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HW #4

PROBLEMS, SECTION 12.9

Expand the following functions in Legendre series.

3.
$f(x) = P_3'(x)$

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$, so $f(x) = P_3'(x) = \frac{1}{2}(15x^2 - 3)$. Suppose there are real coefficients a_i for all $i = 0, 1, 2, \ldots$ such that

(42) $$f(x) = \sum_{l=0}^{\infty} a_l P_l(x),$$

where

(43) $$a_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)\mathrm{d}x.$$

Then,

(44) $$a_0 = \frac{1}{2}\int_{-1}^{1} f(x)P_0(x)\mathrm{d}x = \frac{1}{4}\int_{-1}^{1}(15x^2 - 3)\mathrm{d}x = \frac{1}{4}\left[5x^3 - 3x\right]_{-1}^{1} = 1.$$

(45) $$a_1 = \frac{3}{2}\int_{-1}^{1} f(x)P_1(x)\mathrm{d}x = \frac{3}{2}\int_{-1}^{1}(15x^3 - 3x)\mathrm{d}x = \frac{3}{2}\left[\frac{15}{4}x^4 - \frac{3}{2}x^2\right]_{-1}^{1} = 0.$$

(The next few a_n terms were found by Wolfram Alpha.)

(46) $$a_2 = \frac{5}{2}\int_{-1}^{1} f(x)P_2(x)\mathrm{d}x = \frac{5}{4}\int_{-1}^{1}(15x^2 - 3)P_2(x)\mathrm{d}x = 5.$$

(47) $$a_3 = \frac{7}{2}\int_{-1}^{1} f(x)P_3(x)\mathrm{d}x = \frac{7}{4}\int_{-1}^{1}(15x^2 - 3)P_3(x)\mathrm{d}x = 0.$$

(48) $$a_4 = \frac{9}{2}\int_{-1}^{1} f(x)P_4(x)\mathrm{d}x = \frac{9}{4}\int_{-1}^{1}(15x^2 - 3)P_4(x)\mathrm{d}x = 0.$$

(49) $$a_5 = \frac{11}{2}\int_{-1}^{1} f(x)P_5(x)\mathrm{d}x = \frac{11}{4}\int_{-1}^{1}(15x^2 - 3)P_5(x)\mathrm{d}x = 0.$$

(50) $$a_6 = \frac{13}{2}\int_{-1}^{1} f(x)P_6(x)\mathrm{d}x = \frac{13}{4}\int_{-1}^{1}(15x^2 - 3)P_6(x)\mathrm{d}x = 0.$$

The fact that $a_3 = a_4 = a_5 = a_6 = 0$ makes sense since $f(x)$ is a polynomial of degree 2, so it should only have to be approximated by polynomials of degree 2 or less. So, we have

(51) $$f(x) = \sum_{l=0}^{\infty} a_l P_l(x) = P_0(x) + 5P_2(x), \text{ and}$$

(52) $$P_0(x) + 5P_2(x) = 1 + 5\left[\frac{1}{2}(3x^2 - 1)\right] = \frac{15}{2}x^2 - \frac{3}{2} = \frac{1}{2}(15x^2 - 3) = f(x).$$

Therefore,

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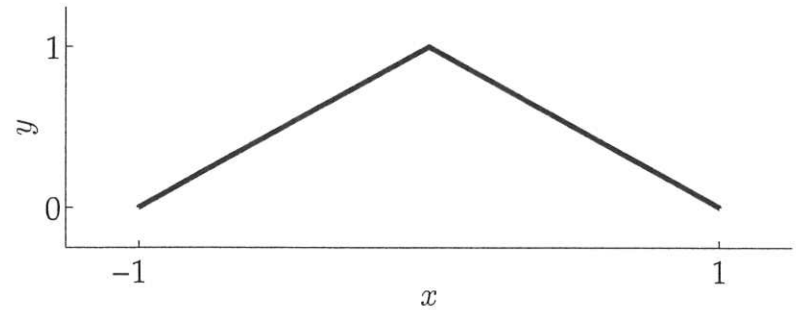
(53) $$\boxed{f(x) = P_0(x) + 5P_2(x).}$$

5.



Our given function $f : (-1, 1) \to \mathbb{R}$ is defined by $f(x) = 1 - |x|$. Suppose there are real coefficients a_i for all $i = 0, 1, 2, \ldots$ such that

$$f(x) = \sum_{l=0}^{\infty} a_l P_l(x), \tag{54}$$

where

(55)
$$a_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x) \mathrm{d}x = \frac{2n+1}{2} \left[\int_{-1}^{0} (1+x) P_n(x) \mathrm{d}x + \int_{0}^{1} (1-x) P_n(x) \mathrm{d}x \right].$$

So,

$$\begin{aligned} a_0 &= \frac{1}{2} \left[\int_{-1}^{0} (1+x) P_0(x) \mathrm{d}x + \int_{0}^{1} (1-x) P_0(x) \mathrm{d}x \right] \\ &= \frac{1}{2} \left[\int_{-1}^{0} (1+x) \mathrm{d}x + \int_{0}^{1} (1-x) \mathrm{d}x \right] \\ &= \frac{1}{2} \left(\left[x + \frac{1}{2} x^2 \right]_{-1}^{0} + \left[x - \frac{1}{2} x^2 \right]_{0}^{1} \right) = \frac{1}{2}. \end{aligned} \tag{56}$$

$$\begin{aligned} a_1 &= \frac{3}{2} \left[\int_{-1}^{0} (1+x) P_1(x) \mathrm{d}x + \int_{0}^{1} (1-x) P_1(x) \mathrm{d}x \right] \\ &= \frac{3}{2} \left[\int_{-1}^{0} (x + x^2) \mathrm{d}x + \int_{0}^{1} (x - x^2) \mathrm{d}x \right] \\ &= \frac{3}{2} \left(\left[\frac{1}{2} x^2 + \frac{1}{3} x^3 \right]_{-1}^{0} + \left[\frac{1}{2} x^2 - \frac{1}{3} x^3 \right]_{0}^{1} \right) = 0. \end{aligned} \tag{57}$$

