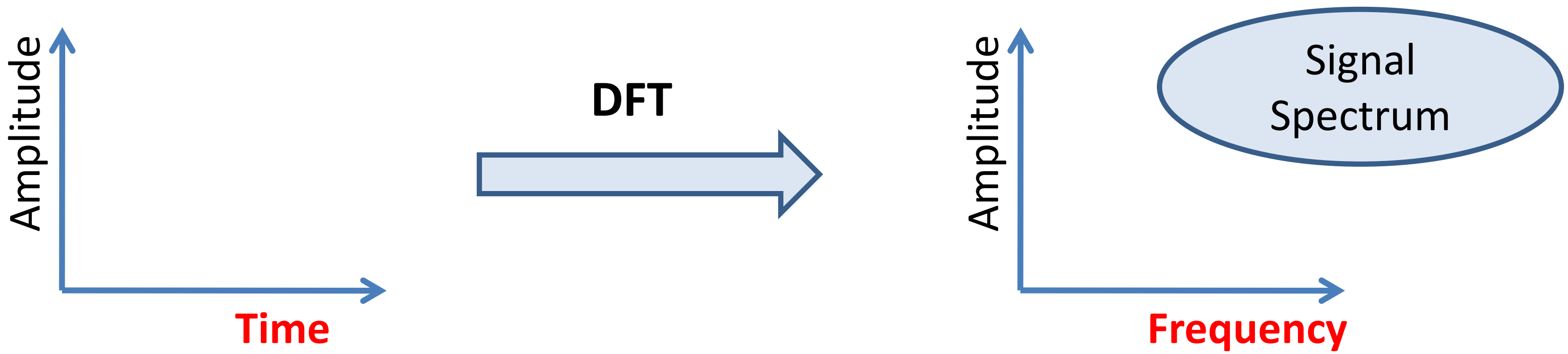
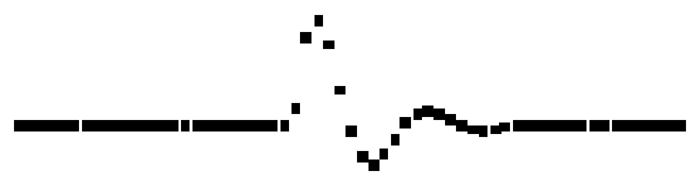
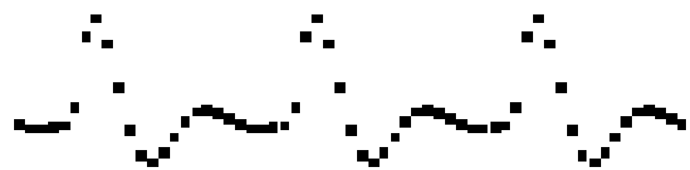
Discrete Fourier Transform (DFT)

DFT transforms the time domain signal samples to the frequency domain components.

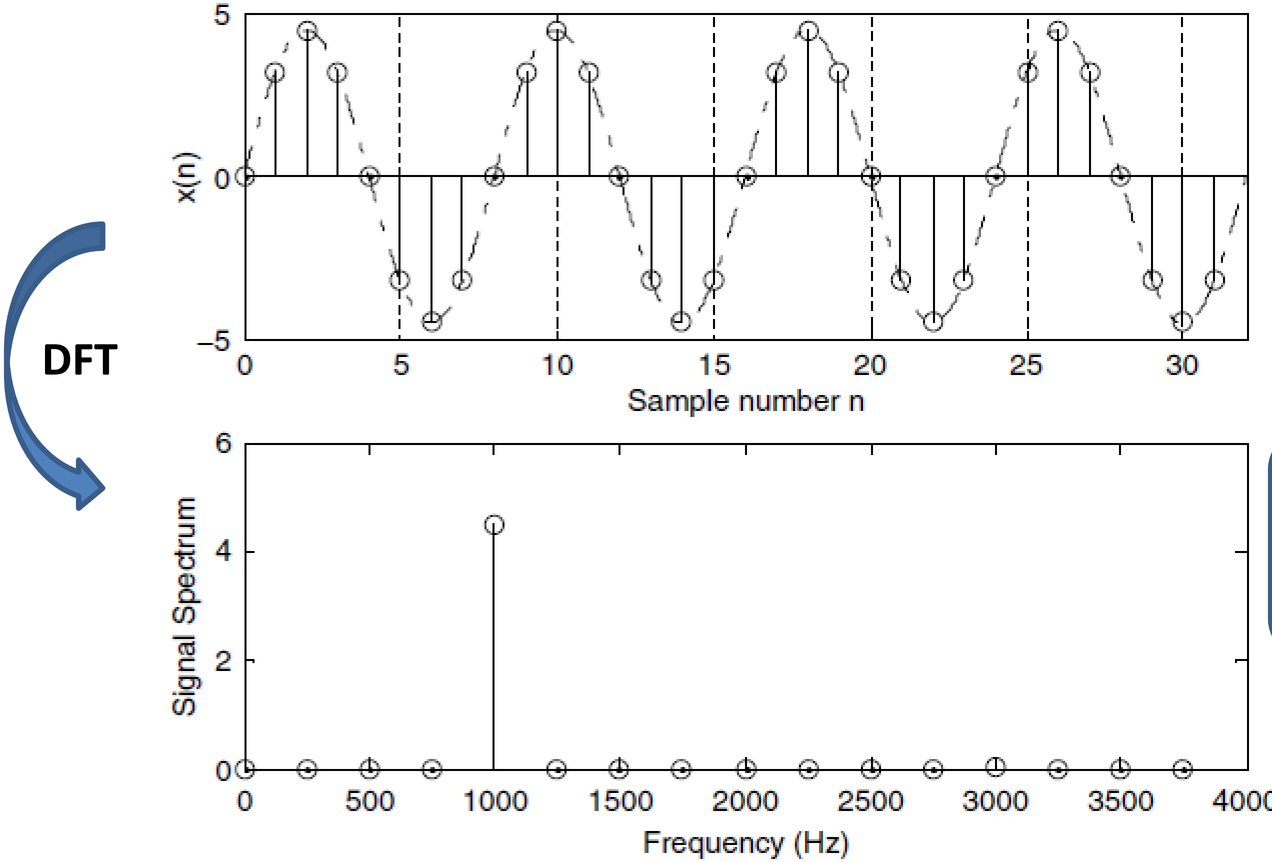


DFT is often used to do frequency analysis of a time domain signal.

Four Types of Fourier Transform

Type of Transform	Example Signal
Fourier Transform *signals that are continuous and aperiodic*	
Fourier Series *signals that are continuous and periodic*	
Discrete Time Fourier Transform *signals that are discrete and aperiodic*	
Discrete Fourier Transform *signals that are discrete and periodic*	

DFT: Graphical Example



1000 Hz sinusoid with 32 samples at 8000 Hz sampling rate.

Sampling rate

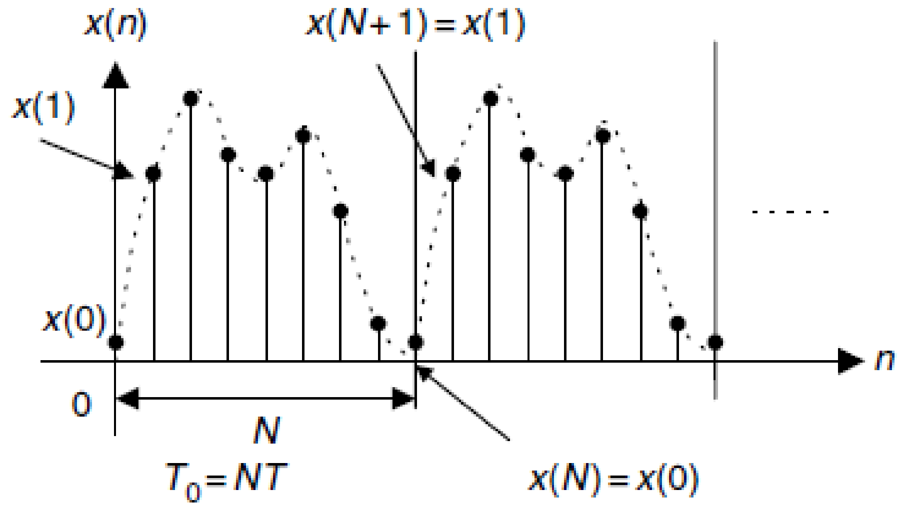
8000 samples = 1 second
32 samples = 32/8000 sec
= 4 millisecond

Frequency

1 second = 1000 cycles
32/8000 sec =
(1000*32/8000=) 4 cycles

DFT Coefficients of Periodic Signals

Periodic Digital Signal



Equation of DFT coefficients: $$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi kn}{N}}, \quad -\infty < k < \infty$$

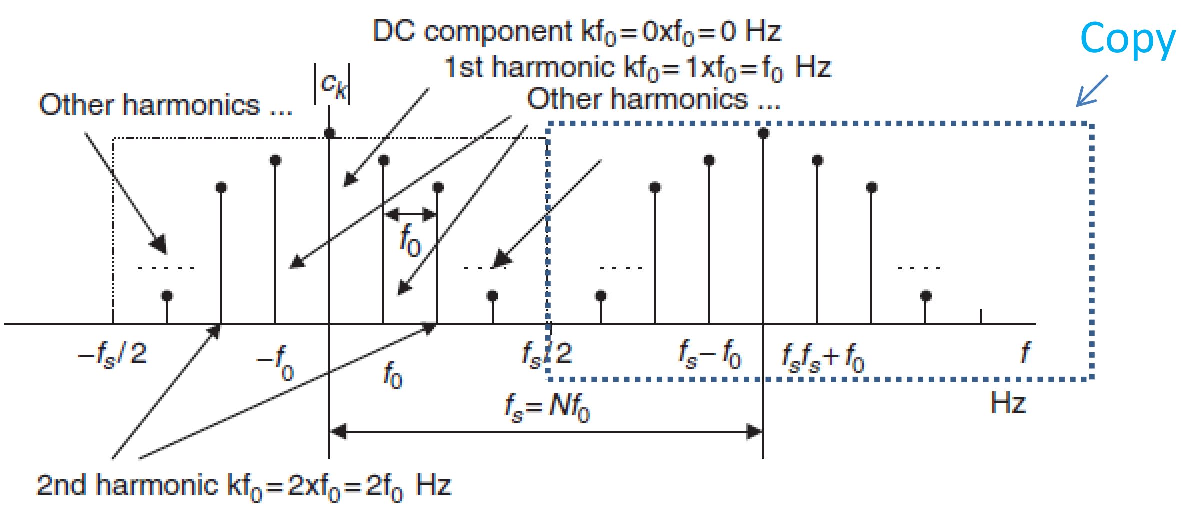
DFT Coefficients of Periodic Signals

Fourier series coefficient c_k is periodic of *N*

$$c_{k+N} = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi(k+N)n}{N}} = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi kn}{N}}e^{-j2\pi n}$$

$$\text{Since } e^{-j2\pi n} = \cos(2\pi n) - j\sin(2\pi n) = 1, \quad \Rightarrow \quad c_{k+N} = c_k.$$

Amplitude spectrum of the periodic digital signal



Example 1

The periodic signal: $\boxed{x(t) = \sin(2\pi t)}$ is sampled at $f_s = 4\,\mathrm{Hz}$

a. Compute the spectrum c_k using the samples in one period.

b. Plot the two-sided amplitude spectrum $|c_k|$ over the range from -2 to 2 Hz.

Solution:

Fundamental frequency

a. We match $x(t) = \sin(2\pi t)$ with $x(t) = \sin(2\pi f t)$ and get ***f* = 1 Hz**.

Therefore the signal has 1 cycle or 1 period in 1 second.

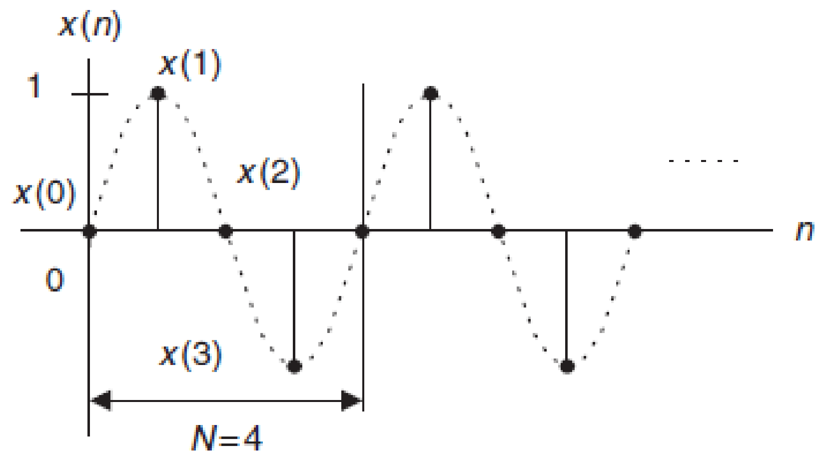
Sampling rate f_s = 4 Hz $\Rightarrow$ 1 second has 4 samples.

Hence, there are 4 samples in 1 period for this particular signal.

$T = 1/f_s = 0.25$ $\xrightarrow{\text{Sampled signal}}$ $x(n) = x(nT) = \sin(2\pi nT) = \sin(0.5\pi n)$

Example 1 - contd. (1)

$x(0) = 0;\ x(1) = 1;\ x(2) = 0;\ \text{and}\ x(3) = -1$



b.

