - 8)] - \frac{1}{3} \sin[0.15\pi(n - 8)]$

90 d phase delay filter output: $y_2(n) = \sin(0.05\pi n - \pi/2) - \frac{1}{3} \sin(0.15\pi n - \pi/2)$

Input:

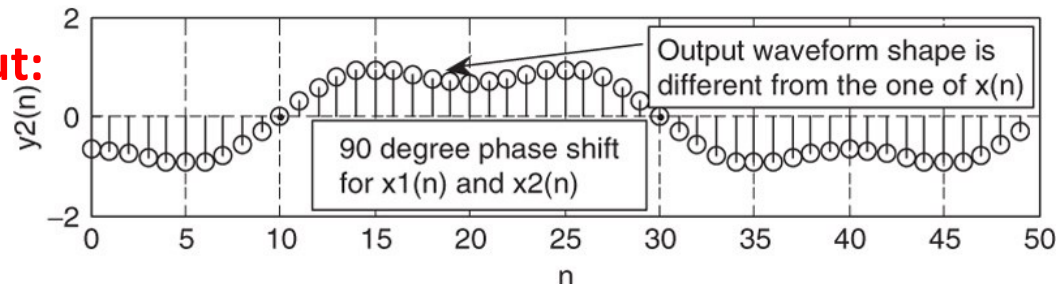


Linear phase filter output:



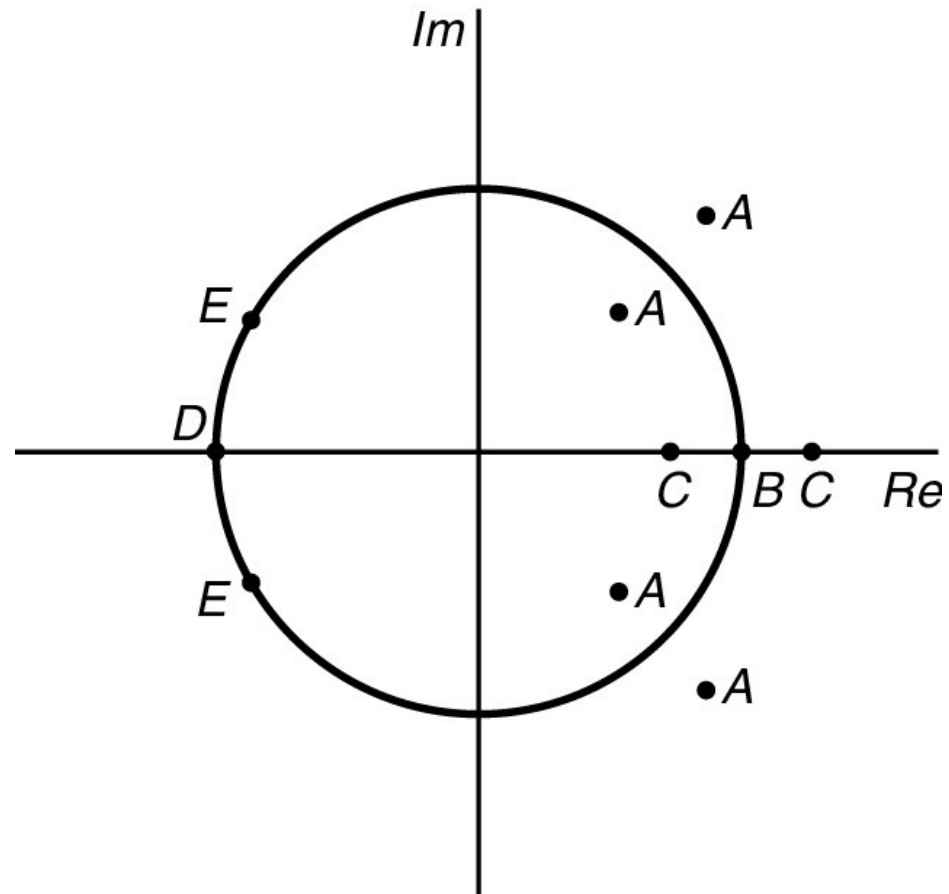
90 d phase delay filter output:

Distorted!



Linear Phase: Zero Placement

- A single zero can be either at $z = 1$ or $z = -1$. (B or D)
- Real zeros not on the unit circle always occur in pairs with r and r^{-1} . (C)
- If the zero is complex, its conjugate is also zero. (E) [on the unit circle]
- Complex zeros not on the unit circle always occur in quadruples with r and r^{-1} . (A)



Example: FIR Filtering With Window Method

Problem: Design a 5-tap FIR band reject filter with a lower cutoff frequency of 2,000 Hz, an upper cutoff frequency of 2,400 Hz, and a sampling rate of 8,000 Hz using the Hamming window method.

Solution:

$$\Omega_L = 2\pi f_L T = 2\pi \times 2000/8000 = 0.5\pi \text{ radians}$$
$$\Omega_H = 2\pi f_H T = 2\pi \times 2400/8000 = 0.6\pi \text{ radians}$$

$$2M + 1 = 5$$

$$h(n) = \begin{cases} \frac{\pi - \Omega_H + \Omega_L}{\pi} & n = 0 \\ -\frac{\sin(\Omega_H n)}{n\pi} + \frac{\sin(\Omega_L n)}{n\pi} & n \neq 0 \end{cases} \quad -2 \leq n \leq 2.$$

$$M = 2$$

$$h(0) = \frac{\pi - \Omega_H + \Omega_L}{\pi} = \frac{\pi - 0.6\pi + 0.5\pi}{\pi} = 0.9$$

$$h(1) = \frac{\sin[0.5\pi \times 1]}{1 \times \pi} - \frac{\sin[0.6\pi \times 1]}{1 \times \pi} = 0.01558$$

$$h(2) = \frac{\sin[0.5\pi \times 2]}{2 \times \pi} - \frac{\sin[0.6\pi \times 2]}{2 \times \pi} = 0.09355$$



$$h(-1) = h(1) = 0.01558$$

$$h(-2) = h(2) = 0.09355$$

Symmetry

Example: Window Method - contd.