The following a_n are computed with Wolfram Alpha:

$$a_2 = \frac{5}{2} \int_{-1}^{1} (1 - |x|) P_2(x) \mathrm{d}x = -\frac{5}{8}. \tag{58}$$

$$a_3 = \frac{7}{2} \int_{-1}^{1} (1 - |x|) P_3(x) \mathrm{d}x = 0. \tag{59}$$

(60)
$$a_4 = \frac{9}{2}\int_{-1}^{1}(1-|x|)P_4(x)\mathrm{d}x = \frac{3}{16}.$$

(61)
$$a_5 = \frac{11}{2}\int_{-1}^{1}(1-|x|)P_5(x)\mathrm{d}x = 0.$$

(62)
$$a_6 = \frac{13}{2}\int_{-1}^{1}(1-|x|)P_6(x)\mathrm{d}x = -\frac{13}{128}.$$

It makes sense that all the odd n terms we found are zero, since our original function has even symmetry. Thus, its Legendre expansion should consist of strictly even polynomials. These are exactly the polynomials $P_n(x)$ where n is even. Thus, we have

(63)
$$\boxed{f(x) \approx \frac{1}{2}P_0(x) - \frac{5}{8}P_2(x) + \frac{3}{16}P_4(x) - \frac{13}{128}P_6(x) + \cdots.}$$

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FIGURE 1 shows the `Matlab`-generated plots of $f(x)$ and some Legendre expansions of $f(x)$. It suggests that the expansion we found is correct since higher-order approximations approach the graph of $f(x)$. The black curve is the real $f(x)$ and the colored curves are Legendre expansions up to various n (cyan is $n = 0$, green is $n = 2$, red is $n = 4$, and blue is $n = 6$).

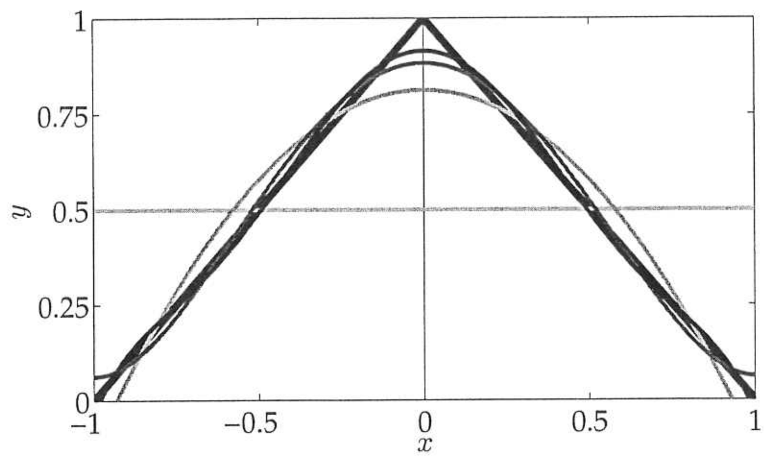

FIGURE 1

11.

Expand the polynomial $7x^4 - 3x + 1$ in a Legendre series. You should get the same results that you got by a different method in the corresponding problem in Section 5.

Let $f(x) = 7x^4 - 3x + 1$. Suppose there are real coefficients a_i for all $i = 0, 1, 2, \ldots$ such that

$$f(x) = \sum_{l=0}^{\infty} a_l P_l(x), \tag{64}$$

where

$$a_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x) \mathrm{d}x. \tag{65}$$

Thus, we have

$$a_0 = \frac{1}{2} \int_{-1}^{1} (7x^4 - 3x + 1) \mathrm{d}x = \frac{1}{2} \left[\frac{7}{5}x^5 - \frac{3}{2}x^2 + x \right]_{-1}^{1} = \frac{12}{5}. \tag{66}$$

$$a_1 = \frac{3}{2} \int_{-1}^{1} (7x^5 - 3x^2 + x) \mathrm{d}x = \frac{3}{2} \left[\frac{7}{6}x^6 - x^3 + \frac{1}{2}x^2 \right]_{-1}^{1} = -3. \tag{67}$$

Again, Wolfram Alpha finds the next a_n values:

$$a_2 = \frac{5}{2} \int_{-1}^{1} (7x^4 - 3x + 1) P_2(x) \mathrm{d}x = 4. \tag{68}$$

$$a_3 = \frac{7}{2} \int_{-1}^{1} (7x^4 - 3x + 1) P_3(x) \mathrm{d}x = 0. \tag{69}$$

$$a_5 = \frac{7}{2} \int_{-1}^{1} (7x^4 - 3x + 1) P_4(x) \mathrm{d}x = \frac{8}{5}. \tag{70}$$

$$a_5 = \frac{11}{2} \int_{-1}^{1} (7x^4 - 3x + 1) P_5(x) \mathrm{d}x = 0. \tag{71}$$

$$a_6 = \frac{13}{2} \int_{-1}^{1} (7x^4 - 3x + 1) P_6(x) \mathrm{d}x = 0. \tag{72}$$

It was superfluous to compute a_5 and a_6, since every a_n for $n > 4$ is expected to be zero because $7x^4 - 3x + 1$ is a polynomial of degree 4. Therefore, higher degree polynomials can't contribute to the expansion. Thus, we find (as we did in Problem 12.5.12.) that

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$$\boxed{7x^4 - 3x + 1 = \frac{12}{5} P_0(x) - 3P_1(x) + 4P_2(x) + \frac{8}{5} P_4(x).} \tag{73}$$

13.
Find the best (in the least squares case) second-degree approximation to each of the function x^4 over the interval $-1 < x < 1$.

