(2.5) The Fundamental existance and uniqeness Th. for Curves [Pg. 45]

A plane Curve ($\tau \equiv 0$) is a helix. Perform the integration as outlined in Ex (5.4) for $K = \text{constant} = c > 0$,

1st we will parametrize α by a parameter t as follows

$t = \int_0^s k(\rho)\,d\rho = \int_0^s c\,d\rho = c\rho\Big|_0^s = cs \quad \therefore t = cs = f(s)$

$\boxed{\frac{t}{c} = s} = g(t)$ (1) (clear that f and g are differentable functions)

$\frac{dt}{ds} = c = k$ and $\boxed{\frac{ds}{dt} = \frac{1}{c}} = \frac{1}{k}$ $\quad c > 0$

$\frac{dT}{dt} = \frac{dT}{ds}\frac{ds}{dt} = T' \cdot \frac{1}{c} = k \cdot N \cdot \frac{1}{c} = c \cdot N \cdot \frac{1}{c} = N \Rightarrow \boxed{\frac{dT}{dt} = N} \rightarrow (2)$

$\frac{dN}{dt} = \frac{dN}{ds}\cdot\frac{ds}{dt} = (-kT + \tau B)\frac{1}{c} = -T$ (as $\tau = 0 \Rightarrow \tau B = 0$) $\Rightarrow \boxed{\frac{dN}{dt} = -T} \rightarrow (3)$

Differentable (3) w.r.t t to get $\frac{d}{dt}\left(\frac{dN}{dt}\right) = -\frac{dT}{dt} = -N$ (from (2))

$\Rightarrow \frac{d^2N}{dt^2} = -N \Rightarrow \boxed{N'' + N = 0} \rightarrow (4)$

(4) is 2^{nd} order homegonous differential equation with constant cofficent.

Let $N = e^{mt} \Rightarrow N' = m e^{mt}$, $N'' = m^2 e^{mt}$

$N'' + N = 0 \Rightarrow m^2 e^{mt} + e^{mt} = 0 \Rightarrow e^{mt}(m^2+1) = 0 \Rightarrow m^2 + 1 = 0$ (as $e^{mt} \neq 0\ \forall m$)

$\Rightarrow m = \pm i \Rightarrow \boxed{N(t) = \vec{a}\cos t + \vec{b}\sin t} \rightarrow (5)$ for some fixed vector $\vec{a}$ and $\vec{b}$

Now, by integrate (2) we get $T = \int N\,dt = \vec{a}\sin t - \vec{b}\cos t + \vec{c_1}$ (where c_1 constant vector)

$\therefore \alpha(s) = \int T\,dt = \int_0^s (\vec{a}\sin t\ \vec{b}\cos t + \vec{c_1})\,dt$.

Now, will find the vectors $\vec{a}, \vec{b}, \vec{c}$.

Now, as $\|N\| = 1 \Rightarrow \langle N, N\rangle = 1 \Rightarrow \langle N, N\rangle' = 0 \Rightarrow 2\langle N, N'\rangle = 0 \Rightarrow \langle N, N'\rangle = 0$

Hence $0 = \langle \vec{a}\cos t + \vec{b}\sin t, -\vec{a}\sin t + \vec{b}\cos t\rangle = -\|a\|^2 \sin t\cos t + \langle a,b\rangle \cos^2 t - \langle a,b\rangle \sin^2 t + \|b\|^2 \sin t \cos t$

$\therefore (\|b\|^2 - \|a\|^2)\sin t\cos t + \langle a,b\rangle(\cos^2 t - \sin^2 t) = 0 \rightarrow (*)$

put $t = 0 \Rightarrow \sin t = 0$, $\cos t = 1$

So $(*)$ will be $\langle a,b\rangle = 0 \Rightarrow a \perp b \quad \therefore a, b$ are orthoganal

So $(*)$ will be $(\|b\|^2 - \|a\|^2)\sin t\cos t + 0.(\cos^2 t - \sin^2 t) = 0 \Rightarrow \frac{1}{2}(\|b\|^2 - \|a\|^2)\sin 2t = 0$