Hamming
window
function

$$w_{ham}(0) = 0.54 + 0.46 \cos\left(\frac{0 \times \pi}{2}\right) = 1.0$$

$$w_{ham}(1) = 0.54 + 0.46 \cos\left(\frac{1 \times \pi}{2}\right) = 0.54$$

$$w_{ham}(2) = 0.54 + 0.46 \cos\left(\frac{2 \times \pi}{2}\right) = 0.08$$



$$w_{ham}(-1) = w_{ham}(1) = 0.54$$

$$w_{ham}(-2) = w_{ham}(2) = 0.08$$

Symmetry

Windowed
impulse
response

$$h_w(0) = h(0)w_{ham}(0) = 0.9 \times 1 = 0.9$$

$$h_w(1) = h(1)w_{ham}(1) = 0.01558 \times 0.54 = 0.00841$$

$$h_w(2) = h(2)w_{ham}(2) = 0.09355 \times 0.08 = 0.00748$$

$$h_w(-1) = h(-1)w_{ham}(-1) = 0.00841$$

$$h_w(-2) = h(-2)w_{ham}(-2) = 0.00748$$

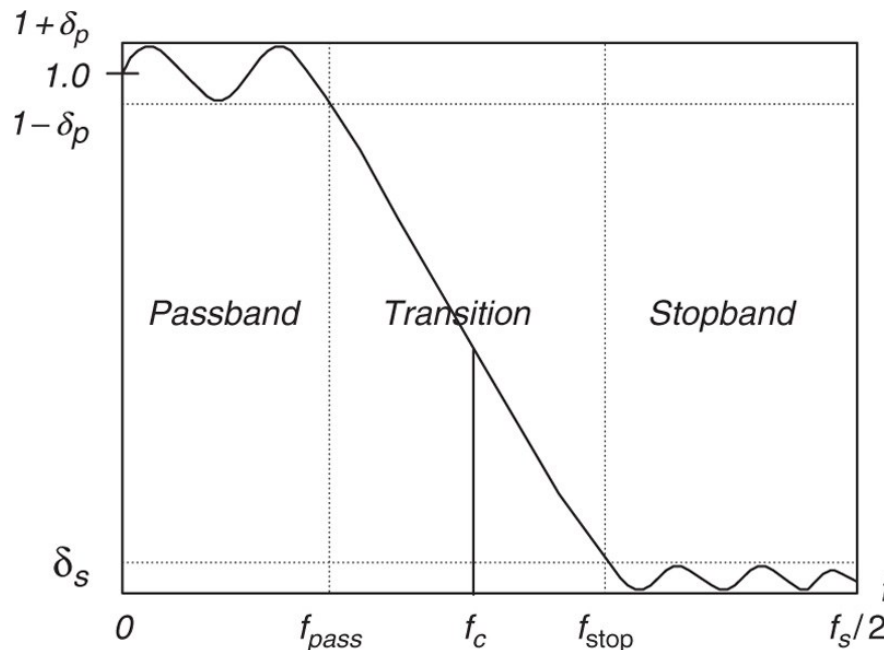
By delaying $h_w(n)$ by $M = 2$ samples,

$$b_0 = b_4 = 0.00748, b_1 = b_3 = 0.00841, \text{ and } b_2 = 0.9$$

$$H(z) = 0.00748 + 0.00841z^{-1} + 0.9z^{-2} + 0.00841z^{-3} + 0.00748z^{-4}$$

FIR Filter Length Estimation

Window Type	Window Function $w(n)$, $-M \leq n \leq M$	Window Length, N	Passband Ripple (dB)	Stopband Attenuation (dB)
Rectangular	1	$N = 0.9/\Delta f$	0.7416	21
Hanning	$0.5 + 0.5 \cos\left(\frac{\pi n}{M}\right)$	$N = 3.1/\Delta f$	0.0546	44
Hamming	$0.54 + 0.46 \cos\left(\frac{\pi n}{M}\right)$	$N = 3.3/\Delta f$	0.0194	53
Blackman	$0.42 + 0.5 \cos\left(\frac{\pi n}{M}\right) + 0.08 \cos\left(\frac{2\pi n}{M}\right)$	$N = 5.5/\Delta f$	0.0017	74



$$\Delta f = |f_{\text{stop}} - f_{\text{pass}}| / f_s$$

$$f_c = (f_{\text{pass}} + f_{\text{stop}}) / 2$$

$$\delta_p \text{ dB} = 20 \cdot \log_{10} (1 + \delta_p)$$

$$\delta_s \text{ dB} = -20 \log_{10} (\delta_s)$$

Example: FIR Filter Length Estimation

Problem:

Design a BPF with

Lower stopband = 0–500 Hz

Passband = 1,600–2,300 Hz

Upper stopband = 3,500–4,000 Hz

Stopband attenuation = 50 dB

Passband ripple = 0.05 dB

Sampling rate = 8,000 Hz

Use Hamming window

Solution:

$$\Delta f_1 = |1600 - 500|/8000 = 0.1375$$

$$\Delta f_2 = |3500 - 2300|/8000 = 0.15$$



$$N_1 = 3.3/0.1375 = 24$$

$$N_2 = 3.3/0.15 = 22$$

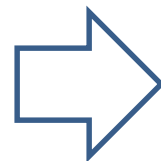
Choose nearest higher odd N = 25

Cutoff frequencies:

$$f_1 = (1600 + 500)/2 = 1050 \text{ Hz}$$

$$f_2 = (3500 + 2300)/2 = 2900 \text{ Hz.}$$

Normalized



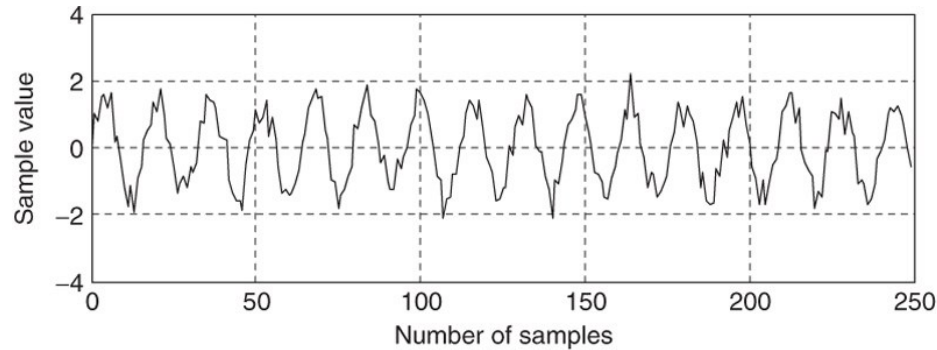
$$\Omega_L = \frac{1050 \times 2\pi}{8000} = 0.2625\pi \text{ radians}$$

$$\Omega_H = \frac{2900 \times 2\pi}{8000} = 0.725\pi \text{ radians}$$

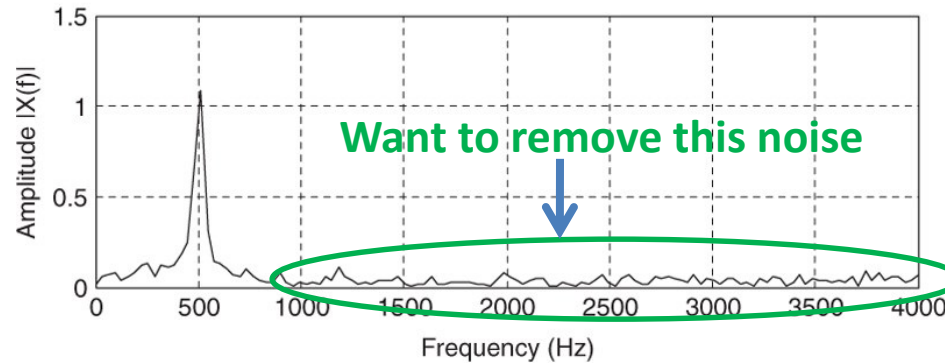
Now design the filter with hint from slide 15.