We want to find coefficients a_0, a_1, a_2 such that

$$x^4 \approx a_0 p_0(x) + a_1 p_1(x) + a_2 p_2(x) \tag{74}$$

where the p_n's are normalized Legendre polynomials:

$$p_n(x) = \sqrt{\frac{2n+1}{2}} P_n(x). \tag{75}$$

Then,

$$a_0 = \int_{-1}^{1} x^4 p_0(x) \mathrm{d}x = \sqrt{\frac{1}{2}} \int_{-1}^{1} x^4 \mathrm{d}x = \frac{2}{5\sqrt{2}} = \frac{\sqrt{2}}{5}. \tag{76}$$

$$a_1 = \int_{-1}^{1} x^4 p_1(x) \mathrm{d}x = \sqrt{\frac{3}{2}} \int_{-1}^{1} x^5 \mathrm{d}x = 0. \tag{77}$$

$$\begin{aligned} a_2 &= \int_{-1}^{1} x^4 p_2(x) \mathrm{d}x = \frac{1}{2}\sqrt{\frac{5}{2}} \int_{-1}^{1} x^4 (3x^2 - 1) \mathrm{d}x \\ &= \frac{1}{2}\sqrt{\frac{5}{2}} \int_{-1}^{1} (3x^6 - x^4) \mathrm{d}x = \frac{1}{2}\sqrt{\frac{5}{2}} \left[\frac{3}{7}x^7 - \frac{1}{5}x^5 \right]_{-1}^{1} \\ &= \frac{1}{2}\sqrt{\frac{5}{2}} \left[\frac{6}{7} - \frac{2}{5} \right] = \frac{4}{7}\sqrt{\frac{2}{5}}. \end{aligned} \tag{78}$$

Thus, we get the least-squares quadratic approximation of x^4:

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$$\boxed{x^4 \approx \frac{\sqrt{2}}{5} p_0(x) + \frac{4}{7}\sqrt{\frac{2}{5}} p_2(x).} \tag{79}$$

FIGURE 2 shows the Matlab-generated plots of x^4 (in black) and the quadratic Legendre approximation of x^4 (in red) on the interval $(-1, 1)$.

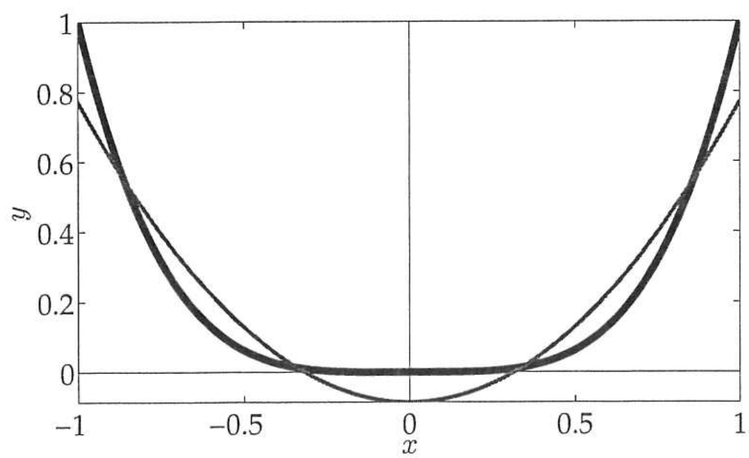


FIGURE 2

15.
Find the best (in the least squares case) second-degree approximation to each of the function $\cos \pi x$ over the interval $-1 < x < 1$.

As in the previous problem, we wish to find coefficients a_0, a_1, a_2 such that

$$\cos(\pi x) \approx a_0 p_0(x) + a_1 p_1(x) + a_2 p_2(x) \tag{80}$$

where the p_n's are normalized Legendre polynomials as in (75). Then,

$$a_0 = \int_{-1}^{1} \cos(\pi x) p_0(x) \mathrm{d}x = \sqrt{\frac{1}{2}} \int_{-1}^{1} \cos(\pi x) \mathrm{d}x = 0. \tag{81}$$

$$a_1 = \int_{-1}^{1} \cos(\pi x) p_1(x) \mathrm{d}x = \sqrt{\frac{3}{2}} \int_{-1}^{1} x \cos(\pi x) \mathrm{d}x = 0. \tag{82}$$

$$a_2 = \int_{-1}^{1} \cos(\pi x) p_2(x) \mathrm{d}x = \frac{1}{2}\sqrt{\frac{5}{2}} \int_{-1}^{1} \cos(\pi x)(3x^2 - 1) \mathrm{d}x = -\sqrt{\frac{5}{2}} \frac{6}{\pi^2} = -\frac{3\sqrt{10}}{\pi^2}. \tag{83}$$

Thus, the least-squares quadratic approximation of $\cos(\pi x)$ is

4

$$\boxed{\cos(\pi x) \approx -\frac{3\sqrt{10}}{\pi^2} p_2(x).} \tag{84}$$

FIGURE 3 shows the `Matlab`-generated plots of $\cos(\pi x)$ (in black) and the quadratic Legendre approximation of $\cos(\pi x)$ (in red) on the interval $(-1, 1)$.