Now, put $t = \pi/4 \Rightarrow \frac{1}{2}(\|b\|^2 - \|a\|^2)\sin\frac{\pi}{2} = 0 \Rightarrow \frac{1}{2}(\|b\|^2 - \|a\|^2) = 0 \Rightarrow \|b\| = \|a\| \rightarrow (**)$

Now, as $\langle N, N\rangle = 1 \Rightarrow 1 = \langle \vec{a}\cos t + \vec{b}\sin t, \vec{a}\cos t + \vec{b}\sin t\rangle$

$\Rightarrow 1 = \|a\|^2\cos^2 t + \|b\|^2 \sin^2 t + 2\langle a,b\rangle \sin t\cos t = \|a\|^2\cos^2 t + \|a\|^2 \sin^2 t$ (as $\|a\| = \|b\|$)

$\Rightarrow \|a\|^2(\cos^2 t + \sin^2 t) = 1 \Rightarrow \|a\|^2 = 1 \Rightarrow \|a\| = 1 = \|b\|$ and $a \perp b$

يتبع →

dotaizas

الموضوع: التاريخ : / / هـ

Now as $\langle T, N \rangle = 0$ we have $\langle \vec{a}\sin t - \vec{b}\cos t + \vec{c}_1, \vec{a}\cos t + \vec{b}\sin t \rangle = 0$

$\Rightarrow \cancel{\|a\|^2 \sin t \cos t} + \overset{a \perp b}{\cancel{\langle a, b\rangle}}\sin^2 t - \overset{a \perp b}{\cancel{\langle a, b\rangle}}\cos^2 t - \overset{\|a\|^2 = \|b\|^2}{\cancel{\|b\|^2 \sin t \cos t}} + \langle a, c_1\rangle \cos t + \langle b, c_1\rangle \sin t = 0$

$\Rightarrow \langle a, c_1\rangle \cos t + \langle b, c_1\rangle \sin t = 0$ let $t = 0$, $\sin t = 0$, $\cos t = 1$

$\Rightarrow \langle a, c_1\rangle \cdot 1 + \cancel{\langle b, c_1\rangle \cdot 0} = 0 \Rightarrow \langle a, c_1\rangle = 0 \Rightarrow \vec{a} \perp \vec{c}$.

and let $t = \frac{\pi}{2}$, $\sin t = 1$, $\cos t = 0 \Rightarrow \langle b, c_1\rangle = 0 \rightarrow \vec{b} \perp \vec{c}$

$1 = \langle T, T\rangle = \|a\|^2 \sin^2 t + \|b\|^2 \cos^2 t + \|c_1\|^2 - 2\overset{0}{\cancel{\langle a, b\rangle \sin t \cos t}} + 2\overset{0}{\cancel{\langle a, c_1\rangle}} \sin t - 2\overset{0}{\cancel{\langle b, c_1\rangle}} \cos t$

$\Rightarrow \sin^2 t + \cos^2 t + \|c_1\|^2 = 1 \Rightarrow \|c_1\|^2 = 0 \Rightarrow \vec{c}_1 = 0$

$\therefore$ let $a = (1,0,0)$, $b = (0,1,0)$, $c_1 = (0,0,0)$

Now $T(t) = \vec{a}\sin t - \vec{b}\cos t + \overset{0}{\vec{c}_1}$

$\therefore T(s) = \vec{a}\sin(cs) - \vec{b}\cos(cs) \Rightarrow T(s) = (\sin cs, -\cos cs, 0)$

$\therefore x(s) = \int_0^s \vec{a}\sin cs - \vec{b}\cos cs \, ds = -\frac{\vec{a}}{c}\cos cs - \frac{\vec{b}}{c}\sin cs \Big]_0^s = -\frac{\vec{a}}{c}\cos cs - \frac{\vec{b}}{c}\sin cs + \frac{\vec{a}}{c}$

$\therefore x(s) = -\frac{1}{c}\left[(\cos cs - 1, 0, 0) + (0, \sin cs, 0)\right] = -\frac{1}{c}(\cos cs - 1, \sin cs, 0)$