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi kn}{N}}, \quad -\infty < k < \infty$$

$$c_0 = \frac{1}{4}\sum_{n=0}^{3} x(n) = \frac{1}{4}(x(0)+x(1)+x(2)+x(3)) = \frac{1}{4}(0+1+0-1) = 0$$

$$c_1 = \frac{1}{4}\sum_{n=0}^{3} x(n)e^{-j2\pi\times 1n/4} = \frac{1}{4}\left(x(0)+x(1)e^{-j\pi/2}+x(2)e^{-j\pi}+x(3)e^{-j3\pi/2}\right)$$

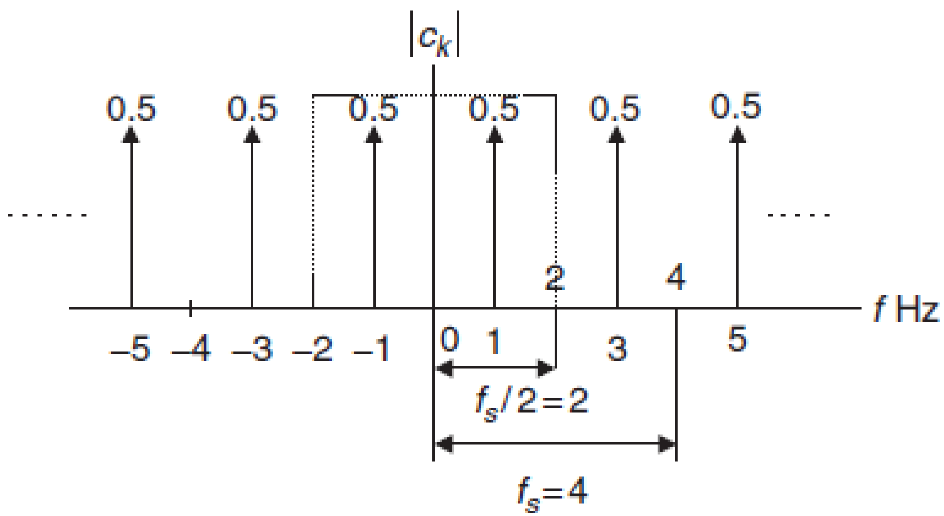
$$= \frac{1}{4}(x(0)-jx(1)-x(2)+jx(3) = 0-j(1)-0+j(-1)) = -j0.5.$$

Example 1 - contd. (2)

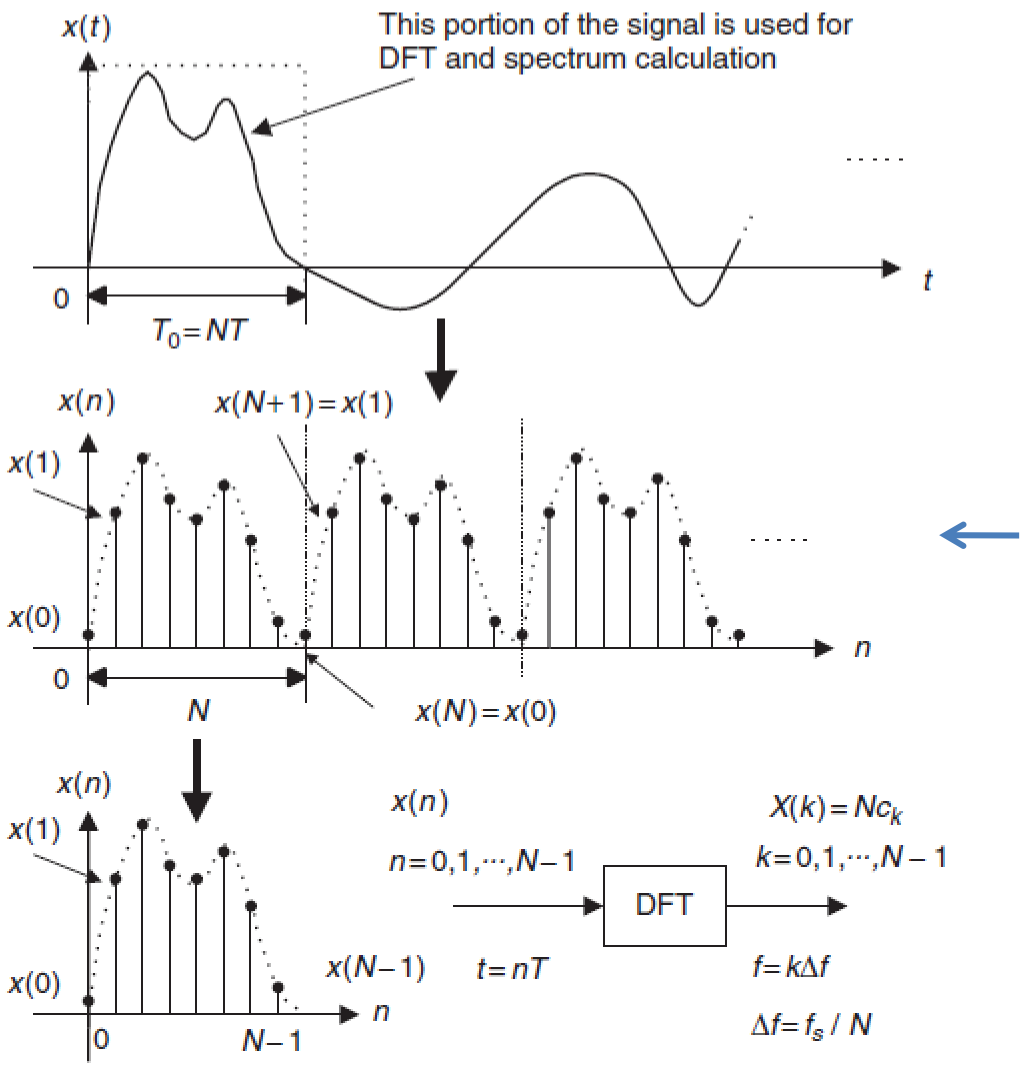
$$c_2 = \frac{1}{4}\sum_{k=0}^{3} x(n)e^{-j2\pi\times 2n/4} = 0, \text{ and } c_3 = \frac{1}{4}\sum_{n=0}^{3} x(k)e^{-j2\pi\times 3n/4} = j0.5.$$

Using periodicity, it follows that

$$c_{-1} = c_3 = j0.5, \text{ and } c_{-2} = c_2 = 0.$$



On the Way to DFT Formulas



Imagine periodicity of *N* samples.

Take first *N* samples (index 0 to *N* -1) as the input to DFT.

DFT Formulas

$$\boxed{X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}}$$

$$= \sum_{n=0}^{N-1} x(n) W_N^{kn}, \text{ for } k = 0,\ 1, \ldots,\ N-1.$$

$$X(k) = x(0)\, W_N^{k0} + x(1)\, W_N^{k1} + x(2) W_N^{k2} + \ldots + x(N-1) W_N^{k(N-1)}$$

Where, $W_N = e^{-j2\pi/N} = \cos\left(\frac{2\pi}{N}\right) - j\sin\left(\frac{2\pi}{N}\right).$

Inverse DFT:

$$\boxed{x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N}} = \frac{1}{N}\sum_{k=0}^{N-1} X(k) W_N^{-kn}, \text{ for } n = 0{,}1, \ldots,\ N-1$$

MATLAB Functions

FFT: Fast Fourier Transform

MATLAB FFT functions.

X = fft(x)	% Calculate DFT coefficients
x = ifft(X)	% Inverse DFT
x = input vector	
X = DFT coefficient vector	

Example 2

Given a sequence $x(n)$ for $0 \le n \le 3$, where $x(0) = 1$, $x(1) = 2$, $x(2) = 3$, and $x(3) = 4$,

a. Evaluate its DFT $X(k)$.

Solution:

$N = 4$ and $W_4 = e^{-j\frac{\pi}{2}}$ $\Rightarrow$ $X(k) = \sum_{n=0}^{3} x(n) W_4^{kn} = \sum_{n=0}^{3} x(n) e^{-j\frac{\pi k n}{2}}$

Thus, for $k = 0$

$$X(0) = \sum_{n=0}^{3} x(n) e^{-j0} = x(0)e^{-j0} + x(1)e^{-j0} + x(2)e^{-j0} + x(3)e^{-j0}$$
$$= x(0) + x(1) + x(2) + x(3)$$
$$= 1 + 2 + 3 + 4 = 10$$

$$X(1) = \sum_{n=0}^{3} x(n) e^{-j\frac{\pi n}{2}} = x(0)e^{-j0} + x(1)e^{-j\frac{\pi}{2}} + x(2)e^{-j\pi} + x(3)e^{-j\frac{3\pi}{2}}$$
$$= x(0) - jx(1) - x(2) + jx(3)$$
$$= 1 - j2 - 3 + j4 = -2 + j2$$

Example 2 - contd.

$$X(2) = \sum_{n=0}^{3} x(n)e^{-j\pi n} = x(0)e^{-j0} + x(1)e^{-j\pi} + x(2)e^{-j2\pi} + x(3)e^{-j3\pi}$$
$$= x(0) - x(1) + x(2) - x(3)$$
$$= 1 - 2 + 3 - 4 = -2$$

$$X(3) = \sum_{n=0}^{3} x(n)e^{-j\frac{3\pi n}{2}} = x(0)e^{-j0} + x(1)e^{-j\frac{3\pi}{2}} + x(2)e^{-j3\pi} + x(3)e^{-j\frac{9\pi}{2}}$$
$$= x(0) + jx(1) - x(2) - jx(3)$$
$$= 1 + j2 - 3 - j4 = -2 - j2$$

Using MATLAB,

$$\gg \mathrm{X} = \mathrm{fft}([1\ \ 2\ \ 3\ \ 4])$$
$$\mathrm{X} = 10.0000 \quad -2.0000 + 2.0000\mathrm{i} \quad -2.0000 \quad -2.0000 - 2.0000\mathrm{i}$$

Example 3

Inverse DFT of the previous example.

$N = 4$ and $W_4^{-1} = e^{j\frac{\pi}{2}}$, $\longrightarrow$ $x(n) = \frac{1}{4}\sum_{k=0}^{3} X(k)W_4^{-nk} = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{j\frac{\pi kn}{2}}.$

$$x(0) = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{j0} = \frac{1}{4}\left(X(0)e^{j0} + X(1)e^{j0} + X(2)e^{j0} + X(3)e^{j0}\right)$$
$$= \frac{1}{4}(10 + (-2 + j2) - 2 + (-2 - j2)) = 1$$

$$x(1) = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{j\frac{k\pi}{2}} = \frac{1}{4}\left(X(0)e^{j0} + X(1)e^{j\frac{\pi}{2}} + X(2)e^{j\pi} + X(3)e^{j\frac{3\pi}{2}}\right)$$
$$= \frac{1}{4}(X(0) + jX(1) - X(2) - jX(3))$$
$$= \frac{1}{4}(10 + j(-2 + j2) - (-2) - j(-2 - j2)) = 2$$

Example 3 – contd.

$$x(2) = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{jk\pi} = \frac{1}{4}\left(X(0)e^{j0} + X(1)e^{j\pi} + X(2)e^{j2\pi} + X(3)e^{j3\pi}\right)$$
$$= \frac{1}{4}(X(0) - X(1) + X(2) - X(3))$$
$$= \frac{1}{4}(10 - (-2 + j2) + (-2) - (-2 - j2)) = 3$$

$$x(3) = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{j\frac{k\pi 3}{2}} = \frac{1}{4}\left(X(0)e^{j0} + X(1)e^{j\frac{3\pi}{2}} + X(2)e^{j3\pi} + X(3)e^{j\frac{9\pi}{2}}\right)$$
$$= \frac{1}{4}(X(0) - jX(1) - X(2) + jX(3))$$
$$= \frac{1}{4}(10 - j(-2 + j2) - (-2) + j(-2 - j2)) = 4$$

Using MATLAB,

$$\gg \mathrm{x} = \mathrm{ifft}([10 \quad -2+2j \quad -2 \quad -2-2j])$$
$$\mathrm{x} = 1 \quad 2 \quad 3 \quad 4.$$

Relationship Between Frequency Bin k and Its Associated Frequency in Hz

$$\boxed{f = \frac{kf_s}{N} \text{ (Hz)}}$$

Frequency step or frequency resolution: $\Delta f = \frac{f_s}{N}$ (Hz)

Example 4

In the previous example, if the sampling rate is 10 Hz,

a. Determine the sampling period, time index, and sampling time instant for a digital sample $x(3)$ in time domain.

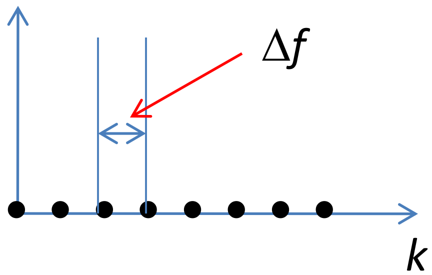
b. Determine the frequency resolution, frequency bin number, and mapped frequency for each of the DFT coefficients $X(1)$ and $X(3)$ in frequency domain.

Example 4 - contd.

a.

Sampling period: $T = 1/f_s = 1/10 = 0.1$ second

For x(3), time index is n = 3, and sampling time instant is $t = nT = 3 \cdot 0.1 = 0.3$ second.



b.

Frequency resolution: $\Delta f = \dfrac{f_s}{N} = \dfrac{10}{4} = 2.5\ \text{Hz}.$

Frequency bin number for X(1) is k = 1, and its corresponding frequency is $f = \dfrac{kf_s}{N} = \dfrac{1 \times 10}{4} = 2.5\ \text{Hz}.$

Similarly, for X(3) is k = 3, and its corresponding frequency is $f = \dfrac{kf_s}{N} = \dfrac{3 \times 10}{4} = 7.5\ \text{Hz}.$

Amplitude and Power Spectrum

Since each calculated DFT coefficient is a complex number, it is not convenient to plot it versus its frequency index

Amplitude Spectrum:

$$A_k = \frac{1}{N}|X(k)| = \frac{1}{N}\sqrt{(\text{Real}[X(k)])^2 + (\text{Imag}[X(k)])^2},$$
$$k = 0, 1, 2, \ldots, N-1.$$

To find one-sided amplitude spectrum, we double the amplitude.

$$\bar{A}_k = \begin{cases} \frac{1}{N}|X(0)|, & k = 0 \\ \frac{2}{N}|X(k)|, & k = 1, \ldots, N/2 \end{cases}.$$

Amplitude and Power Spectrum –contd.