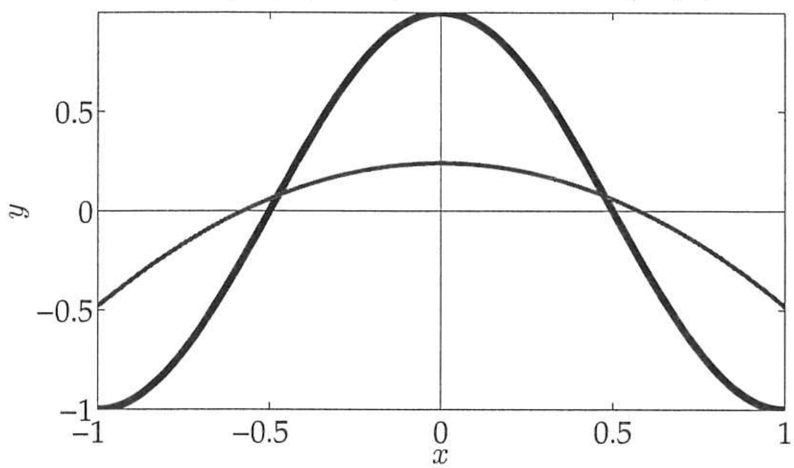


FIGURE 3

HW #4

$\boxed{12.11.1}$ Solving $x^2y'' + 4xy' + (x^2+2)y = 0$

Text shows with $y(x) = \sum_{n=0}^{\infty} a_n x^{n+s}$ obtain

Indicial eqn. $s^2 + 3s + 2 = (s+1)(s+2) = 0$

Recurrence Relation $[(n+s)(n+s-1) + 4(n+s) + 2]a_n = -a_{n-2}$

$$a_n = \frac{-a_{n-2}}{(n+s)^2 + 3(n+s) + 2}$$

For $s=-1$, obtains soln. $y(x) = a_0 x^{-2}\left(x - \frac{x^3}{3!} + \ldots\right) = a_0 \frac{\sin(x)}{x^2}$

For $s=-2$ must seek soln.

$$y_2(x) = a\, y_1(x) \ln|x| + x^{-2}\left[1 + \sum_{n=1}^{\infty} c_n(r_2) x^n\right]$$

$$y_2' = a\, y_1' \ln|x| + \frac{a}{x} y_1 + x^{-2}\left[\sum_{n=1}^{\infty} n c_n x^{n-1}\right] - 2x^{-3}\left[1 + \sum_{n=1}^{\infty} c_n x^n\right]$$

$$y_2'' = a\, y_1'' \ln|x| + \frac{2a}{x} y_1' - \frac{a}{x^2} y_1 + x^{-2}\left[\sum_{n=1}^{\infty} n(n-1) c_n x^{n-2}\right] - 4x^{-3}\left[\sum_{n=1}^{\infty} n c_n x^{n-1}\right]$$
$$+ 6x^{-4}\left[1 + \sum_{n=1}^{\infty} c_n x^n\right]$$

Substitute into d.e.

$$a \ln|x| \underbrace{\left(x^2 y_1'' + 4xy_1' + (x^2+2)y'\right)}_{=0} + 2axy_1' + 3ay_1$$
$$+ x^{-2}\left[\sum_{n=1}^{\infty}[n(n-1)c_n - 4nc_n + 6c_n]x^n + 6 + \sum_{n=1}^{\infty}[4nc_n - 8c_n]x^n - 8\right.$$
$$\left. \cancel{2} + \sum_{n=1}^{\infty} 2c_n x^n\right] + \left[1 + \sum_{n=1}^{\infty} c_n x^n\right] = 0$$

$$= \frac{a}{x^2}\left(2x\cos(x) - \sin(x)\right) + x^{-2}\left[\sum_{n=1}^{\infty}[n(n-1)c_n]x^n + x^2 + \sum_{n=3}^{\infty} c_{n-2} x^n\right] = 0$$

For terms x^{-1}, only appear in first expression $\Rightarrow a = 0$

Recurrence relation $c_n = -\frac{c_{n-2}}{n(n-1)}$ odd coef. generate $y_1(x)$

Terms x^0: $2c_2 + 1 = 0 \Rightarrow c_2 = -1/2$,

$c_4 = -\frac{c_2}{4 \cdot 3} = \frac{1}{4!}$, $c_6 = -\frac{1}{6!}$, ...

$$y_2(x) = x^{-2}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}\right] = x^{-2}\cos(x)$$

5

$\boxed{12.11.5}$ Solve $2xy'' + y' + 2y = 0$

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+s}, \quad y' = \sum_{n=0}^{\infty} (n+s) a_n x^{n+s-1}, \quad y'' = \sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s-2}$$

Substituting

$$2\sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s-1} + \sum_{n=0}^{\infty} (n+s) a_n x^{n+s-1} + 2\sum_{n=1}^{\infty} a_{n-1} x^{n+s-1} = 0$$

Indicial Eqn. $\quad 2s(s-1) + s = 0 \Rightarrow 2s^2 - s = (2s-1)s = 0$

$s = 0, \frac{1}{2}$

Recurrence Relation $\quad [2(n+s)(n+s-1) + n+s] a_n + 2a_{n-1} = 0$

$$a_n = \frac{-2a_{n-1}}{(2n+2s-1)(n+s)}$$

For $s_1 = \frac{1}{2}$, $\quad a_0(\frac{1}{2})$ arb. $\quad a_n = \frac{-2a_{n-1}}{2n(n+\frac{1}{2})} = \frac{-2a_{n-1}}{n(2n+1)}$

$$y_1(x) = a_0 x^{1/2}\left[1 - \tfrac{2}{3}x + \tfrac{2}{15}x^2 - \tfrac{4}{315}x^3 + \dots\right]$$

For $s_2 = 0$, $\quad a_0(0)$ arb. $\quad a_n = \frac{-2a_{n-1}}{(2n-1)n}$

5 $\quad y_2(x) = b_0\left(1 - 2x + \tfrac{2}{3}x^2 - \tfrac{4}{45}x^3 + \dots\right)$

