Lecture 5.1 Properties of Determinant

Property 1: If one row of a matrix consists entirely of zeros, then the determinant is zero.

Example1: $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 2 & 4 & 8 \end{bmatrix}, \quad \det A = 0,$

Property 2: If two rows of a matrix are identical, the determinant is zero.

Example2: $A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 2 & 4 & 8 \end{bmatrix}, \quad \det A = 0$, Row 1 and Row 2 are identical

Property 3: If in a square matrix A two rows proportional, then det A = 0**.**

Example3: $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 4 & 6 \end{bmatrix}, \quad \det(A) = 0.$

Row 1 and Row 3 are proportional, as $R_3 = 2\,R_1$

Property 3: $\det(\mathbf{A}) = \det(\mathbf{A}^T)$.

Example4:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 4 & 8 \end{bmatrix}, \quad \det A = 2, \qquad A^t = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 4 \\ 3 & 2 & 8 \end{bmatrix}, \quad \det(A^t) = 2$$

Property 4: For an $n \times n$ matrix **A** and any scalar λ, $\det(\lambda\mathbf{A}) = \lambda^n \det(\mathbf{A})$.

Note:

When we multiply a matrix with a number, each entry of matrix is multiplied with the same number.

When we take common from determinant, it is taken from each row or each column.

Example5:

$$A=\begin{bmatrix}1&2&3\\0&1&2\\2&4&8\end{bmatrix},\quad 4A=\begin{bmatrix}4&8&9\\0&4&8\\8&16&32\end{bmatrix},$$

$$\det(4A)=\begin{vmatrix}4&8&9\\0&4&8\\8&16&32\end{vmatrix}=(4)(4)(4)\begin{vmatrix}1&2&3\\0&1&2\\2&4&8\end{vmatrix}=4^3\begin{vmatrix}1&2&3\\0&1&2\\2&4&8\end{vmatrix}=64x2=128$$

Property 5: If A and B are of the same order, then det(A+B) $\neq$ det(A) + det(B).

Property 6: If A and B are of the same order, then det(AB)=det(A) det(B).

Property 7: det($\mathbf{A}^{-1}$) = $\frac{1}{\det(A)}$.

Property 8: If det(**A**) = 0, then matrix A is singular matrix

Property 9: Homogeneous system of linear equations AX = 0, will have non- trivial solution if and only if detA = 0 .

Example6. Given that

$$\textbf{Let } A=\begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix}\ \textbf{and}\ \det(A)=-7,\text{find}$$

$$\text{(a)}\det(3A),\quad \text{(b)}\det(2A)^{-1},\quad \text{(c)}\det(2A^{-1}),\ \text{(d) det } A^T=\det A,\quad \text{(e)}\begin{vmatrix}a&g&d\\b&h&e\\c&i&f\end{vmatrix}$$

Solution:

(a) det(3A) = 3^3 det A= (27)(-7) = -189

(b) $\det(2A)^{-1}=\frac{1}{\det(2A)}=\frac{1}{2^3 detA}=\frac{1}{(8)(-7)}=-\frac{1}{56}$

(c) $\det(2A^{-1})=2^3\det(A^{-1})=\frac{8}{\det A}=\frac{8}{-7}=-\frac{8}{7}$

(d) $\det(A^T)$=$\det(A)=-7$

(e) $\begin{vmatrix} a & g & d \\ b & h & e \\ c & i & f \end{vmatrix} = \begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix} = -\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -(-7) = 7$

Taking Transpose — Interchanging R_2 and R_3

Example7. Use row reduction to show that

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b-c)(c-a)(c-b)$$

Solution:

Using property that det A^t = detA

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} = (b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix}$$

$R_2 - R_1, R_3 - R_1$ — **taking common from R_2 and R_3**

$$=(b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & c-b \end{vmatrix}$$ **It is triangular matrix**

$R_3 - R_2$

$$=(b-a)(c-a)(c-b)$$

Example 8. Using properties of determinants show that

$$\begin{vmatrix} b+c & c+a & a+b \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = 0$$

Solution:

$$\begin{vmatrix} b+c & c+a & a+b \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} \quad \boxed{R_1+R_2}$$

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} \quad \text{Taking common from } R_1$$

$$= 0. \quad R_1 \text{ and } R_3 \text{ are equal}$$

Lecture 5.2 Elementary Row operations and Determinant

Let A and B be square matrices

1. *If B is obtained by interchanging two rows of A,*
   then det B = - det A
2. *If B is obtained by multiplying row of A by a nonzero constant k,*
   then det B = k det A
3. *If B is obtained from A by adding a multiple of a row A to another row of A, then det B = det A*

Example 9.