Power Spectrum:

$$P_k = \frac{1}{N^2}|X(k)|^2 = \frac{1}{N^2}\left\{(\text{Real}[X(k)])^2 + (\text{Imag}[X(k)])^2\right\},$$
$$k = 0,\ 1,\ 2, \ldots,\ N-1.$$

For, one-sided power spectrum:

$$\bar{P}_k = \begin{cases} \frac{1}{N^2}|X(0)|^2 & k = 0 \\ \frac{2}{N^2}|X(k)|^2 & k = 0,\ 1, \ldots,\ N/2 \end{cases}$$

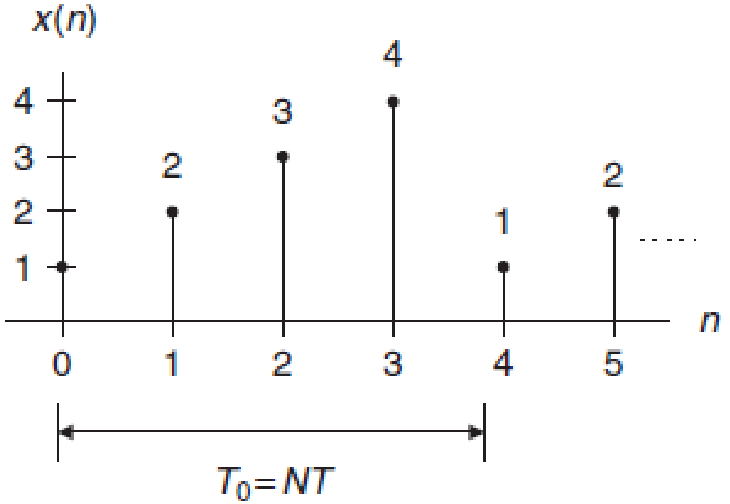
Phase Spectrum:

$$\varphi_k = \tan^{-1}\left(\frac{\text{Imag}[X(k)]}{\text{Real}[X(k)]}\right),\ k = 0,\ 1,\ 2, \ldots,\ N-1.$$

Example 5

Assuming that $f_s = 100\,\text{Hz}$,

a. Compute the amplitude spectrum, phase spectrum, and power spectrum.



Solution:

$$X(0) = 10$$
$$X(1) = -2 + j2$$
$$X(2) = -2$$
$$X(3) = -2 - j2.$$

See Example 2.

For $k = 0$, $f = k \cdot f_s / N = 0 \times 100/4 = 0\,\text{Hz}$,

$$A_0 = \frac{1}{4}|X(0)| = 2.5,\ \varphi_0 = \tan^{-1}\left(\frac{\text{Imag}[X(0)]}{\text{Real}([X(0)]}\right) = 0^0,$$

$$P_0 = \frac{1}{4^2}|X(0)|^2 = 6.25.$$

Example 5 - contd. (1)

For $k = 1$, $f = 1 \times 100/4 = 25\,\text{Hz}$,

$$A_1 = \frac{1}{4}|X(1)| = 0.7071,\ \varphi_1 = \tan^{-1}\left(\frac{\text{Imag}[X(1)]}{\text{Real}[X(1)]}\right) = 135^0,$$

$$P_1 = \frac{1}{4^2}|X(1)|^2 = 0.5000.$$

For $k = 2$, $f = 2 \times 100/4 = 50\,\text{Hz}$,

$$A_2 = \frac{1}{4}|X(2)| = 0.5,\ \varphi_2 = \tan^{-1}\left(\frac{\text{Imag}[X(2)]}{\text{Real}[X(2)]}\right) = 180^0,$$

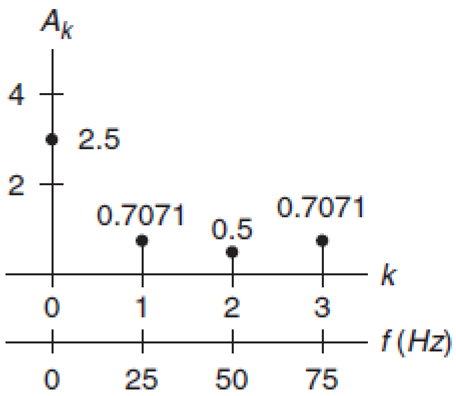
$$P_2 = \frac{1}{4^2}|X(2)|^2 = 0.2500.$$

Similarly, for $k = 3$, $f = 3 \times 100/4 = 75\,\text{Hz}$,

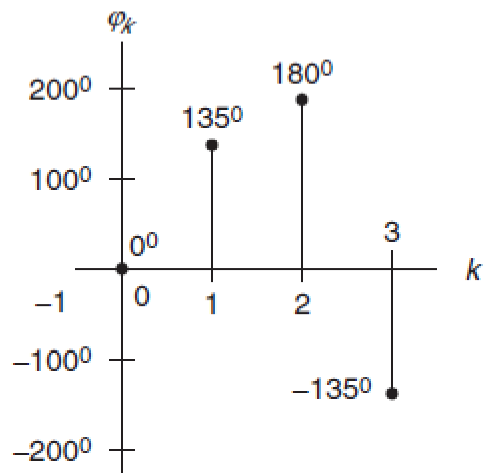
$$A_3 = \frac{1}{4}|X(3)| = 0.7071,\ \varphi_3 = \tan^{-1}\left(\frac{\text{Imag}[X(3)]}{\text{Real}[X(3)]}\right) = -135^0,$$

$$P_3 = \frac{1}{4^2}|X(3)|^2 = 0.5000.$$

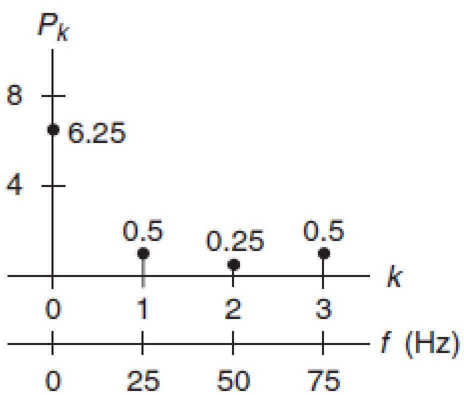
Example 5 – contd. (2)



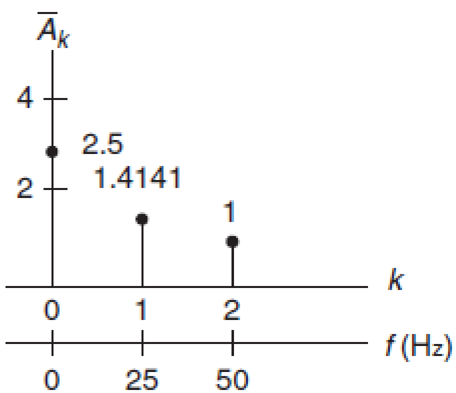
Amplitude Spectrum



Phase Spectrum



Power Spectrum



One sided Amplitude Spectrum

Example 6

Consider a digital sequence sampled at the rate of 10 kHz. If we use a size of 1,024 data points and apply the 1,024-point DFT to compute the spectrum,

a. Determine the frequency resolution.

b. Determine the highest frequency in the spectrum.

Solution:

a. $\Delta f = \frac{f_s}{N} = \frac{10000}{1024} = 9.776\,\text{Hz}.$

b. The highest frequency is the folding frequency, given by

$$f_{\max} = \frac{N}{2}\Delta f = \frac{f_s}{2}$$

$$= 512 \cdot 9.776 = 5000\,\text{Hz}$$

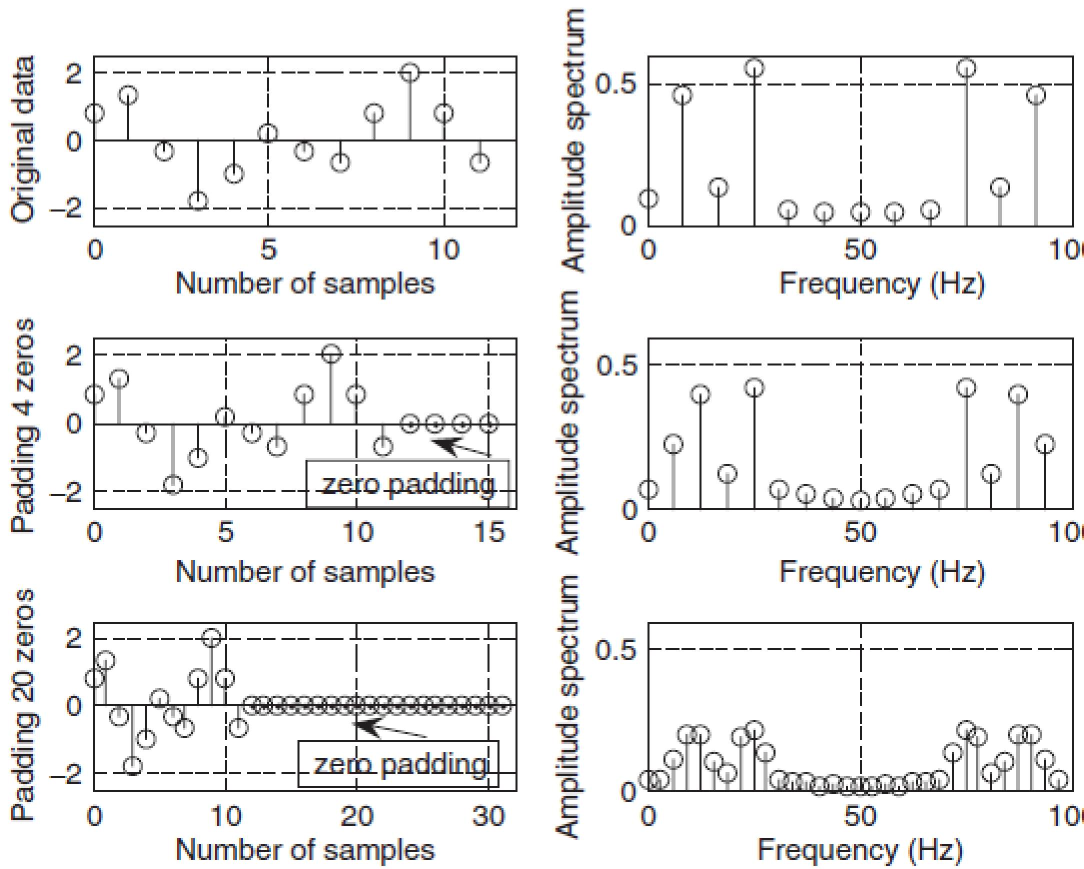
Zero Padding for FFT

FFT: Fast Fourier Transform.

→ A fast version of DFT; It requires signal length to be power of 2.

Therefore, we need to pad zero at the end of the signal.

However, it does not add any new information.



Example 7

Consider a digital signal has sampling rate = 10 kHz. For amplitude spectrum we need frequency resolution of less than 0.5 Hz. For FFT how many data points are needed?

Solution:

$$\Delta f = 0.5\,\text{Hz}$$

$$N = \frac{f_s}{\Delta f} = \frac{10000}{0.5} = 20000$$

For FFT, we need *N* to be power of 2.

$2^{14} = 16384 < 20000$ And $\boxed{2^{15} = 32768 > 20000}$

Recalculated frequency resolution,

$$\Delta f = \frac{f_s}{N} = \frac{10000}{32768} = 0.31\,\text{Hz}.$$

MATLAB Example - 1

$$x(n) = 2 \cdot \sin\left(2000\pi \frac{n}{8000}\right)$$

f_s

Use the MATLAB DFT to compute the signal spectrum with the frequency resolution to be equal to or less than 8 Hz.

$$N = \frac{f_s}{\Delta f} = \frac{8000}{8} = 1000$$

```
% Generate the sine wave sequence
fs = 8000;              %Sampling rate
N = 1000;               % Number of data points
x = 2* sin (2000* pi*[0:1:N - 1]/fs);
```

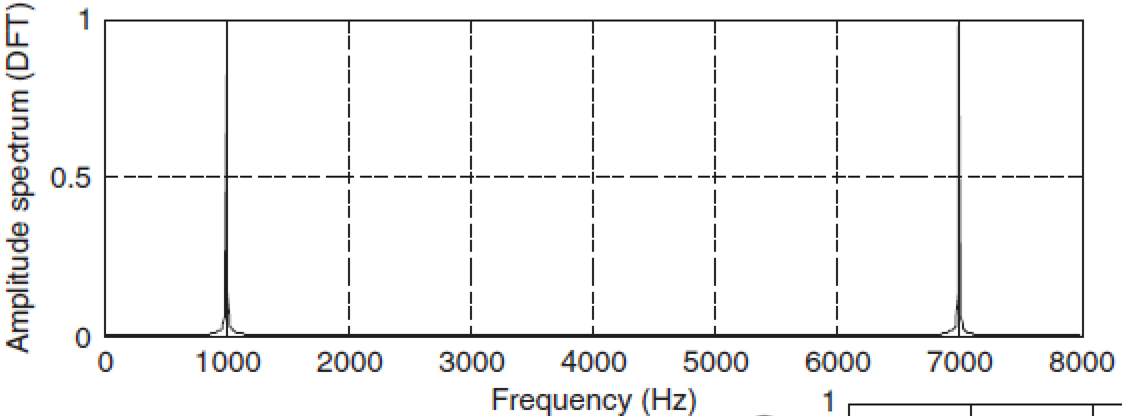
```
figure(1), plot(x);
```

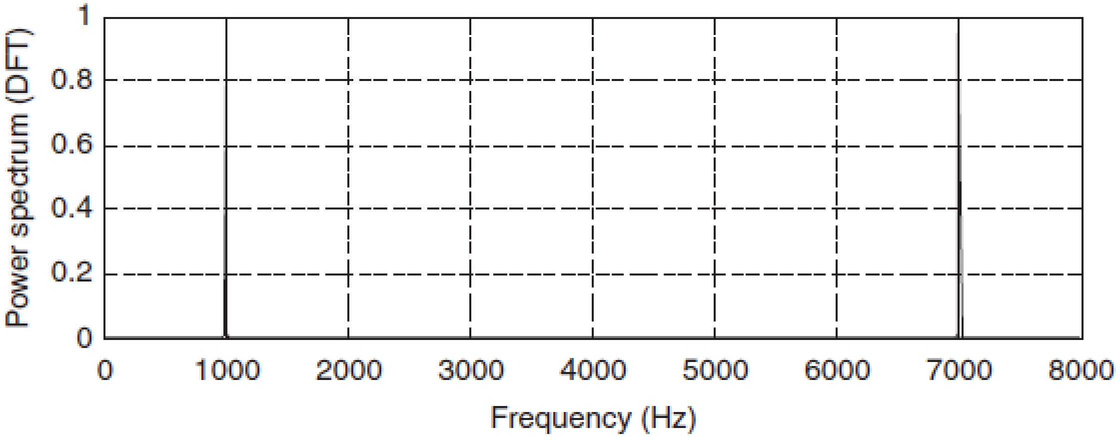
```
xf = abs(fft(x))/N;   %Compute the amplitude spectrum
```

```
P = xf.*xf;              %Compute the power spectrum
f = [0:1:N - 1]*fs/N;         %Map the frequency bin to the frequency (Hz)
```

