Tangent Plane of survace at (a,b)

① Normal factor

$\langle f_x(a,b),\ f_y(a,b),\ -1 \rangle$

② Point at plane $(a, b, f(a,b))$

③ Equation of tangent Plane

$$\boxed{\nu \cdot \langle x-a,\ y-b,\ z-f(a,b) \rangle = 0}$$

For example :

Let $f(x,y) = 6 - x^2 - y^2$. Find the tangent. Plane equation at $(1,2,1)$. Find the equation of Normal Line :

Solution :

$f_x = -2x \qquad f_y = -2y$

$f_x(1,2) = -2 \qquad f_y(1,2) = -4$

So, $\vec{n} = (-2, -4, \boxed{-1})$

★ Now the Equation of tangent Plane :

$\langle -2, -4, -1 \rangle \cdot \langle x+2,\ y+4,\ z+1 \rangle = 0$

$\Leftrightarrow (-2)(x+2) + (-4)(y+4) + (-1)(z+1) = 0$

$\Leftrightarrow -2x - 4y - z - 21 = 0$

★ The equation of Normal Line :

$$\frac{x-1}{-2} = \frac{y-2}{-4} = \frac{z-1}{-1}$$

Example : Find the equations of tangent Plane and the normal Line to

$$f(x,y) = x^3 + y^3 + \frac{x^2}{y} \quad \text{at} \quad (2,1) \ ?$$

Solution :

Let $z = x^3 + y^3 + \frac{x^2}{y}$.

The tangent Point $= (2, 1, f(2,1))$

$= (2, 1, 13)$

The normal vector $= \vec{n}$

$= \langle f_x(2,1), \ f_y(2,1), \ -1 \rangle$

Now $\quad f_x = 3x^2 + \frac{2}{y} x$

$\Rightarrow f_x(2,1) = 16$

$f_y = 3y^2 + (x^2)\cdot\left(\frac{-1}{y^2}\right)$

$\Rightarrow f_y(2,1) = 3(1)^2 + (2)^2\left(\frac{-1}{1}\right) = -1$

$\therefore \vec{n} = \langle 16, -1, -1 \rangle$

$\therefore$ The Equation of tangent plane at $(2,1,13)$:

$$\langle 16, -1, -1 \rangle \cdot \langle x-2, \ y-1, \ z-13 \rangle = 0$$

$\therefore$ The equation of ~~tangent~~ normal line at $(2,1,13)$:

$$\frac{x-2}{16} = \frac{y-1}{-1} = \frac{z-13}{-1}$$

** Let $z = f(x,y)$

① The <u>increment</u> of f at (a,b) :

$$\Delta z = f(a+\Delta x\ ,\ b+\Delta y) - f(a,b)$$

② Suppose $z = f(x,y)$ is defined on

$D = \{ (x,y) : x_0 < x < x_1 \text{ and } y_0 < y < y_1 \}$ and

f_x & f_y are defind on D. Then

$$\boxed{\Delta z = f_x(a,b)\,\Delta x + f_y(a,b) + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y}$$

where

ε_1 and ε_1 are functions of $\Delta x, \Delta y$

and $(\Delta x, \Delta y) \rightsquigarrow (0,0)$.

Example: Find the increment Δz where $z = x^2 - 5xy$?

<u>Solution</u> : $\Delta z = f(x+\Delta x\ ,\ y+\Delta y) - f(x,y)$

$$= (x+\Delta x)^2 - 5(x+\Delta x)(y+\Delta y) - [x^2 - 5xy]$$

$$= x^2 + 2\Delta x\, x + (\Delta x)^2 - 5xy - 5x\Delta y - 5y\Delta x$$
$$- 5\Delta x\,\Delta y - x^2 + 5xy$$

$$= \underbrace{(2x-5)}_{f_x}\Delta x + \underbrace{(-5x)}_{f_y}\Delta y + \underbrace{(\Delta x)}_{\varepsilon_1}\Delta x + \underbrace{(-5\Delta x)}_{\varepsilon_2}\Delta y$$

<u>Remark</u>

If $\varepsilon_1, \varepsilon_2 \rightsquigarrow 0$ when $(\Delta x, \Delta y) \rightsquigarrow (0,0)$ Then f is called <u>differentiable</u> at (a,b). ∎

** The <u>Linear approximation</u> of $f(x,y,z)$ at (a,b,c) is given by :

$$\boxed{\mathcal{L} = f(a,b,c) + f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c)}$$

⊕ Example : A box with dimensions 3, 4, 5 with a Possible error $\pm \frac{1}{16}$ in each measurment. use the differentials to approximate the maximum error in Computing of

(1) surface area (2) volume.

<u>Solution</u>

The surface area = $S = 2(xy + yz + xz)$

So,

$$ds = 2(y+z)\,dx + 2(x+z)\,dy + 2(x+y)\,dz$$

As $dx = dy = dz = \pm \frac{1}{16}$

$$\Rightarrow ds = 2(4+5)\left(\pm\frac{1}{16}\right) + 2(3+5)\left(\pm\frac{1}{16}\right) + 2(4+3)\left(\pm\frac{1}{16}\right).$$

* The same story at volume :

$$v = xyz$$

$$dv = yz\,dx + xz\,dy + xy\,dz$$

⊕ Example : Find the Linear approximation of $v = xyz$ at $(1,1,1) = P$

$$\mathcal{L} = f(1,1,1) + f_x^P(x-a) + f_y^P(y-b) + f_z^P(z-c)$$

$$\mathcal{L} = 1 + 1(x-1) + 1(y-1) + 1(z-1)$$ ■

** Chain Rule :

1 Let $z = f(x(t), y(t))$ where $x(t)$ and $y(t)$ are differentiable Then

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{dt}}$$

Example: $z = x^2 e^y$ where $x = t^2 - 1$ and $y = \sin t$

find $\frac{dz}{dt}$?

Solution

$$\frac{\partial z}{\partial x} = 2 e^y x$$

$$\frac{dx}{dt} = 2t$$

$$\frac{\partial z}{\partial y} = x^2 e^y$$

$$\frac{dy}{dt} = \cos t$$

Therefore, $\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$

$$= (2t)(2e^y x) + (\cos t)(x^2 e^y)$$

$$= \underbrace{(2t)(2e^{\sin t}(t^2-1)) + (\cos t)((t^2-1)^2 e^{\sin t})}$$

Notice that The final statement is presented by $\underline{\underline{t}}$.

$\boxed{2}$ Let $z = f(x(s,t),\ y(s,t))$ then

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s} \qquad \text{and}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$

⊕ Example:

Let $f = e^{xy}$ where $x = 3u$ and $y = 4v^2 u$.

Find $\dfrac{\partial f}{\partial u}$ and $\dfrac{\partial f}{\partial v}$?

Solution
$$\frac{\partial f}{\partial x} = y e^{xy}$$

$$\frac{\partial f}{\partial y} = x e^{xy}$$

$$\frac{\partial x}{\partial u} = 3 \qquad \frac{\partial x}{\partial v} = 0 \qquad \frac{\partial y}{\partial u} = 4v^2$$

$$\frac{\partial y}{\partial v} = 8uv$$

Now

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$= (ye^{xy}) \cdot (3) + (xe^{xy})(4v^2)$$

$$= [(4v^2 y)\, e^{3u(4v^2u)}](3) + (3u\, e^{3u(4v^2u)})(4v^2)$$

The same story for $\dfrac{\partial f}{\partial v}$