1. Show that the differential equation below has a regular singular point at $x = 0$, and determine two linearly independent solutions for $x > 0$:

$$x^2y'' + xy' + 2xy = 0.$$

$y(x) = \sum_{n=0}^{\infty} a_n x^{n+s}$, $y' = \sum_{n=0}^{\infty} (n+s)a_n x^{n+s-1}$, $y'' = \sum_{n=0}^{\infty} (n+s)(n+s-1)a_n x^{n+s-2}$

substituting (+ shifting indices)

$$\sum_{n=0}^{\infty} (n+s)(n+s-1)a_n x^{n+s} + \sum_{n=0}^{\infty} (n+s)a_n x^{n+s} + 2\sum_{n=1}^{\infty} a_{n-1} x^{n+s} = 0$$

<u>Indicial Eqn</u>: $s(s-1) + s = s^2 = 0$

<u>Recurrence Relation</u>

$(n+s)^2 a_n + 2a_{n-1} = 0$, so $a_n = -\dfrac{2a_{n-1}}{(n+s)^2}$

For $s_1 = 0$, a_0 arb. $a_n = \dfrac{-2a_{n-1}}{n^2} = \dfrac{(-1)^n 2^n a_0}{(n!)^2}$

$$y_1(x) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{(n!)^2} x^n$$

For $s_2 = 0$, must seek soln.

$y_2(x) = y_1(x)\ln|x| + \sum_{n=1}^{\infty} b_n x^n$ $\qquad y_2' = y_1'\ln|x| + \frac{1}{x}y_1 + \sum_{n=1}^{\infty} n b_n x^{n-1}$

$y_2'' = y_1''\ln|x| + \frac{2}{x}y_1' - \frac{1}{x^2}y_1 + \sum_{n=1}^{\infty} n(n-1)b_n x^{n-2}$

Substituting

$$\ln|x|\underbrace{\left[x^2y_1'' + xy_1' + 2xy_1\right]}_{=0} + 2xy_1' + \sum_{n=1}^{\infty} n(n-1)b_n x^n + \sum_{n=1}^{\infty} n b_n x^n + 2\sum_{n=2}^{\infty} b_{n-1}x^n = 0$$

$$\Rightarrow \sum_{n=1}^{\infty} n^2 b_n x^n + 2\sum_{n=2}^{\infty} b_{n-1} x^n = -2\sum_{n=1}^{\infty} \frac{(-1)^n 2^n n x^n}{(n!)^2}$$

x^1: $b_1 = 4$ $\qquad b_n = -\dfrac{2b_{n-1}}{n^2} - \dfrac{(-1)^n 2^{n+1}}{n(n!)^2}$

$$y_2(x) = y_1(x)\ln(x) + 4x - 3x^2 + \frac{22}{27}x^3 - \frac{25}{216}x^4 + \dots$$

$$= y_1(x)\ln(x) - 2\sum_{n=1}^{\infty} \frac{(-1)^n 2^n H_n}{(n!)^2} x^n \qquad \text{with} \quad H_n = \frac{1}{n} + \frac{1}{n-1} + \dots + \frac{1}{2} + 1$$

5

$\boxed{12.12,2}$ Show $J_2(x) = \frac{2}{x} J_1(x) - J_0(x)$

(Eqn. 12.9) gives $J_2(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+3)} \left(\frac{x}{2}\right)^{2n+2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+2)!} \left(\frac{x}{2}\right)^{2n+2}$

$J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1)!} \left(\frac{x}{2}\right)^{2n+1}$ $\qquad J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,n!} \left(\frac{x}{2}\right)^{2n}$

$$\frac{2}{x} J_1(x) - J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1)!} \left(\frac{x}{2}\right)^{2n} - \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,n!} \left(\frac{x}{2}\right)^{2n}$$

$$= \sum_{n=0}^{\infty} \left[\frac{1}{(n+1)!} - \frac{1}{n!}\right] \frac{(-1)^n}{n!} \left(\frac{x}{2}\right)^{2n} = \sum_{n=1}^{\infty} \frac{-n}{(n+1)!} \frac{(-1)^n}{n!} \left(\frac{x}{2}\right)^{2n}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n+1)!(n-1)!} \left(\frac{x}{2}\right)^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+2)!\,n!} \left(\frac{x}{2}\right)^{2(n+1)} = J_2(x)$$

3

$\boxed{12.13,3}$ $J_{-\frac{1}{2}}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+\frac{1}{2})} \left(\frac{x}{2}\right)^{2n-1/2} = \sqrt{\frac{2}{x}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+\frac{1}{2})} \left(\frac{x}{2}\right)^{2n}$

$\Gamma(\frac{1}{2}) = \sqrt{\pi}$ $\qquad \Gamma(\frac{3}{2}) = \frac{1}{2}\sqrt{\pi}$ $\qquad \Gamma(\frac{5}{2}) = \frac{3}{2}(\frac{1}{2}\sqrt{\pi}) \ldots$ $\qquad \Gamma(n+\frac{1}{2}) = \left(\frac{1}{2}\right)^n \sqrt{\pi}\,(2n-1)(2n-3)\cdots 1$