MATLAB Example – contd. (1)

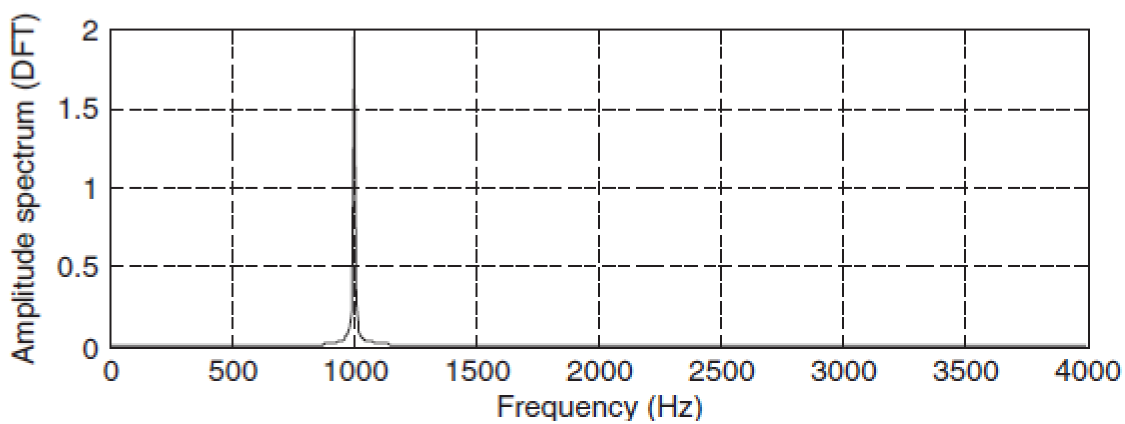
```
subplot(2,1,1); plot(f,xf);grid
xlabel('Frequency (Hz)'); ylabel('Amplitude spectrum (DFT)');
subplot(2,1,2);plot(f,P);grid
xlabel('Frequency (Hz)'); ylabel('Power spectrum (DFT)');
```



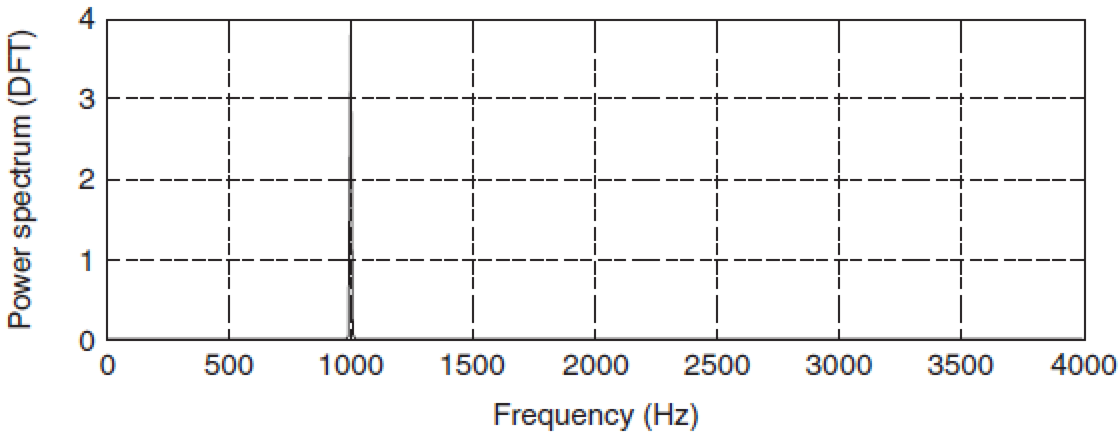


MATLAB Example – contd. (2)

```
% Convert it to one-sided spectrum
xf(2:N) = 2*xf(2:N);            % Get the single-sided spectrum
P = xf.*xf; % Calculate the power spectrum
f = [0:1:N/2]*fs/N % Frequencies up to the folding frequency
subplot(2,1,1); plot(f,xf(1:N/2+1));grid
xlabel('Frequency (Hz)'); ylabel('Amplitude spectrum (DFT)');
subplot(2,1,2);plot(f,P(1:N/2+1));grid
xlabel('Frequency (Hz)'); ylabel('Power spectrum (DFT)');
```



MATLAB Example – contd. (3)




```
% Zero padding to the length of 1024
x = [x, zeros(1,24)];
N = length(x);
xf = abs(fft(x))/N;          %Compute the amplitude spectrum with zero padding
P = xf.*xf;                  %Compute the power spectrum
f = [0:1:N-1]*fs/N;          %Map frequency bin to frequency (Hz)
subplot(2,1,1); plot(f,xf);grid
xlabel('Frequency (Hz)'); ylabel('Amplitude spectrum (FFT)');
subplot(2,1,2);plot(f,P);grid
xlabel('Frequency (Hz)'); ylabel('Power spectrum (FFT)');
```

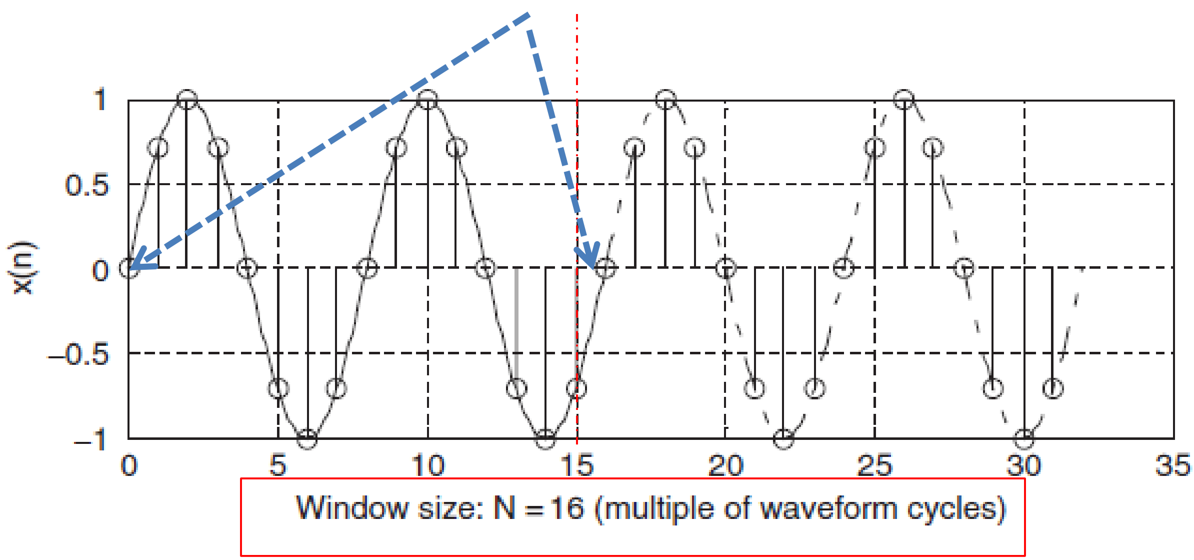
Effect of Window Size

When applying DFT, we assume the following:

1. Sampled data are periodic to themselves (repeat).

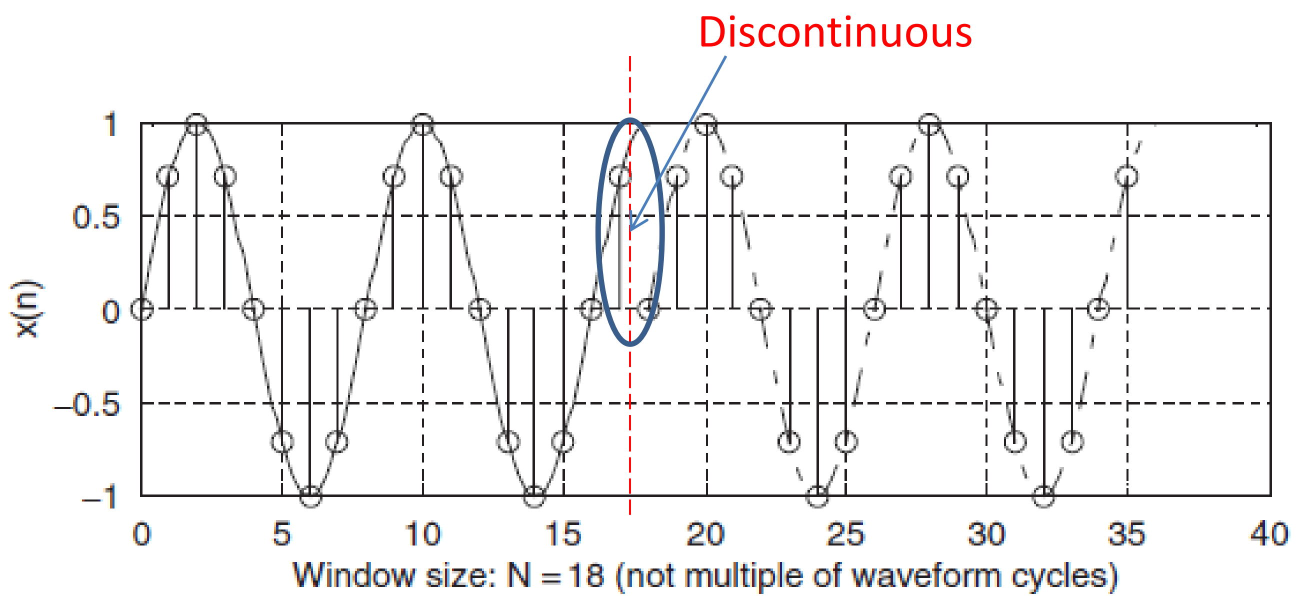
2. Sampled data are continuous to themselves and band limited to the folding frequency.

1 Hz sinusoid, with 32 samples



Effect of Window Size –contd. (1)

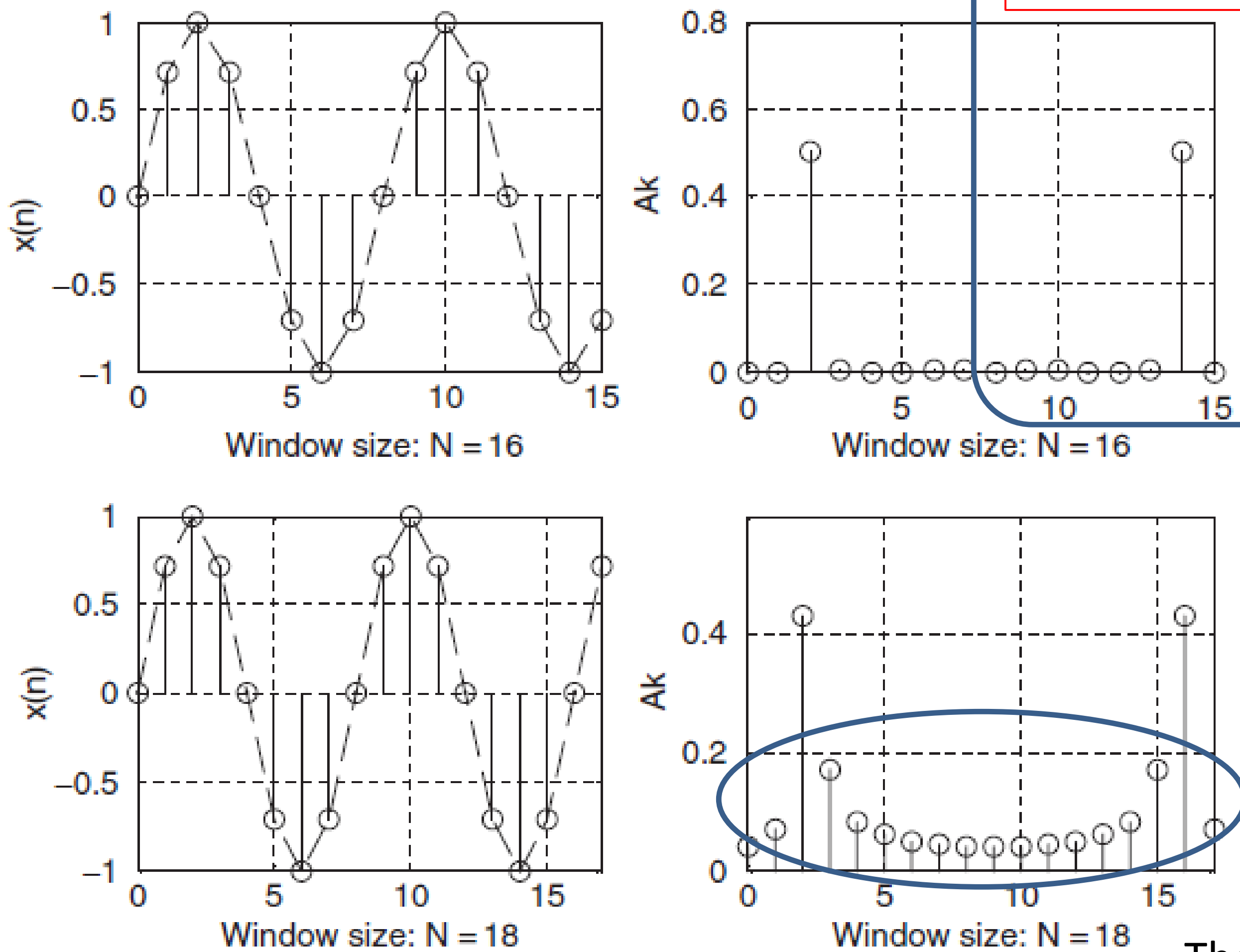
If the window size is not multiple of waveform cycles:



Effect of Window Size –contd. (2)

2- cycles

Mirror Image



Produces single frequency

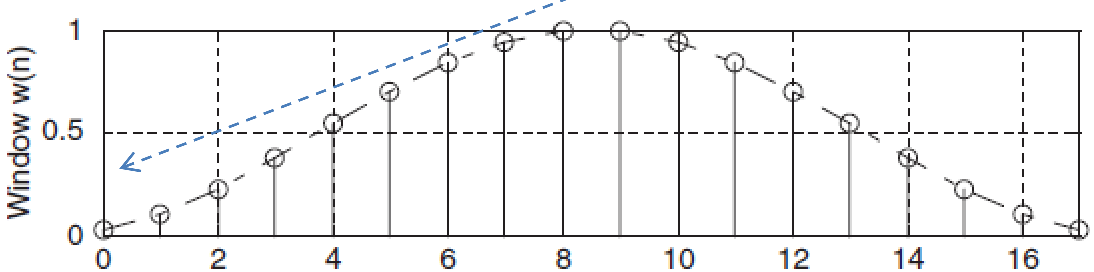
Produces many harmonics as well.