Application: Noise Reduction

Input waveform:
sinusoid +
broadband
noise



Spectrum:



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Specification: **LPF**

Pass band frequency [0 – 800 Hz]

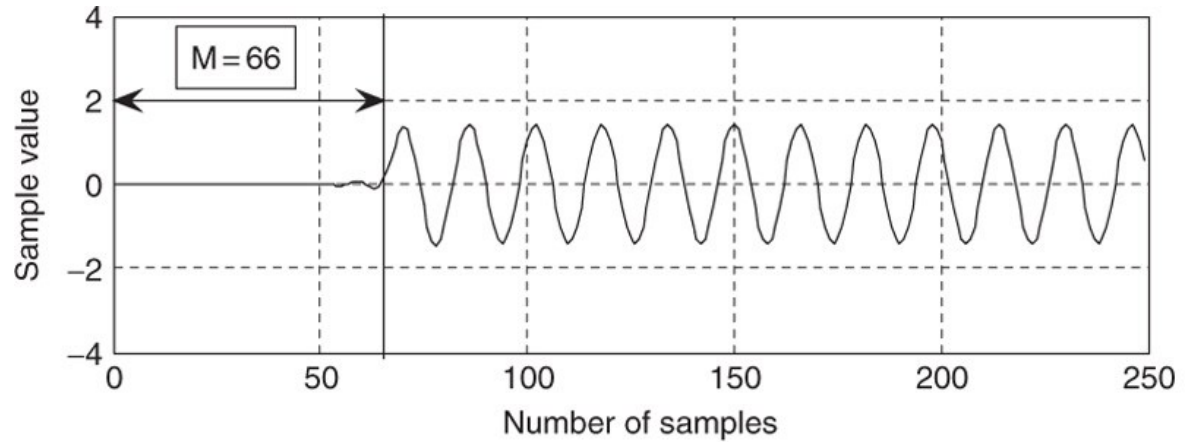
Stop band frequency [1000 – 4000 Hz]

Pass band ripple < 0.02 dB

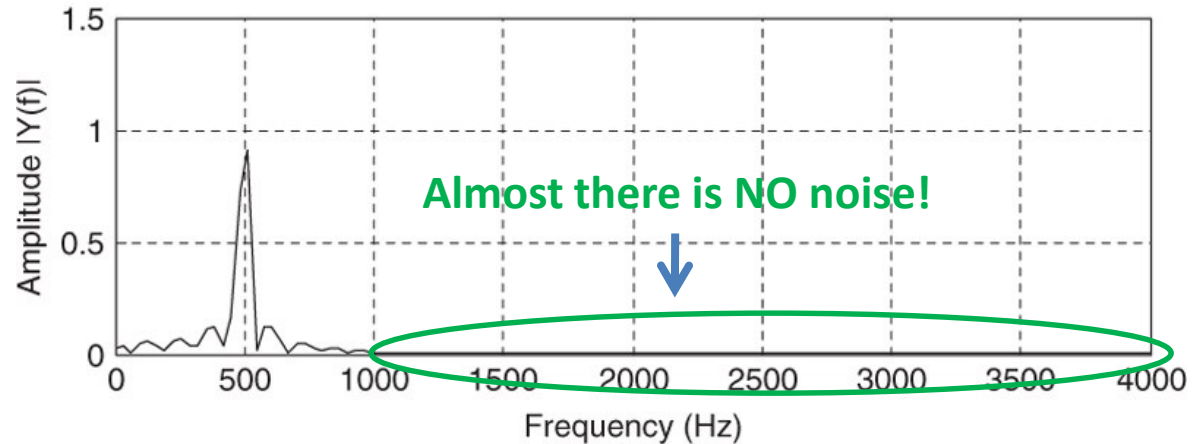
Stop band attenuation = 50 dB

Application: Noise Reduction -contd.

133- tap FIR filter,
so a delay of 66



However, noise
reduction in real
world scenario is
not so easy!



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Frequency Sampling Design Method

Simple to design

Filter length = $2M + 1$

$$H_k \text{ at } \Omega_k = \frac{2\pi k}{(2M + 1)} \text{ for } k = 0, 1, \dots, M$$

Magnitude response in the range $[0 \sim \pi]$

Calculate FIR filter coefficients:

$$h(n) = \frac{1}{2M + 1} \left\{ H_0 + 2 \sum_{k=1}^M H_k \cos\left(\frac{2\pi k(n - M)}{2M + 1}\right) \right\}$$

for $n = 0, 1, \dots, M$.

Use the symmetry:

$$h(n) = h(2M - n) \text{ for } n = M + 1, \dots, 2M.$$

Example: Frequency Sampling Design Method

Problem: Design a linear phase lowpass FIR filter with 7 taps and a cutoff frequency of $\Omega_c = 0.3\pi$ radian using the frequency sampling method.

Solution:

$$N = 2M + 1 = 7 \rightarrow M = 3, \quad \Rightarrow \quad \Omega_k = \frac{2\pi}{7}k \text{ radians, } k = 0, 1, 2, 3.$$

for $\Omega_0 = 0$ radians, $H_0 = 1.0$

for $\Omega_1 = \frac{2}{7}\pi$ radians, $H_1 = 1.0$

for $\Omega_2 = \frac{4}{7}\pi$ radians, $H_2 = 0.0$

for $\Omega_3 = \frac{6}{7}\pi$ radians, $H_3 = 0.0$.

$$h(n) = \frac{1}{7} \left\{ 1 + 2 \sum_{k=1}^3 H_k \cos [2\pi k(n-3)/7] \right\}, \quad n = 0, 1, \dots, 3.$$
$$= \frac{1}{7} \{ 1 + 2 \cos [2\pi(n-3)/7] \}$$

$$h(0) = \frac{1}{7} \{ 1 + 2 \cos(-6\pi/7) \} = -0.11456$$

$$h(1) = \frac{1}{7} \{ 1 + 2 \cos(-4\pi/7) \} = 0.07928$$

$$h(2) = \frac{1}{7} \{ 1 + 2 \cos(-2\pi/7) \} = 0.32100$$

$$h(3) = \frac{1}{7} \{ 1 + 2 \cos(-0 \times \pi/7) \} = 0.42857.$$

By symmetry:

$$h(4) = h(2) = 0.32100$$

$$h(5) = h(1) = 0.07928$$

$$h(6) = h(0) = -0.11456.$$

Coefficient Quantization Effect

Filter coefficients are usually truncated or rounded off for the application.