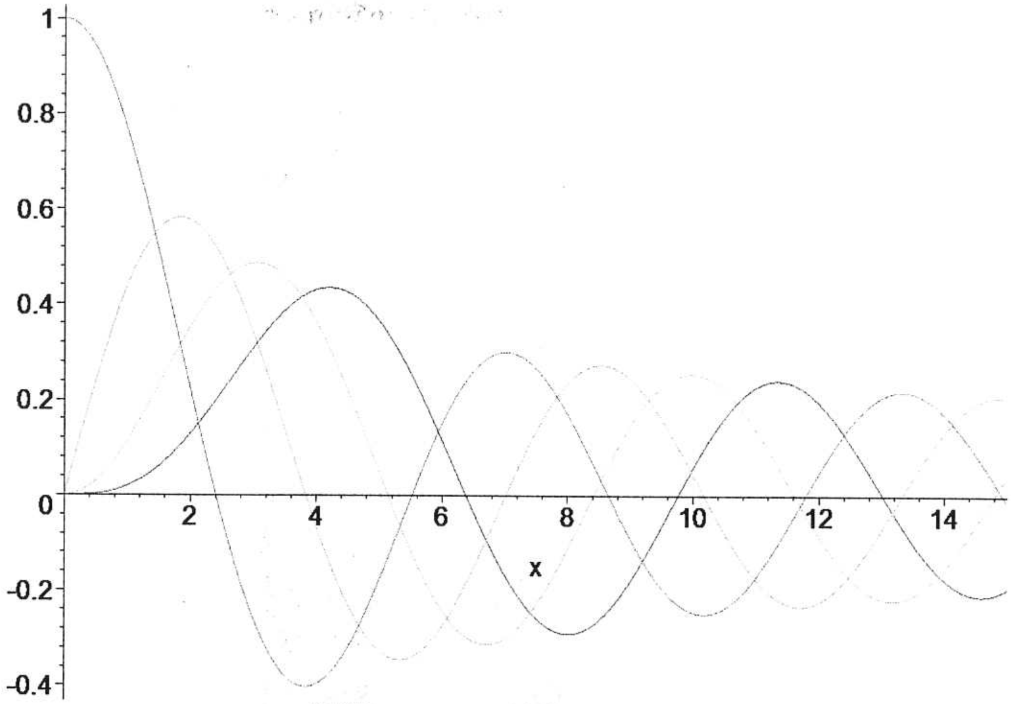
$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\right) = \sqrt{\frac{2}{\pi x}} \cos(x)$$

so $\cos(x) = \sqrt{\frac{\pi x}{2}}\, J_{-\frac{1}{2}}(x)$

3

12.14.1 Plot of Bessel functions

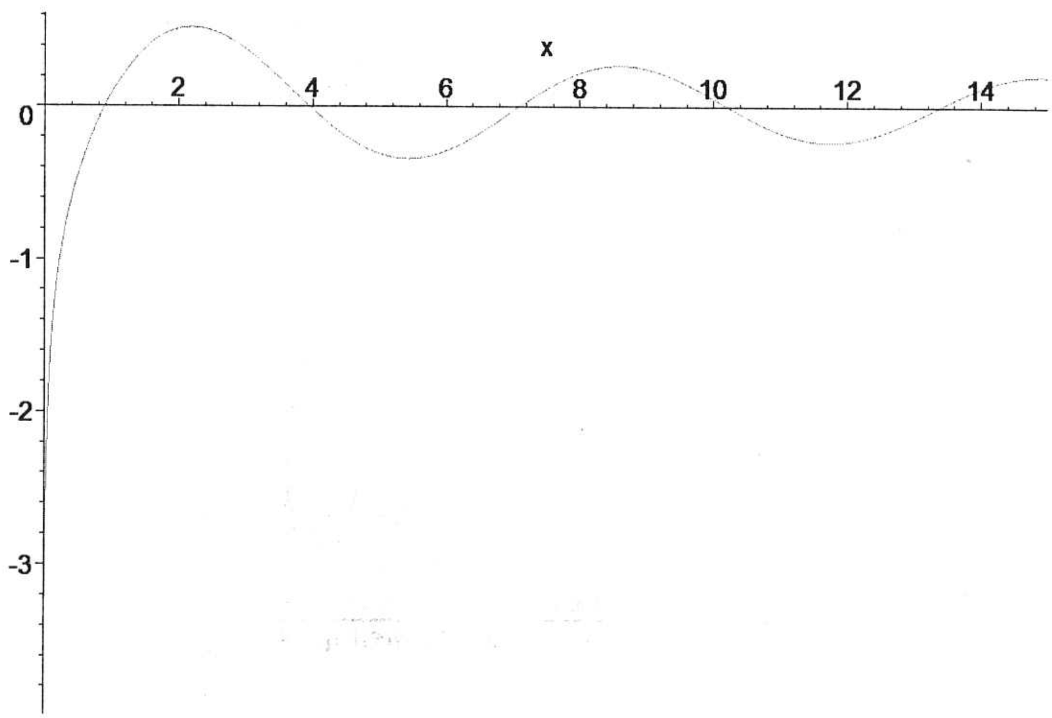
```
> plot({BesselJ(0,x),BesselJ(1,x),BesselJ(2,x),BesselJ(3,x)},x=0..15
  );
```