Spectral Leakage

The bigger the discontinuity, the more the leakage

Reducing Leakage Using Window

To reduce the effect of spectral leakage, a window function can be used whose amplitude tapers smoothly and gradually toward zero at both ends.



$$x_w(n) = x(n)w(n), \text{ for } n = 0, 1, \ldots, N-1.$$

Window function, $w(n)$
Data sequence, $x(n)$
Obtained windowed sequence, $x_w(n)$

Example 8

Given,

$x(2) = 1$ and $w(2) = 0.2265$;

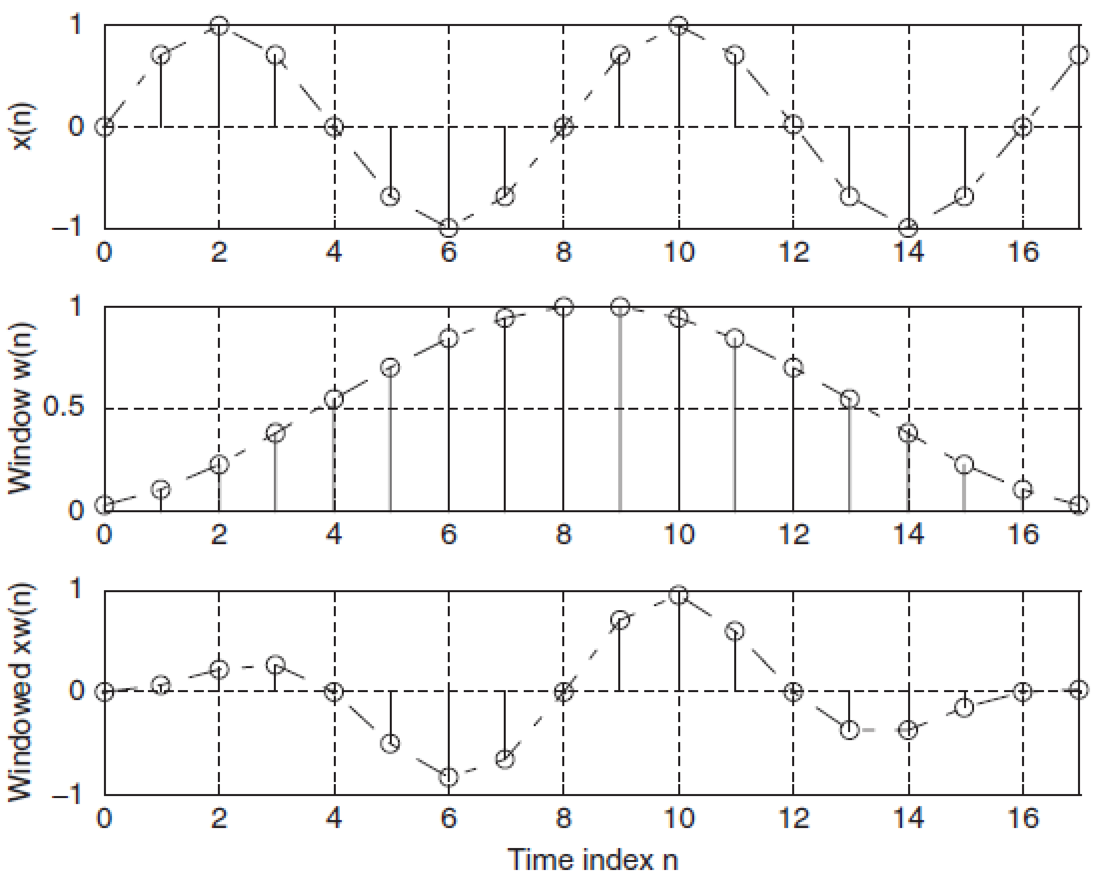
$x(5) = -0.7071$ and $w(5) = 0.7008$,

Calculate,

$x_w(2)$ and $x_w(5)$.

$$x_w(2) = x(2) \times w(2)$$
$$= 1 \times 0.2265 = 0.2265$$

$$x_w(5) = x(5) \times w(5)$$
$$= -0.7071 \times 0.7008 = -0.4956$$



Different Types of Windows

Rectangular Window (no window): $w_R(n) = 1 \qquad 0 \le n \le N-1$

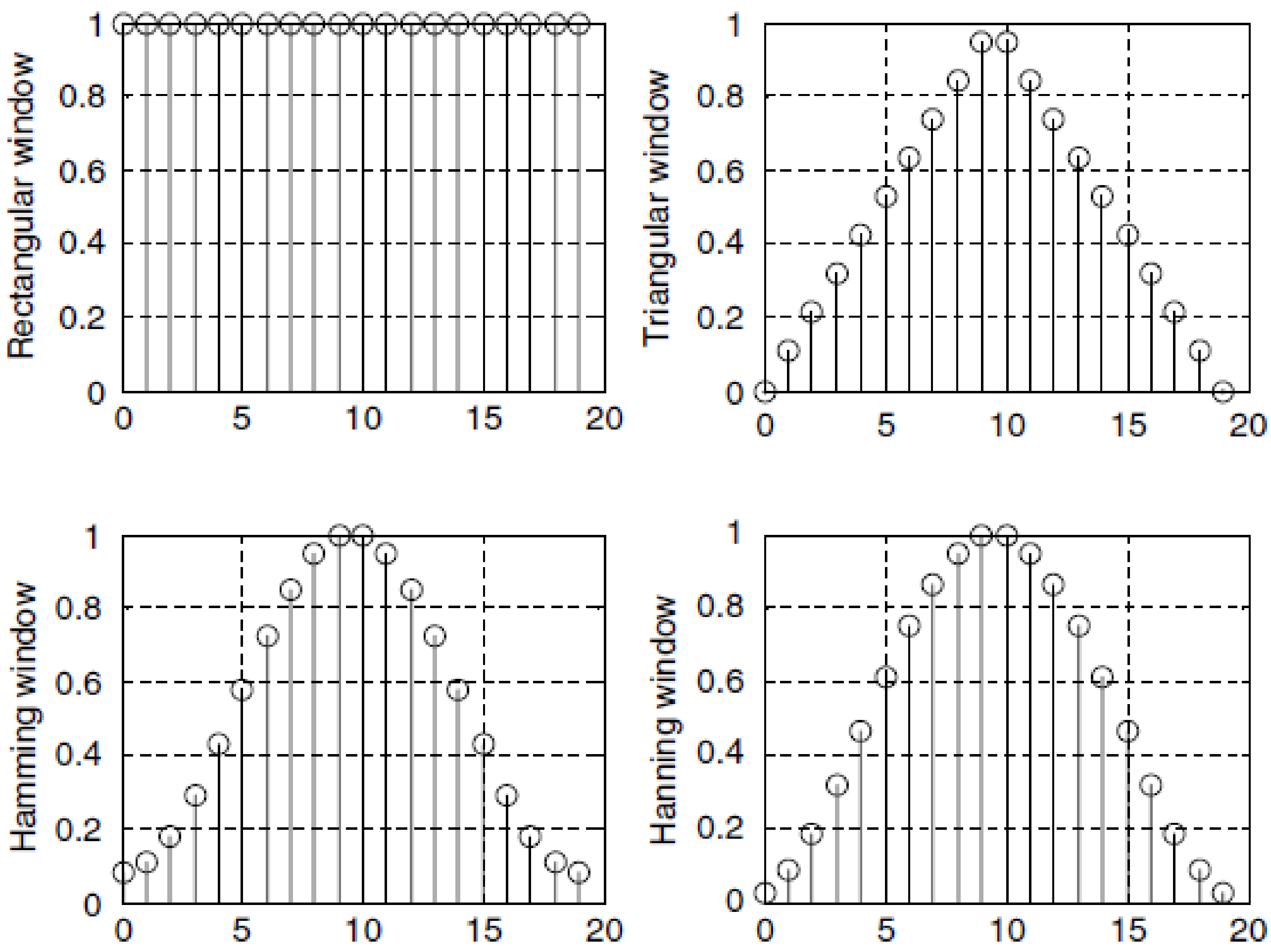
Triangular Window: $$w_{tri}(n) = 1 - \frac{|2n - N + 1|}{N-1}, \; 0 \le n \le N-1$$

Hamming Window: $$w_{hm}(n) = 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right), \; 0 \le n \le N-1$$

Hanning Window: $$w_{hn}(n) = 0.5 - 0.5\cos\left(\frac{2\pi n}{N-1}\right), \; 0 \le n \le N-1$$

Different Types of Windows -contd.

Window size of 20 samples



Example 9

Problem:

Considering the sequence $x(0) = 1$, $x(1) = 2$, $x(2) = 3$, and $x(3) = 4$, and given $f_s = 100$ Hz, $T = 0.01$ seconds, compute the amplitude spectrum, phase spectrum, and power spectrum

Using the Hamming window function.

Solution:

Since N = 4, Hamming window function can be found as:

$$w_{hm}(0) = 0.54 - 0.46\cos\left(\frac{2\pi \times 0}{4-1}\right) = 0.08$$

$$w_{hm}(1) = 0.54 - 0.46\cos\left(\frac{2\pi \times 1}{4-1}\right) = 0.77.$$

Similarly, $w_{hm}(2) = 0.77$, $w_{hm}(3) = 0.08$.

Example 9 - contd. (1)

Windowed sequence:

$$x_w(0) = x(0) \times w_{hm}(0) = 1 \times 0.08 = 0.08$$
$$x_w(1) = x(1) \times w_{hm}(1) = 2 \times 0.77 = 1.54$$
$$x_w(2) = x(2) \times w_{hm}(2) = 3 \times 0.77 = 2.31$$
$$x_w(0) = x(3) \times w_{hm}(3) = 4 \times 0.08 = 0.32.$$

DFT Sequence:

$$X(k) = x(0)\, W_N^{k0} + x(1)\, W_N^{k1} + x(2) W_N^{k2} + \ldots + x(N-1)\, W_N^{k(N-1)}$$

$$\Rightarrow \quad X(k) = x_w(0) W_4^{k\times 0} + x(1) W_4^{k\times 1} + x(2) W_4^{k\times 2} + x(3) W_4^{k\times 3}.$$

$$\Rightarrow \quad \begin{cases} X(0) = 4.25 \\ X(1) = -2.23 - j1.22 \\ X(2) = 0.53 \\ X(3) = -2.23 + j1.22 \end{cases} \qquad \Delta f = \frac{1}{NT} = \frac{1}{4 \cdot 0.01} = 25\,\mathrm{Hz}$$

Example 9 - contd. (2)

$$A_0 = \frac{1}{4}|X(0)| = 1.0625,\ \varphi_0 = \tan^{-1}\left(\frac{0}{4.25}\right) = 0^0,$$

$$P_0 = \frac{1}{4^2}|X(0)|^2 = 1.1289$$

$$A_1 = \frac{1}{4}|X(1)| = 0.6355,\ \varphi_1 = \tan^{-1}\left(\frac{-1.22}{-2.23}\right) = -151.32^0,$$

$$P_1 = \frac{1}{4^2}|X(1)|^2 = 0.4308$$

$$A_2 = \frac{1}{4}|X(2)| = 0.1325,\ \varphi_2 = \tan^{-1}\left(\frac{0}{0.53}\right) = 0^0,$$

$$P_2 = \frac{1}{4^2}|X(2)|^2 = 0.0176.$$

$$A_3 = \frac{1}{4}|X(3)| = 0.6355,\ \varphi_3 = \tan^{-1}\left(\frac{1.22}{-2.23}\right) = 151.32^0,$$

$$P_3 = \frac{1}{4^2}|X(3)|^2 = 0.4308.$$

MATLAB Example - 2

$$x(n) = 2 \cdot \sin\left(2000\pi \frac{n}{8000}\right)$$

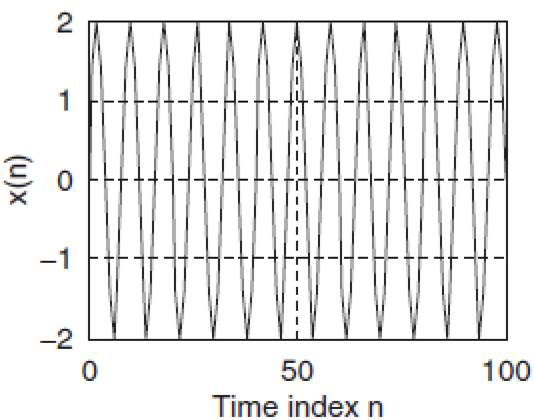
Compute the spectrum of a Hamming window function with a window size = 100.