Transfer function with infinite precision:

$$H(z) = \sum_{n=0}^K b_n z^{-n} = b_0 + b_1 z^{-1} + \dots + b_K z^{-K}$$

Transfer function with quantized precision:

$$H^q(z) = \sum_{n=0}^K b_n^q z^{-n} = b_0^q + b_1^q z^{-1} + \dots + b_K^q z^{-K}$$

Error of the magnitude frequency response:

$$|H(e^{j\Omega}) - H^q(e^{j\Omega})| = \sum_{n=0}^K |(b_n - b_n^q) e^{-jn\Omega}|$$

$$< \sum_{n=0}^K |b_n - b_n^q| < (K + 1) \cdot 2^{-B-1}$$

$K = \text{tap}$

Example

25 – tap FIR filter; 7 bits used for fraction

Let infinite precision coeff. = 0.00759455135346

$$0.00759455135346 \times 2^7 = 0.9721 = 1(\text{rounded up to the integer})$$

Quantized coeff. = $1 / 2^7 = 0.0078125$

Error is bounded by
 $< 25 / 256 = 0.0977$

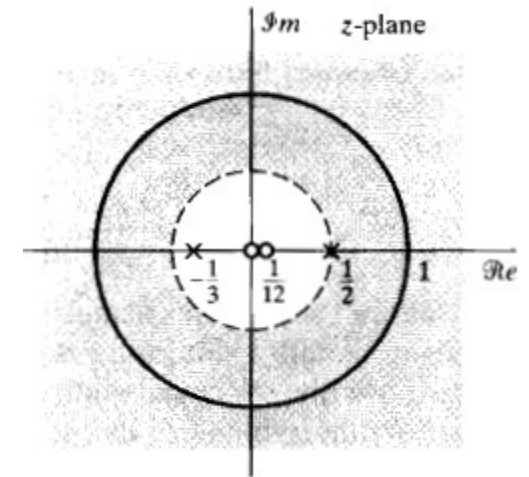
Complementary Example - I

Consider a signal that is the sum of two real exponentials:

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n].$$

The z -transform is then

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} \left\{ \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n] \right\} z^{-n} \\ &= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u[n] z^{-n} + \sum_{n=-\infty}^{\infty} \left(-\frac{1}{3}\right)^n u[n] z^{-n} \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^n + \sum_{n=0}^{\infty} \left(-\frac{1}{3} z^{-1}\right)^n \\ &= \frac{1}{1 - \frac{1}{2} z^{-1}} + \frac{1}{1 + \frac{1}{3} z^{-1}} = \frac{2(1 - \frac{1}{12} z^{-1})}{(1 - \frac{1}{2} z^{-1})(1 + \frac{1}{3} z^{-1})} \\ &= \frac{2z(z - \frac{1}{12})}{(z - \frac{1}{2})(z + \frac{1}{3})}. \end{aligned}$$



$$z \text{ transform of } a^n u(n) = \frac{z}{z - a}$$

$$= \frac{1}{1 - az^{-1}}$$

Complementary Example - II

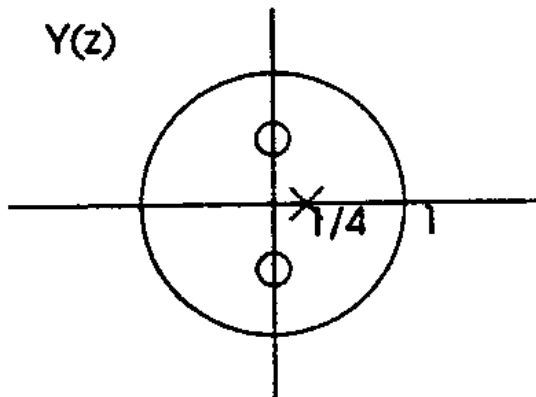
Given:

$$y[n] = \left(\frac{1}{2}\right)^n x[n]$$

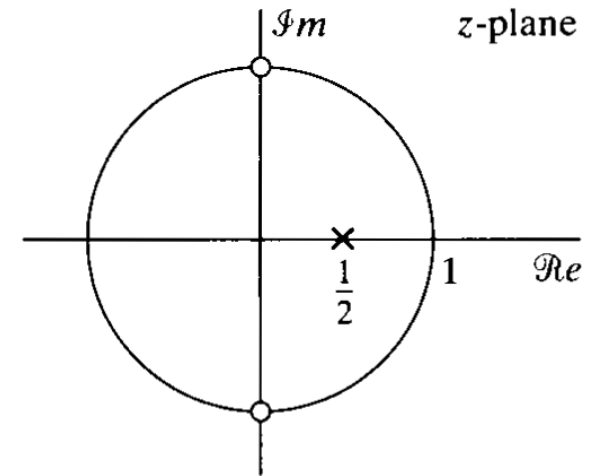
$$y[n] = \left(\frac{1}{2}\right)^n x[n] \Rightarrow Y(z) = X(2z) = \frac{4z^2 + 1}{2z - \frac{1}{2}}$$

zeros $\pm \frac{1}{2}j$

poles $\frac{1}{4}, \infty$

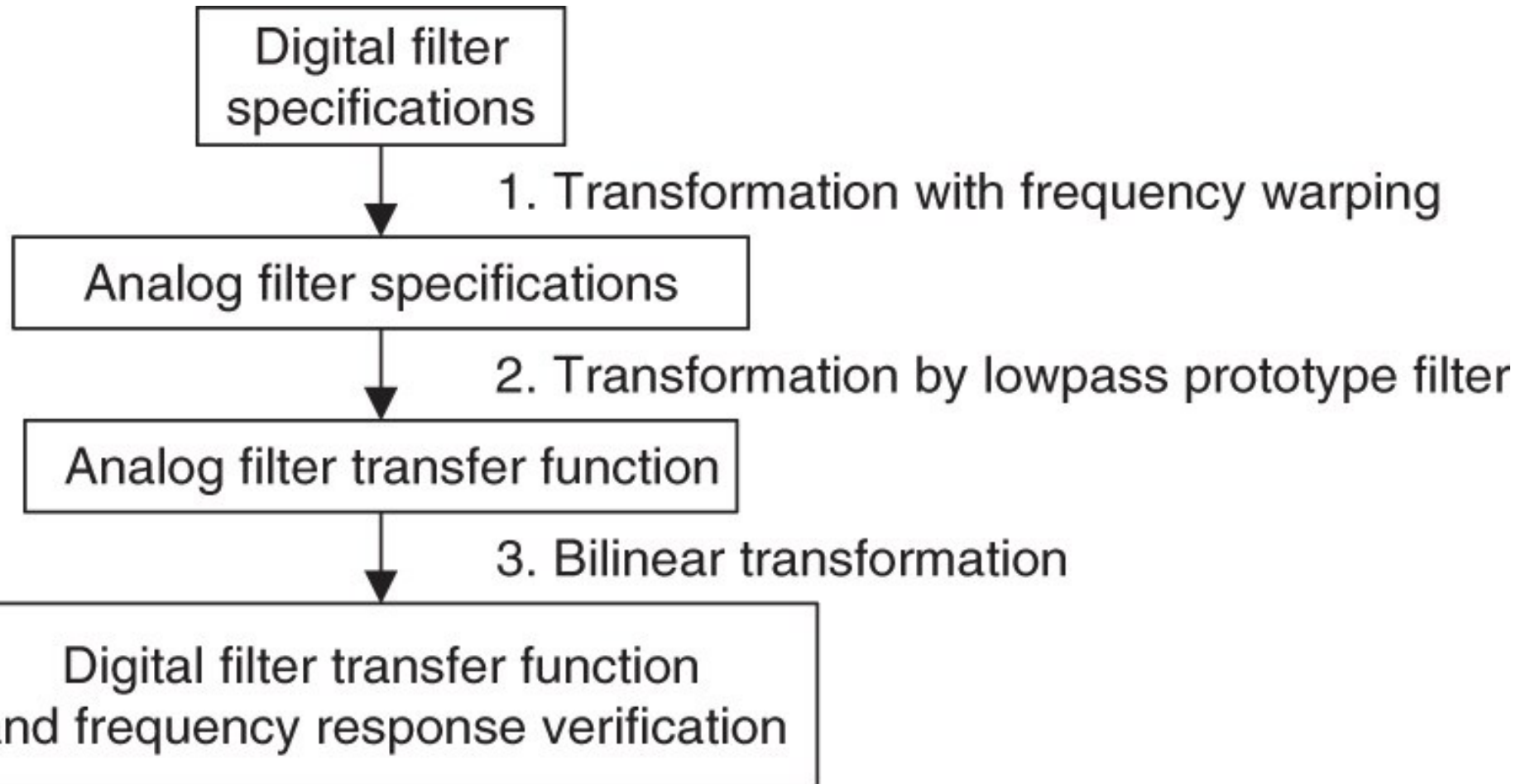


Given:



$$X(z) = \frac{z^2 + 1}{z - \frac{1}{2}}$$

IIR Filter Design: Bilinear Transformation Method



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Bilinear Transformation Method

For LPF and HPF: $\omega_a = \frac{2}{T} \tan\left(\frac{\omega_d T}{2}\right)$

For BPF and BRF: $\omega_{al} = \frac{2}{T} \tan\left(\frac{\omega_l T}{2}\right)$, $\omega_{ah} = \frac{2}{T} \tan\left(\frac{\omega_h T}{2}\right)$

$$\omega_0 = \sqrt{\omega_{al}\omega_{ah}}, \quad W = \omega_{ah} - \omega_{al}$$

Frequency
Warping

From LPF to LPF: $H(s) = H_P(s)|_{s=\frac{s}{\omega_a}}$

From LPF to HPF: $H(s) = H_P(s)|_{s=\frac{\omega_a}{s}}$

From LPF to BPF: $H(s) = H_P(s)|_{s=\frac{s^2+\omega_0^2}{sW}}$

From LPF to BRF: $H(s) = H_P(s)|_{s=\frac{sW}{s^2+\omega_0^2}}$

Prototype
Transformation

Obtained Transfer Function: $H(z) = H(s)|_{s=\frac{2z-1}{Tz+1}}$

Example 1: Bilinear Transformation Method

Problem: Design a first-order digital highpass Chebyshev filter with a cutoff frequency of 3 kHz and 1 dB ripple on passband using a sampling frequency of 8,000 Hz.

Solution:

$$\omega_d = 2\pi f = 2\pi(3000) = 6000\pi \text{ rad/sec, and } T = 1/f_s = 1/8000 \text{ sec.}$$

$$\omega_a = \frac{2}{T} \tan\left(\frac{\omega_d T}{2}\right) = 16000 \times \tan\left(\frac{6000\pi/8000}{2}\right) = 3.8627 \times 10^4 \text{ rad/sec.}$$

First-order LP Chebyshev filter prototype: $H_P(s) = \frac{1.9652}{s + 1.9625}$

Applying transformation LPF to HPF: $H(s) = H_P(s)\Big|_{\frac{\omega_a}{s}} = \frac{1.9652}{\frac{\omega_a}{s} + 1.9652} = \frac{1.9652s}{1.9652s + 3.8627 \times 10^4}$

$$\Rightarrow H(s) = \frac{s}{s + 1.9656 \times 10^4}$$

Applying BLT: $H(z) = \frac{s}{s + 1.9656 \times 10^4} \Big|_{s=16000(z-1)/(z+1)}$

$$H(z) = \frac{0.4487 - 0.4487z^{-1}}{1 + 0.1025z^{-1}}$$

Example 2: Bilinear Transformation Method

Problem: Design a second-order digital bandpass Butterworth filter with the following specifications:

- an upper cutoff frequency of 2.6 kHz and
- a lower cutoff frequency of 2.4 kHz,
- a sampling frequency of 8,000 Hz.

Solution:

$$\omega_h = 2\pi f_h = 2\pi(2600) = 5200\pi \text{ rad/sec}$$

$$\omega_l = 2\pi f_l = 2\pi(2400) = 4800\pi \text{ rad/sec, and } T = 1/f_s = 1/8000 \text{ sec.}$$

$$\omega_{ah} = \frac{2}{T} \tan\left(\frac{\omega_h T}{2}\right) = 16000 \times \tan\left(\frac{5200\pi/8000}{2}\right) = 2.6110 \times 10^4 \text{ rad/sec}$$

$$\omega_{al} = 16000 \times \tan\left(\frac{\omega_l T}{2}\right) = 16000 \times \tan(0.3\pi) = 2.2022 \times 10^4 \text{ rad/sec}$$

$$W = \omega_{ah} - \omega_{al} = 26110 - 22022 = 4088 \text{ rad/sec}$$

$$\omega_0^2 = \omega_{ah} \times \omega_{al} = 5.7499 \times 10^8$$

A first-order LPF prototype will produce second-order BPF prototype.

Example 2: Bilinear Transformation Method

Contd.

1st order LPF prototype: $H_P(s) = \frac{1}{s + 1}$

Applying transformation

LPF to BPF:

$$H(s) = H_P(s) \Big|_{\frac{s^2 + \omega_0^2}{sW}} = \frac{Ws}{s^2 + Ws + \omega_0^2} = \frac{4088s}{s^2 + 4088s + 5.7499 \times 10^8}$$

Applying BLT:

$$H(z) = \frac{4088s}{s^2 + 4088s + 5.7499 \times 10^8} \Big|_{s=16000(z-1)/(z+1)}$$



$$H(z) = \frac{0.0730 - 0.0730z^{-2}}{1 + 0.7117z^{-1} + 0.8541z^{-2}}$$

Pole Zero Placement Method

Second-Order BPF Design

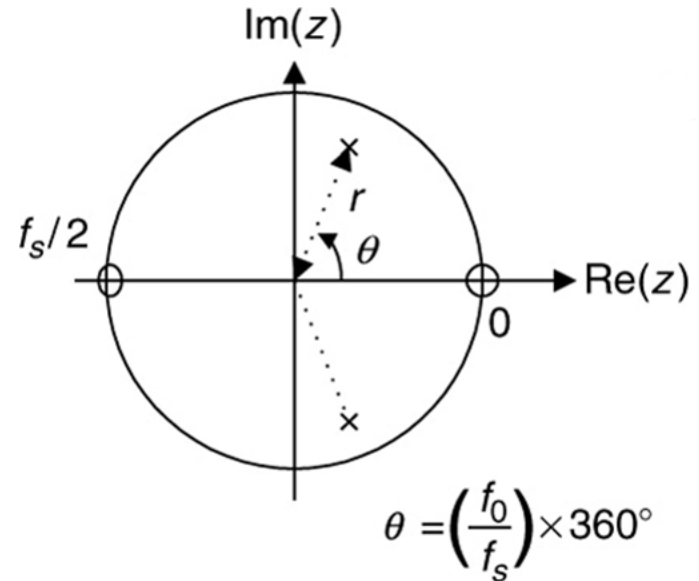
r: controls bandwidth

θ : controls central frequency

Location of poles & zeros:
controls magnitude

Location of pole:
determines stability

Number of zero:
determines phase linearity



$$r \approx 1 - (BW_{3dB} / f_s) \times \pi \qquad \theta = \left(\frac{f_0}{f_s}\right) \times 360^\circ$$

$$H(z) = \frac{K(z-1)(z+1)}{(z-re^{j\theta})(z-re^{-j\theta})} = \frac{K(z^2-1)}{(z^2-2rz\cos\theta+r^2)}$$