3/

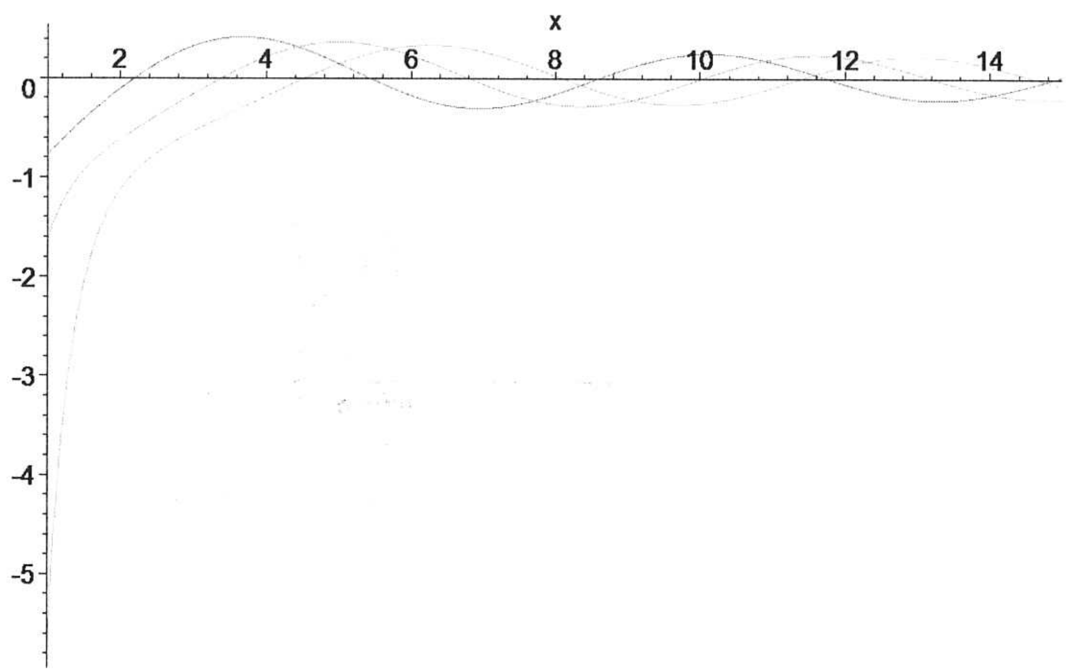
12.14.3 Plot of Weber functions

```
> plot(BesselY(0,x),x=0..15);
```



3/ ↓

```
> plot({BesselY(1,x),BesselY(2,x),BesselY(3,x)},x=1..15);
```


x
2
4
6
8
10
12
14
0
-1
-2
-3
-4
-5

>

Artem Mavrin

Math 342B Homework 7

12.15.1. Prove equation (15.2) by a method similar to the one used above to prove (15.1).

We will prove equation (15.2) in the book:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{-p}J_p(x)\right] = -x^{-p}J_{p+1}(x). \tag{1}$$

We start with equation (12.9):

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+1+p)}\left(\frac{x}{2}\right)^{2n+p}. \tag{2}$$

Multiplying (2) on both sides by x^{-p}, we get

$$x^{-p}J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+1+p)}\frac{x^{2n}}{2^{2n+p}}. \tag{3}$$

Differentiating (3) with respect to x gives

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{-p}J_p(x)\right] &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+1+p)}\frac{x^{2n}}{2^{2n+p}}\right] \\
&= \sum_{n=0}^{\infty} \frac{2n(-1)^n}{\Gamma(n+1)\Gamma(n+1+p)}\frac{x^{2n-1}}{2^{2n+p}} \\
&= \sum_{n=0}^{\infty} \frac{n(-1)^n}{\Gamma(n+1)\Gamma(n+1+p)}\frac{x^{2n-1}}{2^{2n+p-1}} \\
&= 0 + \sum_{n=1}^{\infty} \frac{n(-1)^n}{\Gamma(n+1)\Gamma(n+1+p)}\frac{x^{2n-1}}{2^{2n+p-1}} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^n}{\Gamma(n)\Gamma(n+1+p)}\frac{x^{2n-1}}{2^{2n+p-1}} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{\Gamma(n+1)\Gamma(n+2+p)}\frac{x^{2n+1}}{2^{2n+p+1}} \\
&= -x^{-p}\sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+2+p)}\frac{x^{2n+p+1}}{2^{2n+p+1}} \\
&= -x^{-p}\sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+2+p)}\left(\frac{x}{2}\right)^{2n+p+1} \\
&= -x^{-p}J_{p+1}(x).
\end{aligned} \tag{4}$$

Thus from (4) we have

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{-p}J_p(x)\right] = -x^{-p}J_{p+1}(x).} \tag{5}$$

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12.15.4. Use equations (15.1) to (15.5) to do Problems 12.2 to 12.6.

In the textbook, equations (15.1) to (15.5) are

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^p J_p(x)] = x^p J_{p-1}(x), \tag{6}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{-p} J_p(x)\right] = -x^{-p} J_{p+1}(x), \tag{7}$$

$$J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x} J_p(x), \tag{8}$$

$$J_{p-1}(x) - J_{p+1}(x) = 2J_p'(x), \quad \text{and} \tag{9}$$

$$\begin{aligned} J_p'(x) &= -\frac{p}{x} J_p(x) + J_{p-1}(x) \\ &= \frac{p}{x} J_p(x) - J_{p+1}(x). \end{aligned} \tag{10}$$

Problems 12.2 to 12.6 follow.

12.2.2. Show that $J_2(x) = (2/x)J_1(x) - J_0(x)$.

Let $p = 1$ in equation (8). Then we have

$$J_0(x) + J_2(x) = \frac{2}{x} J_1(x), \tag{11}$$

so that

1

$$\boxed{J_2(x) = \frac{2}{x} J_1(x) - J_0(x).} \tag{12}$$

12.2.3. Show that $J_1(x) + J_3(x) = (4/x)J_2(x)$.

Let $p = 2$ in equation (8). Then we have

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$$\boxed{J_1(x) + J_3(x) = \frac{4}{x} J_2(x).} \tag{13}$$

12.2.4. Show that $(\mathrm{d}/\mathrm{d}x)J_0(x) = -J_1(x)$.

Let $p = 0$ in equation (7), so that we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^0 J_0(x)\right] = -x^0 J_1(x). \tag{14}$$

Thus,

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$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x} J_0(x) = -J_1(x).} \tag{15}$$

12.2.5. Show that $(\mathrm{d}/\mathrm{d}x)[xJ_1(x)] = xJ_0(x)$.

Let $p = 1$ in equation (6). Then, we have

(16) $$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^1 J_1(x)\right] = x^1 J_0(x).$$

Thus,

/

(17) $$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}[xJ_1(x)] = xJ_0(x).}$$