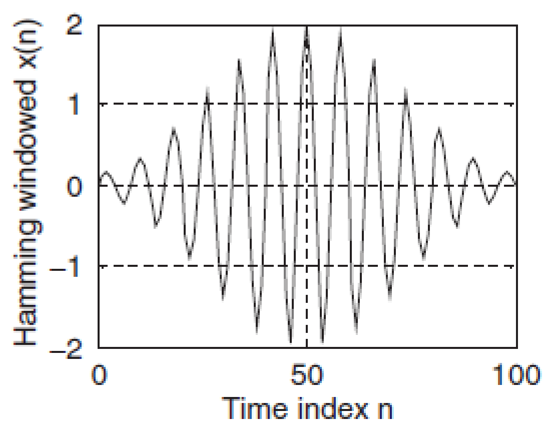
```
% Generate the sine wave sequence
fs = 8000; T = 1/fs;            % Sampling rate and sampling period

% Generate the sine wave sequence
x = 2* sin (2000*pi*[0:1:100]*T);
% Apply the FFT algorithm
N=length(x);
index_t = [0:1:N-1];
f = [0:1:N-1]*fs/N;
xf = abs(fft(x))/N;
```

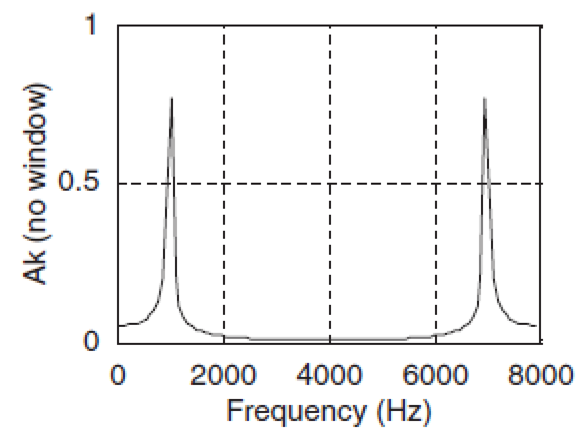
```
%Using the Hamming window
x_hm = x.*hamming(N)';          %Apply the Hamming window function
xf_hm=abs(fft(x_hm))/N;         %Calculate the amplitude spectrum
```

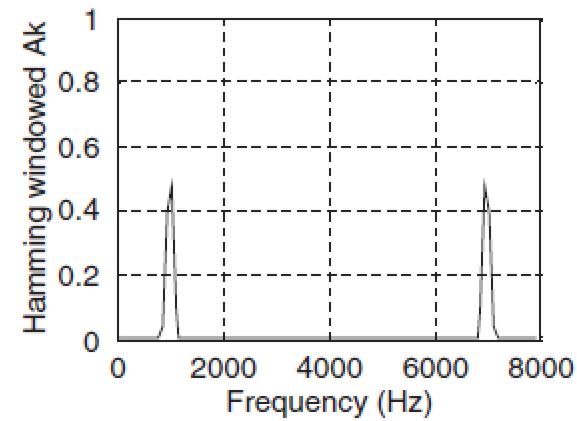
MATLAB Example – 2 contd.

```
subplot(2,2,1);plot(index_t,x);grid
xlabel('Time index n'); ylabel('x(n)');
subplot(2,2,3); plot(index_t,x_hm);grid
xlabel('Time index n'); ylabel('Hamming windowed x(n)');
subplot(2,2,2);plot(f,xf);grid;axis([0 fs 0 1]);
xlabel('Frequency (Hz)'); ylabel('Ak (no window)');
subplot(2,2,4); plot(f,xf_hm);grid;axis([0 fs 0 1]);
xlabel('Frequency (Hz)'); ylabel('Hamming windowed Ak');
```









DFT Matrix

Frequency Spectrum | Multiplication Matrix | Time-Domain samples

$$
\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ \vdots \\ X(N-2) \\ X(N-1) \end{bmatrix}
=
\begin{bmatrix}
1 & 1 & 1 & \cdots & 1 & 1 \\
1 & e^{-j\frac{2\pi}{N}} & e^{-j\frac{4\pi}{N}} & \cdots & e^{-j\frac{2(N-2)\pi}{N}} & e^{-j\frac{2(N-1)\pi}{N}} \\
1 & e^{-j\frac{4\pi}{N}} & e^{-j\frac{8\pi}{N}} & \cdots & e^{-j\frac{4(N-2)\pi}{N}} & e^{-j\frac{4(N-2)\pi}{N}} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & e^{-j\frac{2(N-2)\pi}{N}} & e^{-j\frac{4(N-2)\pi}{N}} & \cdots & e^{-j\frac{2(N-2)^2\pi}{N}} & e^{-j\frac{2(N-2)(N-1)\pi}{N}} \\
1 & e^{-j\frac{2(N-1)\pi}{N}} & e^{-j\frac{4(N-1)\pi}{N}} & \cdots & e^{-j\frac{2(N-1)(N-2)\pi}{N}} & e^{-j\frac{(N-1)^2\pi}{N}}
\end{bmatrix}
\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N-2) \\ x(N-1) \end{bmatrix}
$$

DFT Matrix

Let, $w_N = e^{-j2\pi / N}$

Then

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ \vdots \\ X(N-1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w & w^2 & \cdots & w^{(N-1)} \\ 1 & w^2 & w^4 & \cdots & w^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & w^{(N-1)} & w^{2(N-1)} & \cdots & w^{(N-1)^2} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N-1) \end{bmatrix}$$

DFT equation: $X(k) = \sum_{m=0}^{N-1} x(m) w_N^{mk} \qquad k = 0, \ldots, N-1$

DFT requires N^2 complex multiplications.

FFT

FFT: Fast Fourier Transform

A very efficient algorithm to compute DFT; it requires less multiplication.

The length of input signal, x(n) must be 2^m samples, where m is an integer.

Samples N = 2, 4, 8, 16 or so.

If the input length is not 2^m, append (pad) zeros to make it 2^m.

4	5	1	7	1

N = 5

4	5	1	7	1	0	0	0

N = 8, power of 2

DFT to FFT: Decimation in Frequency

DFT: $$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \text{ for } k = 0, 1, \ldots, N-1,$$

$$X(k) = x(0) + x(1) W_N^k + \ldots + x(N-1) W_N^{k(N-1)}$$

$$X(k) = \boxed{x(0) + x(1) W_N^k + \ldots + x\left(\frac{N}{2} - 1\right) W_N^{k(N/2-1)}} + \boxed{x\left(\frac{N}{2}\right) W^{kN/2} + \ldots + x(N-1) W_N^{k(N-1)}}$$

$$X(k) = \sum_{n=0}^{(N/2)-1} x(n) W_N^{kn} + \boxed{\sum_{n=N/2}^{N-1} x(n) W_N^{kn}}$$

$$X(k) = \sum_{n=0}^{(N/2)-1} x(n) W_N^{kn} + W_N^{(N/2)k} \sum_{n=0}^{(N/2)-1} x\left(n + \frac{N}{2}\right) W_N^{kn}.$$

$$W_N^{N/2} = e^{-j\frac{2\pi(N/2)}{N}} = e^{-j\pi} = -1;$$

$$X(k) = \sum_{n=0}^{(N/2)-1} \left(x(n) + (-1)^k x\left(n + \frac{N}{2}\right)\right) W_N^{kn}$$

DFT to FFT: Decimation in Frequency

Now decompose into even (k = 2m) and odd (k = 2m+1) sequences.

$$X(2m) = \sum_{n=0}^{(N/2)-1} \left(x(n) + x\left(n + \frac{N}{2}\right)\right) W_N^{2mn}$$

$$X(2m+1) = \sum_{n=0}^{(N/2)-1} \left(x(n) - x\left(n + \frac{N}{2}\right)\right) W_N^{n} W_N^{2mn}$$

$$\boxed{W_N^2 = e^{-j\frac{2\pi \times 2}{N}} = e^{-j\frac{2\pi}{(N/2)}} = W_{N/2}}$$

$$X(2m) = \sum_{n=0}^{(N/2)-1} a(n) W_{N/2}^{mn} = DFT\{a(n)\ with\ (N/2)\ \text{points}\}$$

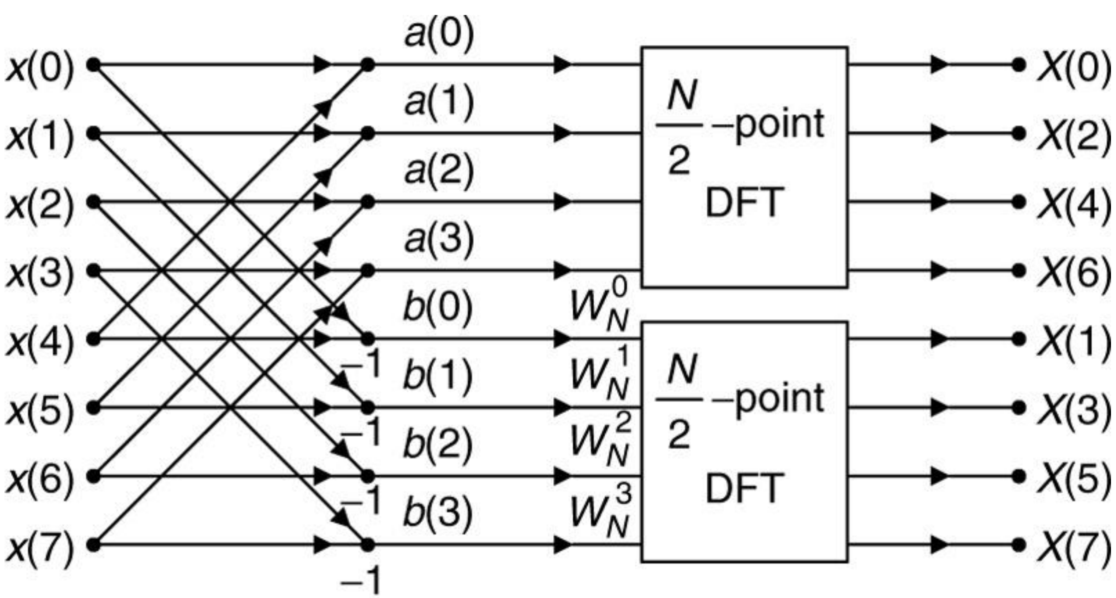
$$X(2m+1) = \sum_{n=0}^{(N/2)-1} b(n) W_N^{n} W_{N/2}^{mn} = DFT\{b(n) W_N^{n}\ with\ (N/2)\ \text{points}\}$$

$$a(n) = x(n) + x\left(n + \frac{N}{2}\right), \text{ for } n = 0,1 \ldots, \frac{N}{2} - 1$$

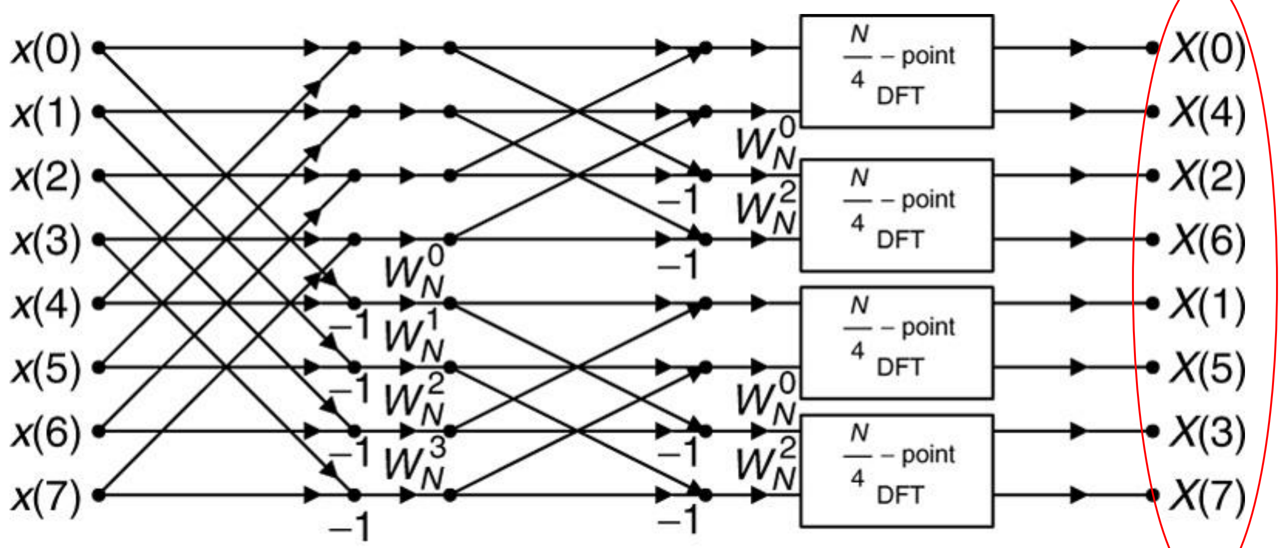
$$b(n) = x(n) - x\left(n + \frac{N}{2}\right), \text{ for } n = 0,1, \ldots, \frac{N}{2} - 1.$$