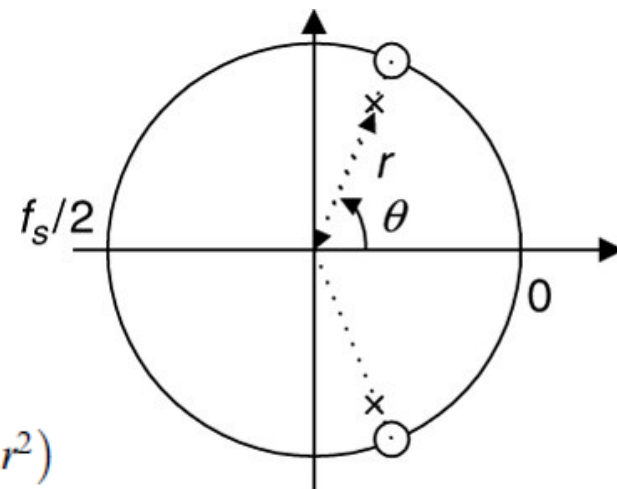
$$K = \frac{(1-r)\sqrt{1-2r\cos 2\theta+r^2}}{2|\sin\theta|}$$

Pole Zero Placement Method

Second-Order **BRF** Design

$$r \approx 1 - (BW_{3dB}/f_s) \times \pi \quad \theta = \left(\frac{f_0}{f_s}\right) \times 360^\circ$$

$$H(z) = \frac{K(z - e^{j\theta})(z + e^{-j\theta})}{(z - re^{j\theta})(z - re^{-j\theta})} = \frac{K(z^2 - 2z \cos \theta + 1)}{(z^2 - 2rz \cos \theta + r^2)}$$



Example

$$K = \frac{(1 - 2r \cos \theta + r^2)}{(2 - 2 \cos \theta)}$$

Sampling rate = 8,000 Hz

3 dB bandwidth: $BW = 100$ Hz

Narrow passband centered at $f_0 = 1,500$ Hz.

$$K = \frac{(1 - 2 \times 0.9607 \cos 67.5^\circ + 0.9607^2)}{(2 - 2 \cos 67.5^\circ)} = 0.9620$$

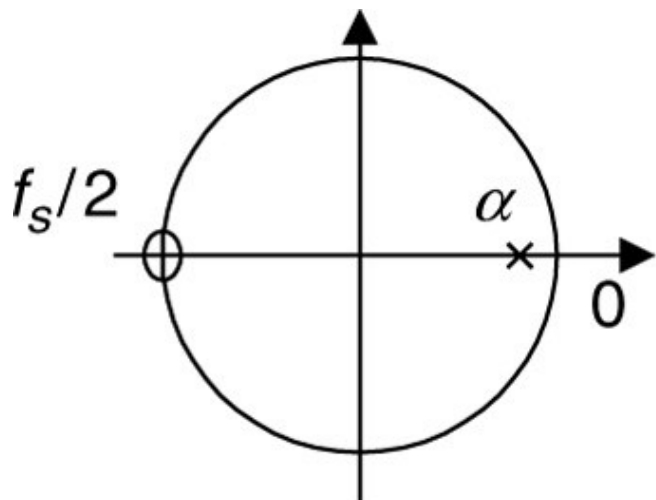
$$r \approx 1 - (100/8000) \times \pi = 0.9607,$$

$$\theta = \left(\frac{1500}{8000}\right) \times 360^\circ = 67.5^\circ$$

$$H(z) = \frac{0.9620(z^2 - 2z \cos 67.5^\circ + 1)}{(z^2 - 2 \times 0.9607z \cos 67.5^\circ + 0.9607^2)} \\ = \frac{0.9620 - 0.7363z^{-1} + 0.9620z^{-2}}{1 - 0.7353z^{-1} + 0.9229z^{-2}}$$

Pole Zero Placement Method

First-Order LPF Design



When $f_c < f_s/4$, $\alpha \approx 1 - 2 \times (f_c/f_s) \times \pi$.

$$H(z) = \frac{K(z+1)}{(z-\alpha)} \quad K = \frac{(1-\alpha)}{2}$$

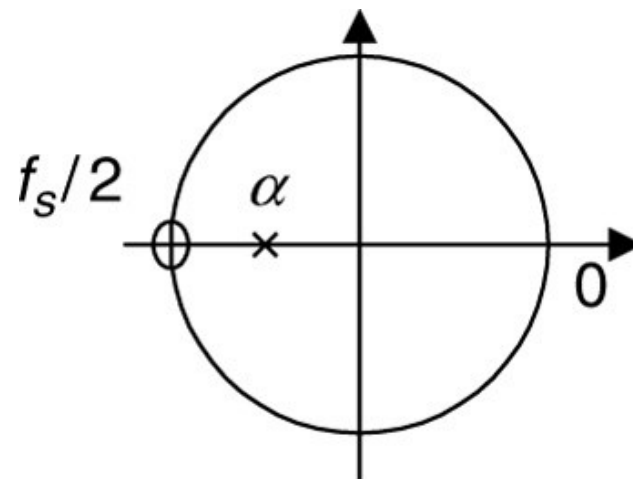
Example

Sampling rate = 8,000 Hz

3 dB cutoff frequency: $f_c = 100$ Hz

$100 \text{ Hz} < f_s/4 = 2,000 \text{ Hz}$

$$\alpha \approx 1 - 2 \times (100/8000) \times \pi = 0.9215$$



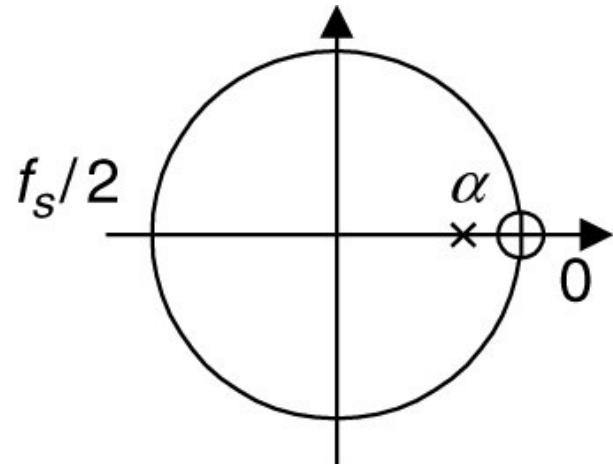
When $f_c > f_s/4$, $\alpha \approx -(1 - \pi + 2 \times (f_c/f_s) \times \pi)$.