12.2.6. Show that $J_0(x) - J_2(x) = 2(\mathrm{d}/\mathrm{d}x)J_1(x)$.

Let $p = 1$ in equation (9). Then, we have

1

(18) $$\boxed{J_0(x) - J_2(x) = 2\frac{\mathrm{d}}{\mathrm{d}x}J_1(x).}$$

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Find the solutions to the following differential equations in terms of Bessel functions by using equations (16.1) and (16.2)

Equations (16.1) and (16.2) in the textbook state that the differential equation

(19) $$f'' + \frac{1-2a}{x}y' + \left[\left(bcx^{c-1}\right)^2 + \frac{a^2 - p^2c^2}{x^2}\right]y = 0$$

has the solution

(20) $$y = x^a Z_p(bx^c).$$

12.16.2. $y'' + 4x^2 y = 0$

We want this differential equation to be in the form of (19) above. Since the differential equation has no y' term, it follows that

(21) $$\frac{1-2a}{x} = 0 \implies a = \frac{1}{2}.$$

Thus,

(22) $$4x^2 = \left(bcx^{c-1}\right)^2 + \frac{\frac{1}{4} - p^2c^2}{x^2}$$

(23) $$\implies 4x^4 = b^2c^2x^{2(c-1)+2} + \frac{1}{4} - p^2c^2$$
$$= b^2c^2x^{2c} + \frac{1}{4} - p^2c^2.$$

Therefore, we have three equations from (23):

(24) $$2c = 4, \quad b^2c^2 = 4, \quad \text{and} \quad \frac{1}{4} - p^2c^2 = 0.$$

So, we get

$$2c = 4 \implies c = 2, \tag{25}$$

$$4b^2 = 4 \implies b^2 = 1 \implies b = \pm 1, \tag{26}$$

and

$$\frac{1}{4} - 4p^2 = 0 \implies 4p^2 = \frac{1}{4} \implies p^2 = \frac{1}{16} \implies p = \pm\frac{1}{4}. \tag{27}$$

Choose positive values for b and p. Then we have the solution of the differential equation in terms of Bessel functions as in (20):

$$\boxed{y = x^{1/2} Z_{1/4}\left(x^2\right).} \tag{28}$$

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12.16.6. $4xy'' + y = 0$

Dividing our differential equation on both sides by $4x$, we get

$$y'' + \frac{y}{4x} = 0 \tag{29}$$

Following the procedure in the last problem, we see that (29) has no y' term, so we again have

$$\frac{1-2a}{x} = 0 \implies a = \frac{1}{2}. \tag{30}$$

Moreover, we equate the y coefficients of (19) and (29):

$$\frac{1}{4x} = \left(bcx^{c-1}\right)^2 + \frac{\frac{1}{4} - p^2c^2}{x^2} \tag{31}$$

$$\implies \frac{1}{4}x = b^2c^2x^{2c} + \frac{1}{4} - p^2c^2. \tag{32}$$

Equation (32) gives us three equations:

$$2c = 1, \quad b^2c^2 = \frac{1}{4}, \quad \text{and} \quad \frac{1}{4} - p^2c^2 = 0. \tag{33}$$

Thus, we get

$$2c = 1 \implies c = \frac{1}{2}, \tag{34}$$

$$\frac{1}{4}b^2 = \frac{1}{4} \implies b^2 = 1 \implies b = \pm 1, \tag{35}$$

and

$$\frac{1}{4} - \frac{1}{4}p^2 = 0 \implies p^2 = 1 \implies p = \pm 1. \tag{36}$$

Choosing positive b and p, we get the solution to (29) in terms of Bessel functions like in (20):

$$\boxed{y = x^{1/2} Z_1\left(x^{1/2}\right).} \tag{37}$$

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12.18.10. The differential equation for transverse vibrations of a string whose density increases linearly from one end to the other is $y'' + (Ax + B)y = 0$, where A and B are constants. Find the general solution of this equation in terms of Bessel functions. *Hint:* Make the change of variable $Ax + B = Au$.

Let $Ax + B = Au$, so that $\frac{\mathrm{d}}{\mathrm{d}u} = \frac{\mathrm{d}}{\mathrm{d}x}$. Then our differential equation becomes

$$y'' + Auy = 0. \tag{54}$$

To solve (54) in terms of Bessel functions, we have to make it look like (19). Then, since there is no y' term, it follows that

$$\frac{1-2a}{u} = 0 \implies a = \frac{1}{2}. \tag{55}$$

Also, we get

$$Au = \left(bcu^{c-1}\right)^2 + \frac{\frac{1}{4} - p^2c^2}{u^2} \implies Au^3 = b^2c^2u^{2c} + \frac{1}{4} - p^2c^2. \tag{56}$$

Equation (56) gives us three equations:

$$2c = 3, \quad b^2c^2 = A, \quad \text{and} \quad \frac{1}{4} - p^2c^2 = 0. \tag{57}$$

So, we have

$$2c = 3 \implies c = \frac{3}{2}, \tag{58}$$

$$\frac{9}{4}b^2 = A \implies b = \pm\frac{2}{3}\sqrt{A}, \tag{59}$$

and

$$\frac{1}{4} - \frac{9}{4}p^2 = 0 \implies p = \pm\frac{1}{3}. \tag{60}$$

Choosing positive b and p, we get a solution in the form of (20):

$$y = u^{1/2} Z_{1/3}\left(\frac{2}{3}\sqrt{A}u^{3/2}\right). \tag{61}$$

In terms of x, the solution to the differential equation is

$$\boxed{y = \left(x + \frac{B}{A}\right)^{1/2} Z_{1/3}\left(\frac{2}{3}\sqrt{A}\left(x + \frac{B}{A}\right)^{3/2}\right).} \tag{62}$$

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