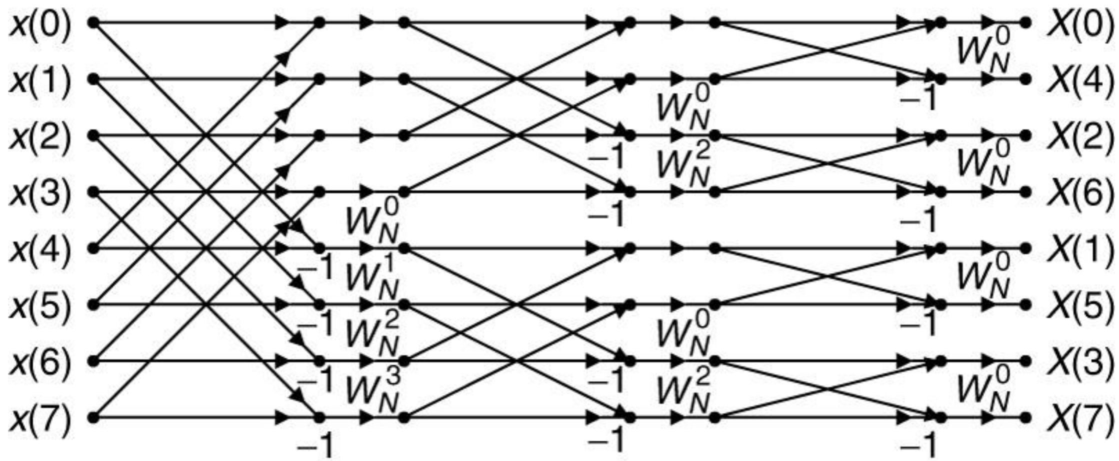
DFT to FFT: Decimation in Frequency

$$DFT\{x(n)\ with\ N\ \text{points}\} = \begin{cases} DFT\{a(n)\ with\ (N/2)\ \text{points}\} \\ DFT\{b(n)W_N^n\ with\ (N/2)\ \text{points}\} \end{cases}$$



DFT to FFT: Decimation in Frequency





12 complex multiplication

DFT to FFT: Decimation in Frequency

Binary	index	1st split	2nd split	3rd split	Bit reversal
000	0	0	0	0	000
001	1	2	4	4	100
010	2	4	2	2	010
011	3	6	6	6	011
100	4	1	1	1	001
101	5	3	5	5	101
110	6	5	3	3	011
111	7	7	7	7	111

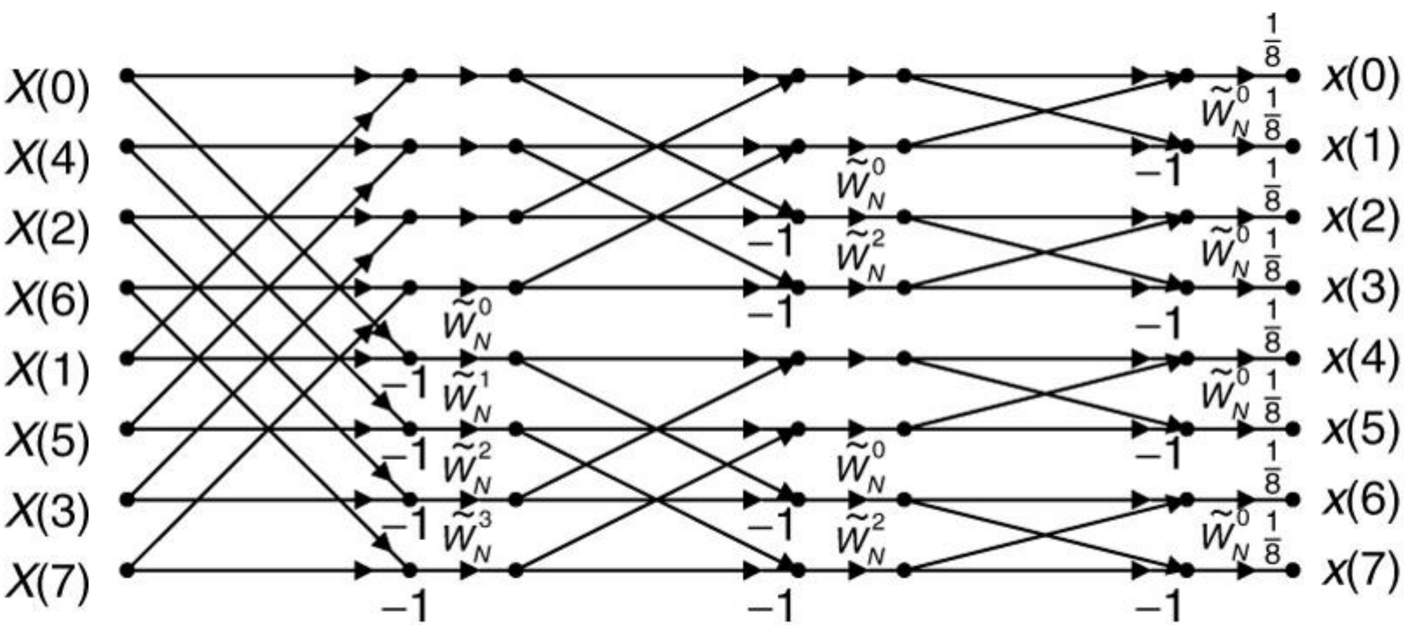
Complex multiplications of DFT $= N^2$, and

Complex multiplications of FFT $= \frac{N}{2}\log_2(N)$

For 1024 samples data sequence, DFT requires 1024×1024 = 1048576 complex multiplications. FFT requires (1024/2)log(1024) = 5120 complex multiplications.

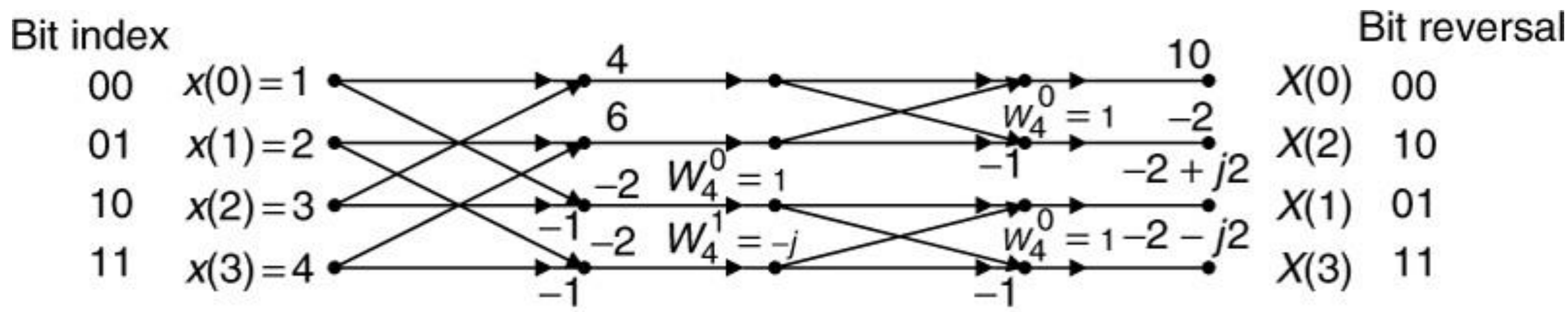
IFFT: Inverse FFT

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)W_N^{-kn} = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\tilde{W}_N^{kn}, \text{ for } k = 0, 1, \ldots, N-1.$$



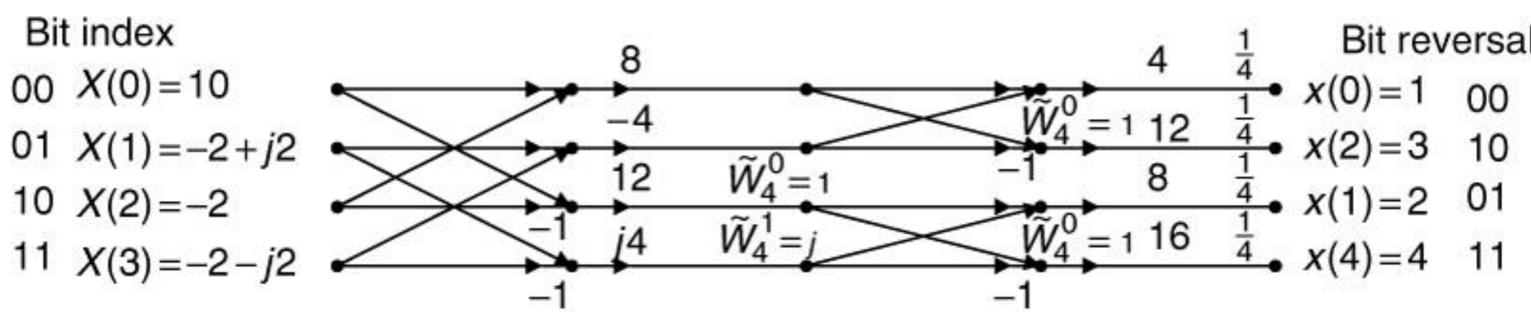
FFT and IFFT Examples

FFT



Number of complex multiplication = $\frac{N}{2}\log_2(N) = \frac{4}{2}\log_2(4) = 4.$

IFFT



DFT to FFT: Decimation in Time

Split the input sequence x(n) into the even indexed x(2m) and x(2m + 1), each with N/2 data points.

$$X(k) = \sum_{m=0}^{(N/2)-1} x(2m) W_N^{2mk} + \sum_{m=0}^{(N/2)-1} x(2m+1) W_N^k W_N^{2mk},$$

for $k = 0, 1, \ldots, N-1.$

Using

$$w_N^2 = \left(e^{-j2\pi/N}\right)^2 = e^{-j2\pi/(N/2)} = w_{N/2}$$

$$X(k) = \sum_{m=0}^{(N/2)-1} x(2m) W_{N/2}^{mk} + W_N^k \sum_{m=0}^{(N/2)-1} x(2m+1) W_{N/2}^{mk},$$

for $k = 0, 1, \ldots, N-1.$

DFT to FFT: Decimation in Time

Define new functions as

$$G(k) = \sum_{m=0}^{(N/2)-1} x(2m) W_{N/2}^{mk} = DFT\{x(2m) \; with \; (N/2) \text{ points}\}$$

$$H(k) = \sum_{m=0}^{(N/2)-1} x(2m+1) W_{N/2}^{mk} = DFT\{x(2m+1) \; with \; (N/2) \text{ points}\}.$$

As,

$$G(k) = G\left(k + \frac{N}{2}\right), \text{ for } k = 0, 1, \ldots, \frac{N}{2} - 1$$

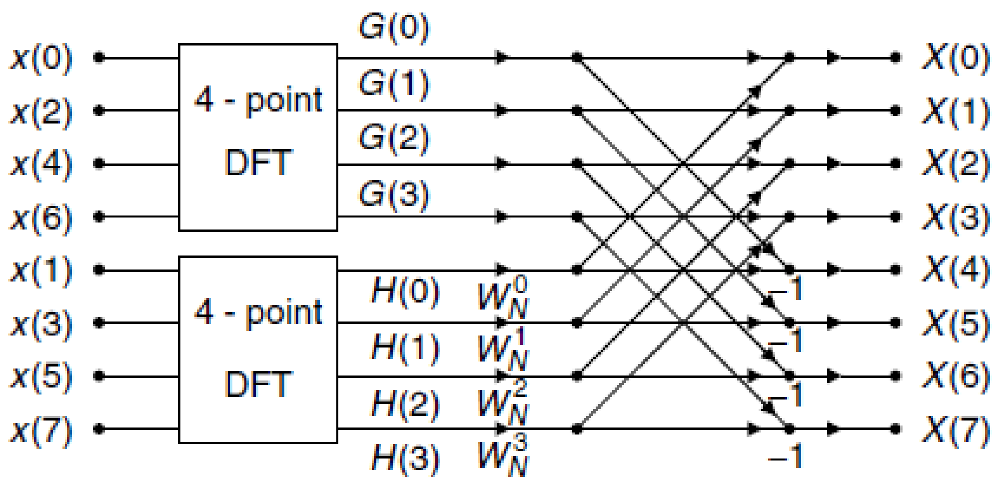
$$H(k) = H\left(k + \frac{N}{2}\right), \text{ for } k = 0, 1, \ldots, \frac{N}{2} - 1.$$

$$\boxed{\begin{aligned} X(k) &= G(k) + W_N^k H(k), \text{ for } k = 0, 1, \ldots, \frac{N}{2} - 1. \\ X\left(\frac{N}{2} + k\right) &= G(k) - W_N^k H(k), \text{ for } k = 0, 1, \ldots, \frac{N}{2} - 1. \end{aligned}}$$

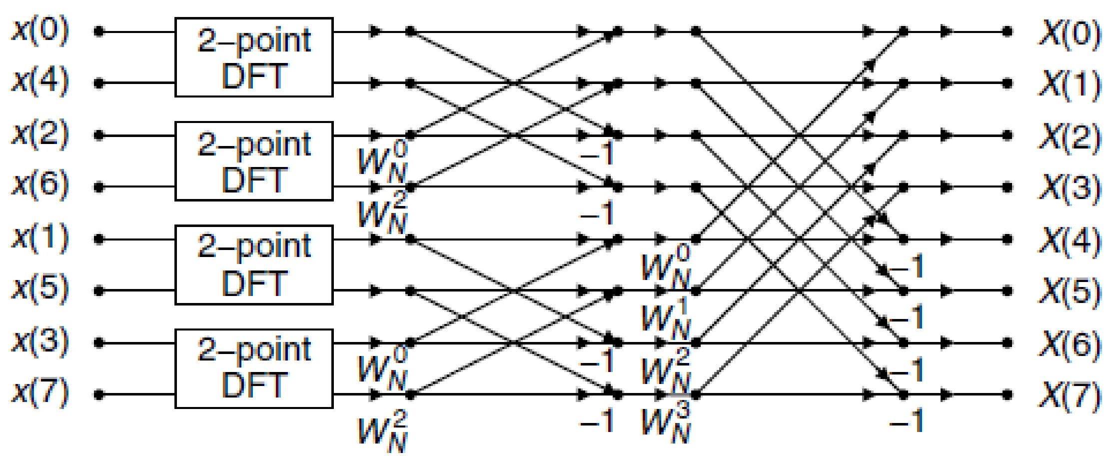
$$W_N^{(N/2+k)} = -W_N^k.$$

DFT to FFT: Decimation in Time

First iteration:

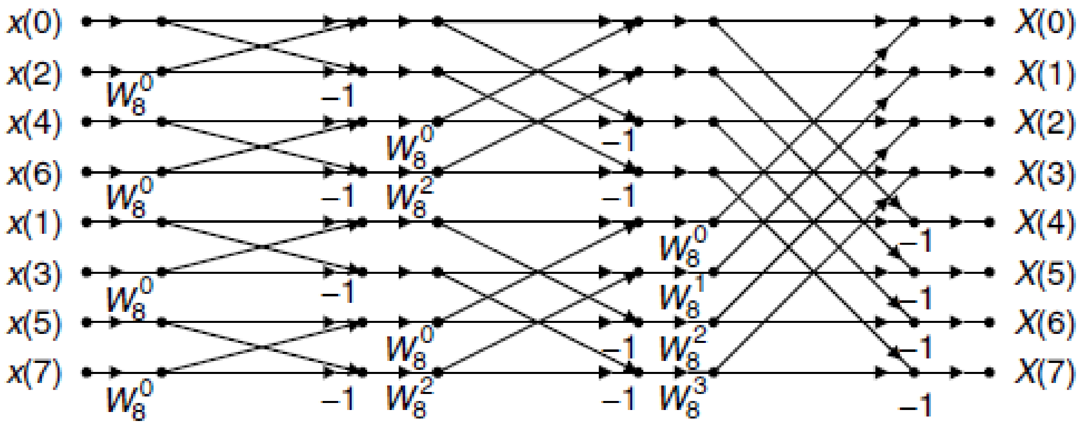


Second iteration:



DFT to FFT: Decimation in Time

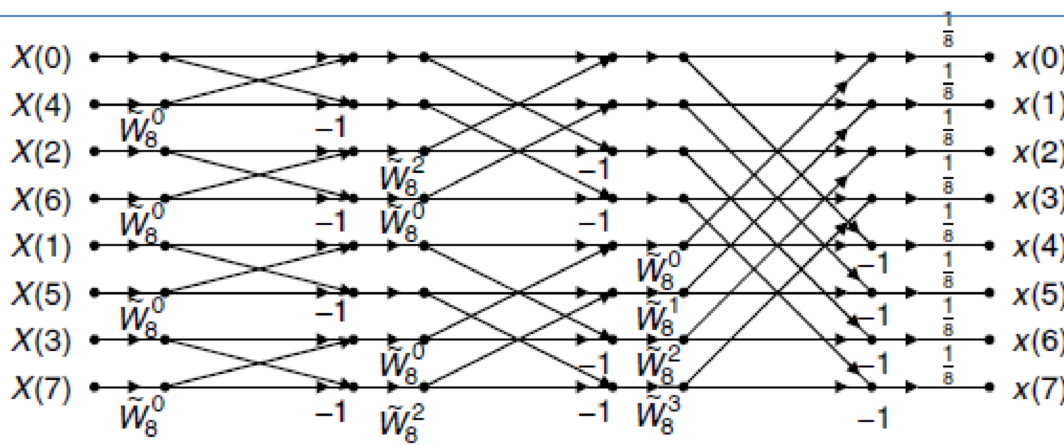
Third iteration:



$$W_N = e^{-\frac{2\pi}{N}} = \cos\left(\frac{2\pi}{N}\right) - j\sin\left(\frac{2\pi}{N}\right)$$

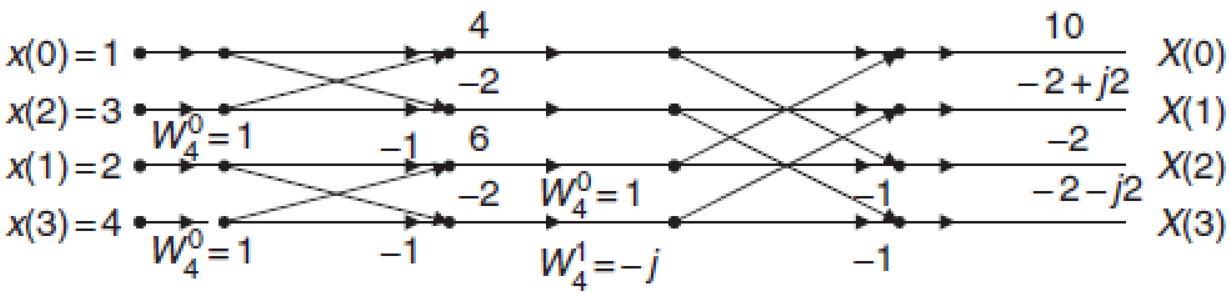
$$W_8^2 = e^{-\frac{2\pi\times 2}{8}} = e^{-\frac{\pi}{2}} = \cos(\pi/2) - j\sin(\pi/2) = -j$$

IFFT

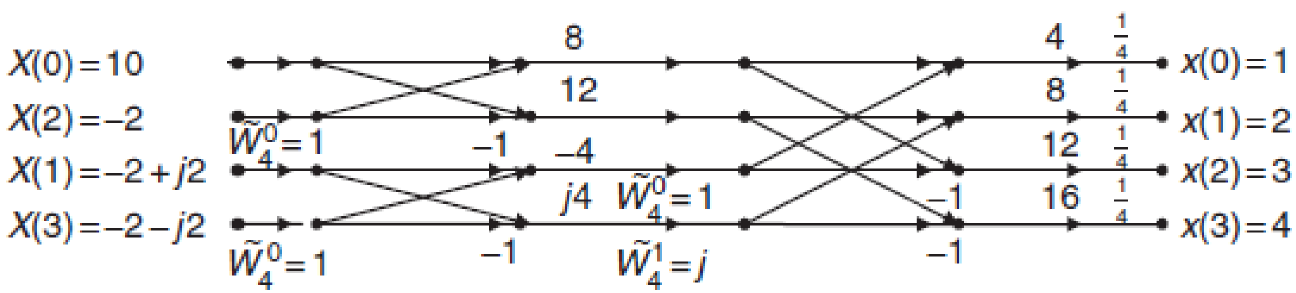


FFT and IFFT Examples

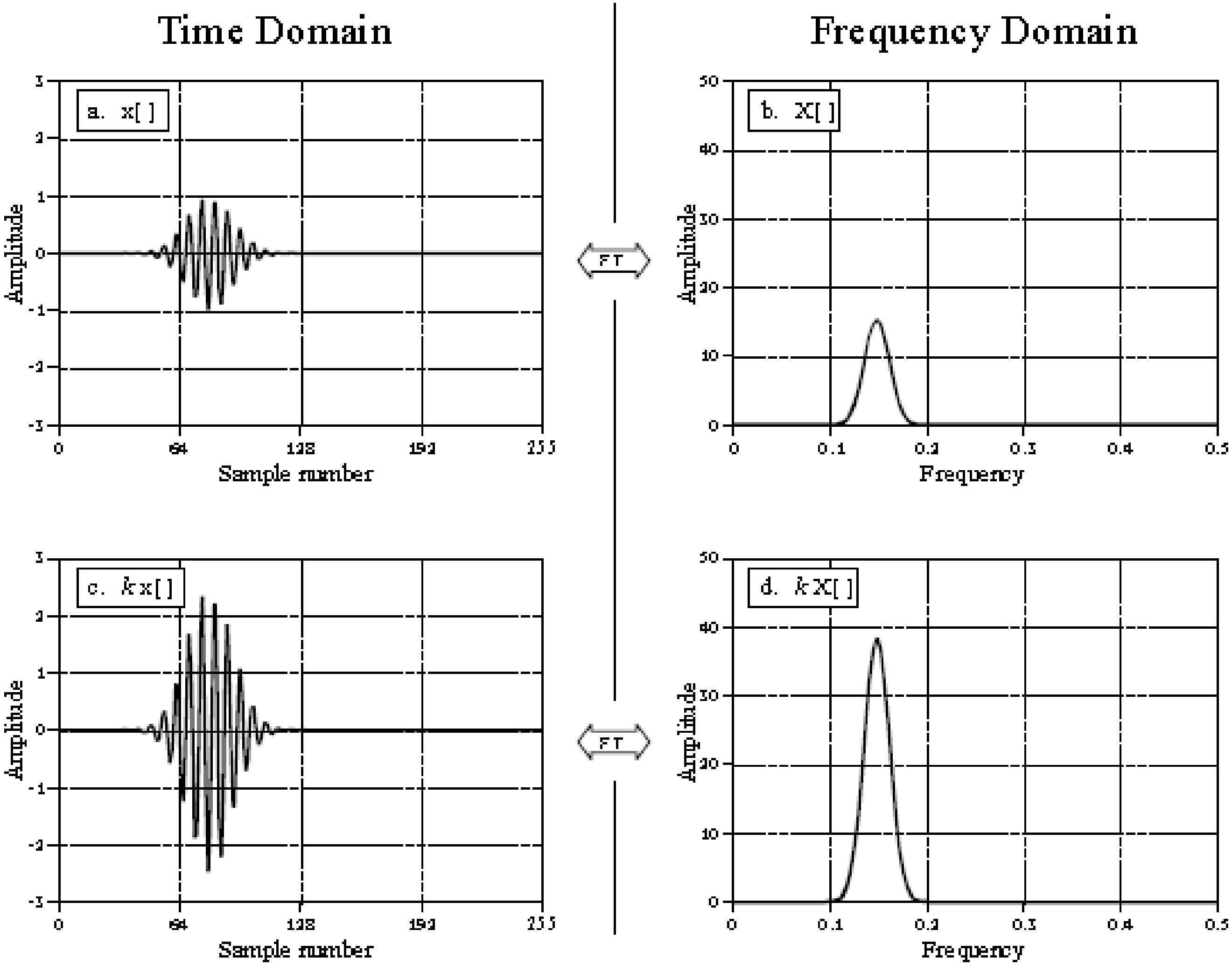
FFT



IFFT



Fourier Transform Properties (1)



FT is linear:

- **Homogeneity**
- **Additivity**

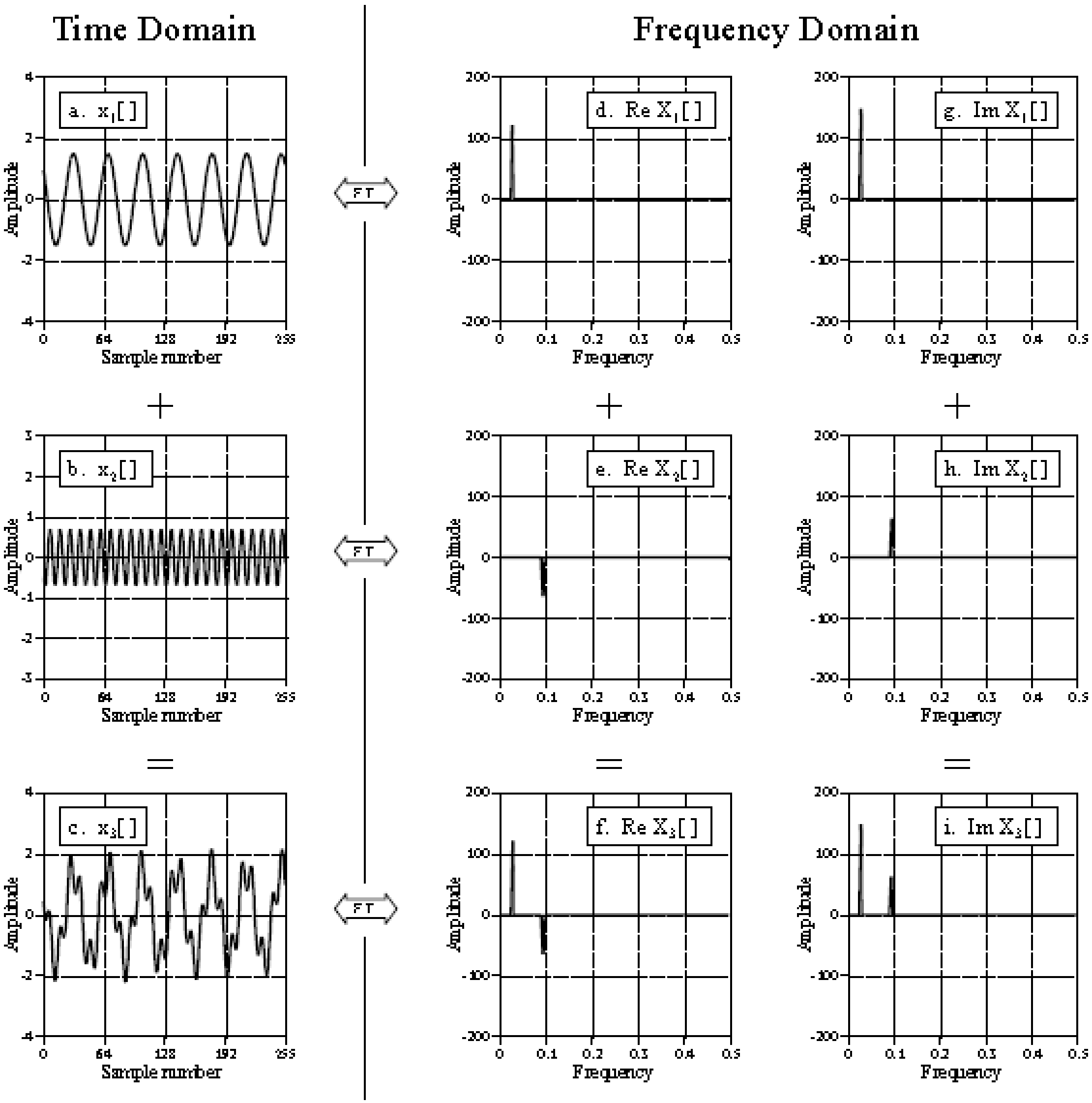
Homogeneity:

$x[\,] \xrightarrow{DFT} X[\,]$

$kx[\,] \xrightarrow{DFT} kX[\,]$

Frequency is not changed.

Fourier Transform Properties (2)



Additivity

$$If : x_1[n] + x_2[n] = x_3[n]$$

$$Then : \operatorname{Re} X_1[f] + \operatorname{Re} X_2[f] = \operatorname{Re} X_3[f]$$

$$and \quad \operatorname{Im} X_1[f] + \operatorname{Im} X_2[f] = \operatorname{Im} X_3[f]$$

Fourier Transform Pairs

Delta Function Pairs in Polar Form

Delta Function

Shifted Delta Function

Same Magnitude, Different Phase

Shifted Delta Function