$$K = \frac{(1 - 0.9215)}{2} = 0.03925$$

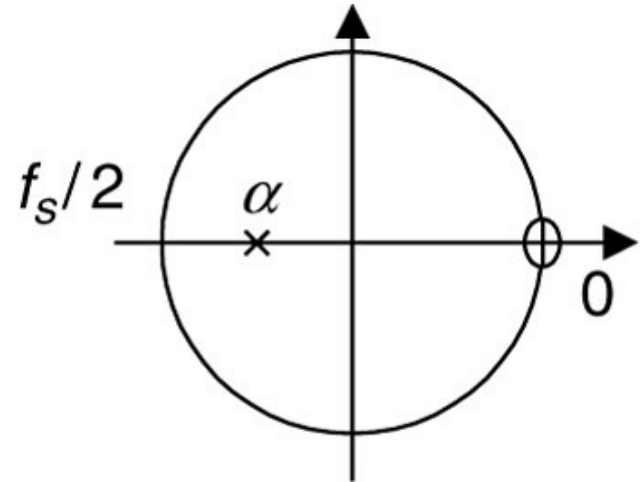
$$H(z) = \frac{0.03925(z+1)}{(z-0.9215)} = \frac{0.03925 + 0.03925z^{-1}}{1 - 0.9215z^{-1}}$$

Pole Zero Placement Method

First-Order HPF Design



When $f_c < f_s/4$, $\alpha \approx 1 - 2 \times (f_c/f_s) \times \pi$



When $f_c > f_s/4$, $\alpha \approx -(1 - \pi + 2 \times (f_c/f_s) \times \pi)$

$$H(z) = \frac{K(z-1)}{(z-\alpha)} \quad K = \frac{(1+\alpha)}{2}$$

Practice examples.

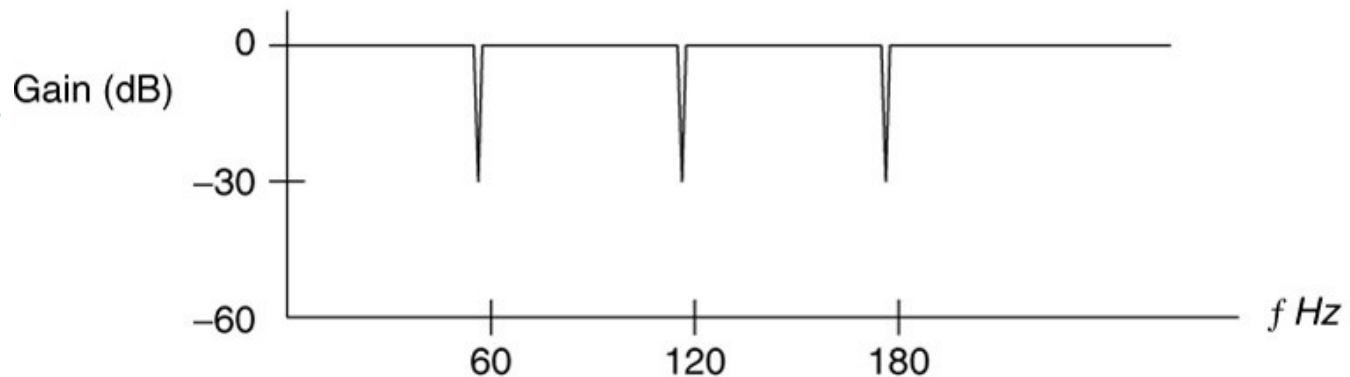
Application: 60 - Hz Hum Eliminator

Hum noise: created by poor power supply or electromagnetic interference and characterized by a frequency of 60 Hz and its harmonics.

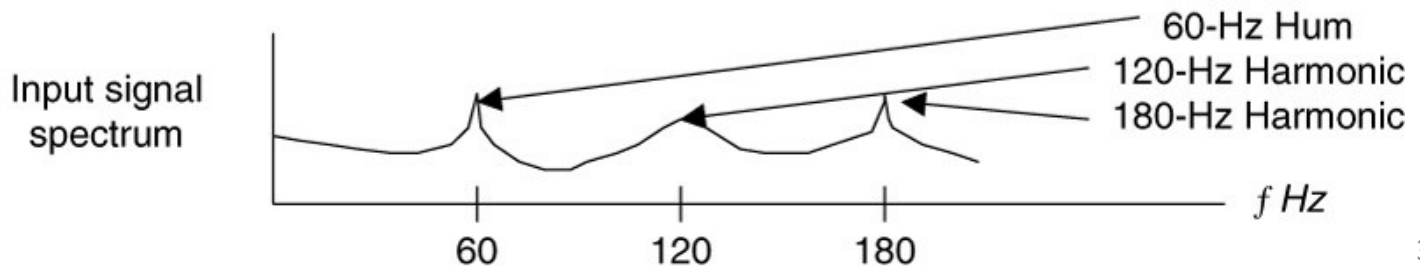
Hum
eliminator



Frequency
response of
Hum
eliminator

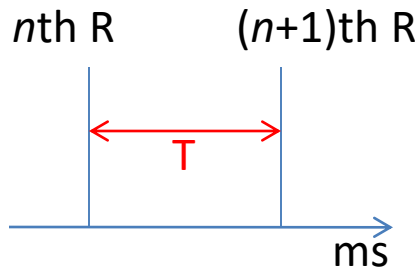


Corrupted
by hum &
harmonics

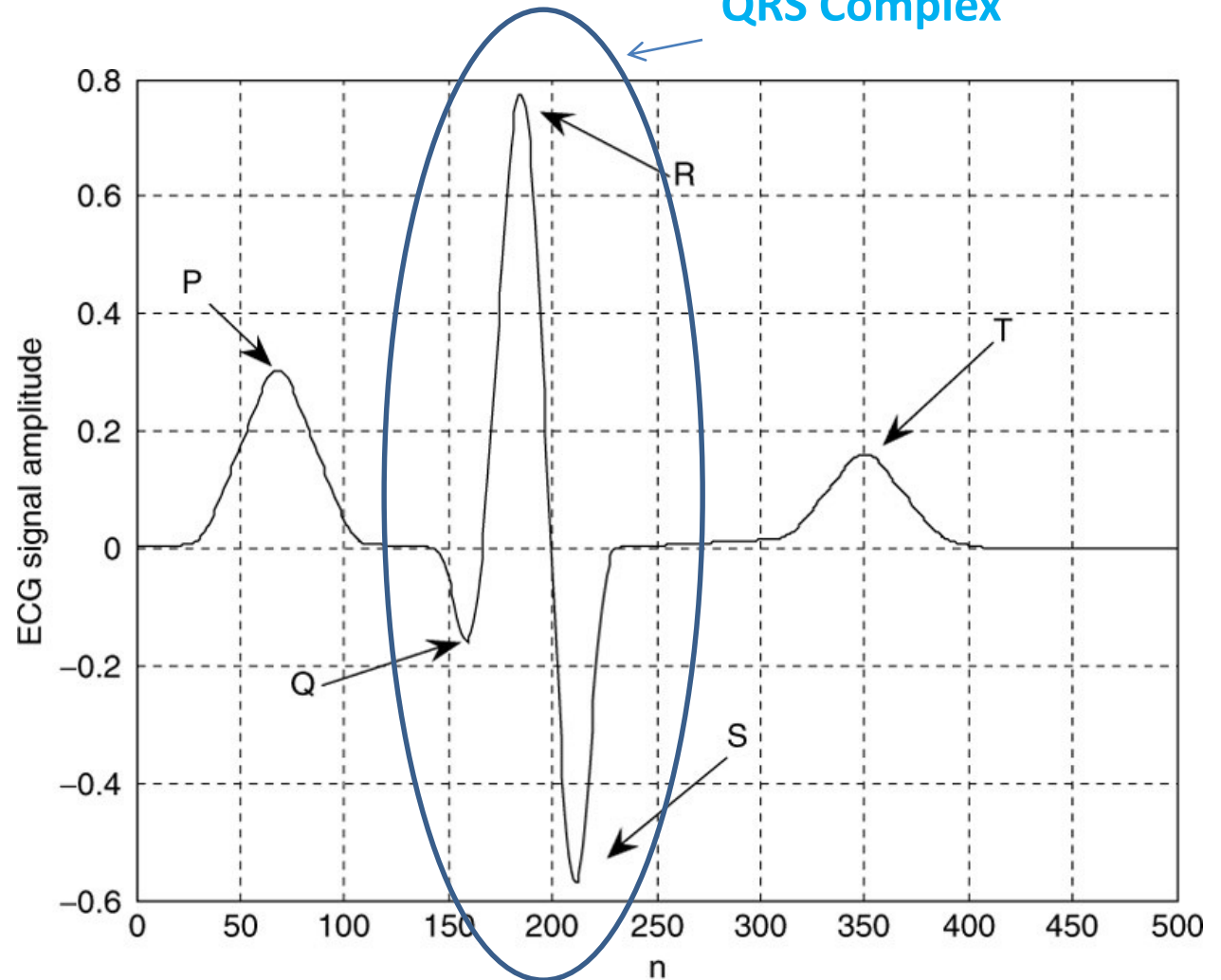


ECG Pulse

ECG + Hum \rightarrow makes difficult to analyze.



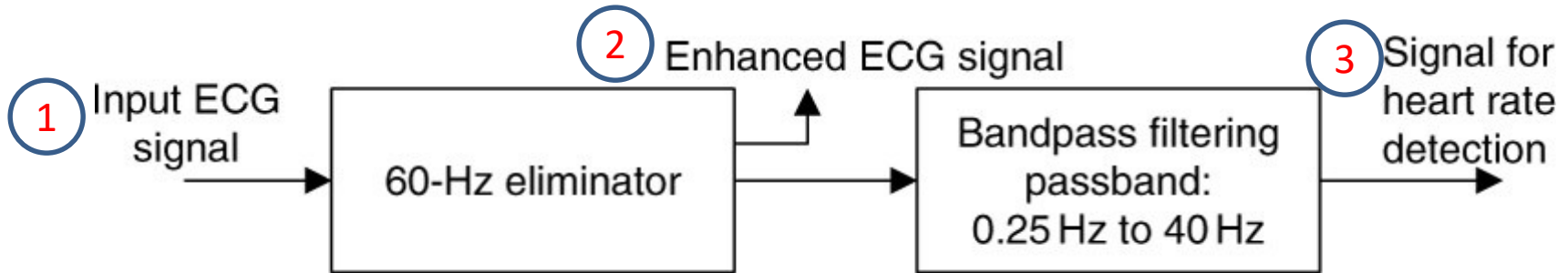
QRS Complex



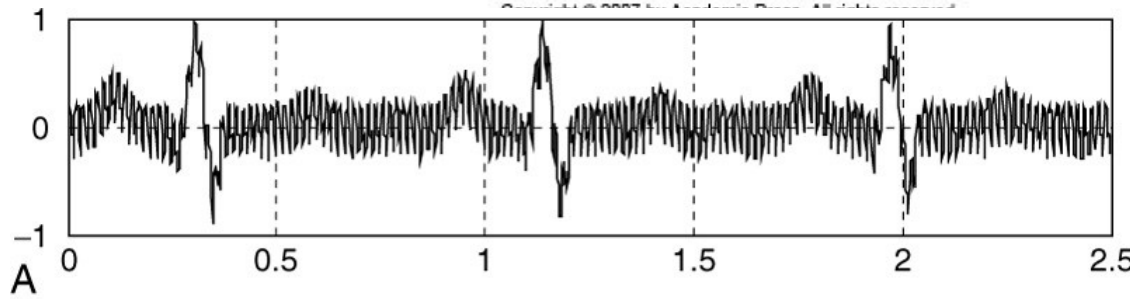
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$$\text{Heart beat /min} = 60000 / T$$

Heart Beat Detection Using ECG Pulse

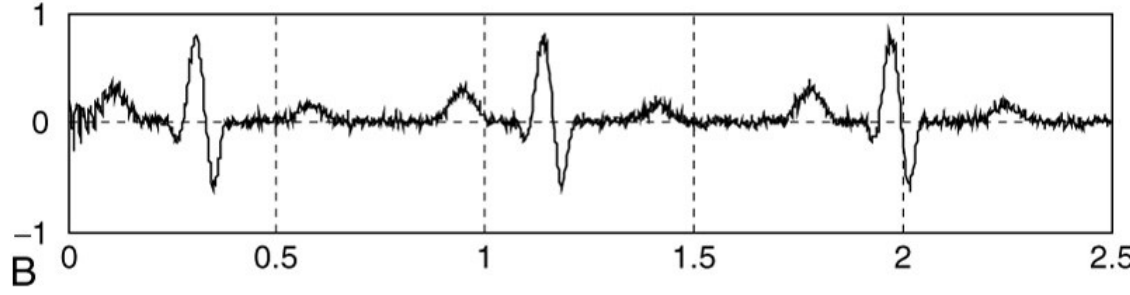


1

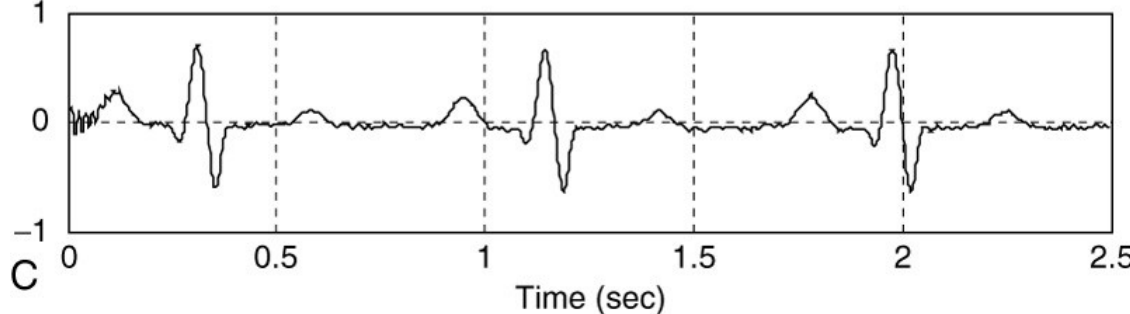


To filter muscle noise ≈ 40 Hz

2



3



Practice example