Find determinant of Matrix by using elementary row operations

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 4 & 8 \end{bmatrix}$$

Solution:

Reducing to triangular matrix, multiply row 1 by (-2) and add to row 3

$$\det A = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{vmatrix} = (1)(1)(2) = 2$$

Example10.

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 4 & 8 \end{bmatrix}$ and det A = 2. Find determinant of matrix

$$\text{(i)}\ A_1 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 8 \\ 0 & 1 & 2 \end{bmatrix},\ \text{(ii)}\ A_2 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix},\ \text{(iii)}\ A_3 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

Solution:

(i) The matrix A_1 can be obtained by interchanging Row 2 and Row 3 of matrix A

det A_1 = - det A = -2

(ii) The matrix A_2 can be obtained by multiplying Row 3 of matrix A by 1/2

det A_2 = ½ det A = ½ (2) = 1

(iii) The matrix A_3 can be obtained by Row operation on matrix A (-2 Row 1 to Row 3)

det A_3 = det A = 2

Example11. Given that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 6,\ \text{find(a)} \begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix}, \text{(b)} \begin{vmatrix} 3a & 3b & 3c \\ -d & -e & -f \\ 4g & 4h & 4i \end{vmatrix}, \text{(c)} \begin{vmatrix} a+g & b+h & c+i \\ d & e & f \\ g & h & i \end{vmatrix}$$

Solution:

$$\text{(a)} \begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix} = -\begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix} = (-1)(-1) \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = (-1)(-1)(6) = 6$$

$R_1 \leftrightarrow R_3$ $\qquad$ $R_2 \leftrightarrow R_3$

$$\text{(b)} \begin{vmatrix} 3a & 3b & 3c \\ -d & -e & -f \\ 4g & 4h & 4i \end{vmatrix} = (3)(-1)(4) \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = (-12)(6) = -72$$

Taking common from each row

R_1-R_3

$$\text{(c)} \begin{vmatrix} a+g & b+h & c+i \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 6$$

Lecture 5.3: Evaluation of Determinant

Finding determinant by using Properties of determinant

Example 12. Given that

$$\textbf{Let } A=\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \textbf{ and } \det(A)=-7, \text{find}$$

$$\text{(a)} \det(3A), \quad \text{(b)} \det(2A)^{-1}, \quad \text{(c)} \det(2A^{-1}), \quad \text{(d)} \det A^T=\det A, \quad \text{(e)} \begin{vmatrix} a & g & d \\ b & h & e \\ c & i & f \end{vmatrix}$$

Solution:

(a) $\det(3A) = 3^3 \det A = (27)(-7) = -189$

(b) $\det(2A)^{-1} = \frac{1}{\det(2A)} = \frac{1}{2^3 detA} = \frac{1}{(8)(-7)} = -\frac{1}{56}$

(c) $\det(2A^{-1}) = 2^3 \det(A^{-1}) = \frac{8}{\det A} = \frac{8}{-7} = -\frac{8}{7}$

(d) $\det(A^T)=\det(A) = -7$

(e) $$\begin{vmatrix} a & g & d \\ b & h & e \\ c & i & f \end{vmatrix} = \begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix} = -\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -(-7)=7$$

Taking Transpose

Interchanging R_2 and R_3

Finding determinant by using Elementary row operations

Example12.

Use row reduction to show that

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b-c)(c-a)(c-b)$$

Solution:

Using property that det A^t = detA

$$\begin{vmatrix}1 & a & a^2\\1 & b & b^2\\1 & c & c^2\end{vmatrix} = \begin{vmatrix}1 & a & a^2\\0 & b-a & b^2-a^2\\0 & c-a & c^2-a^2\end{vmatrix} = (b-a)(c-a)\begin{vmatrix}1 & a & a^2\\0 & 1 & b+a\\0 & 1 & c+a\end{vmatrix}$$

$\boxed{R_2 - R_1, R_3 - R_1}$ **taking common from R_2 and R_3**

$$=(b-a)(c-a)\begin{vmatrix}1 & a & a^2\\0 & 1 & b+a\\0 & 0 & c-b\end{vmatrix}$$ **It is triangular matrix**

$\boxed{R_3 - R_2}$

$$=(b-a)(c-a)(c-b)$$

Finding determinant by using Properties of determinant

Example13.

Using properties of determinants show that

$$\begin{vmatrix}b+c & c+a & a+b\\a & b & c\\1 & 1 & 1\end{vmatrix} = 0$$

Solution:

$$\begin{vmatrix}b+c & c+a & a+b\\a & b & c\\1 & 1 & 1\end{vmatrix} = \begin{vmatrix}a+b+c & a+b+c & a+b+c\\a & b & c\\1 & 1 & 1\end{vmatrix}$$ $\boxed{R_1+R_2}$

$$= (a+b+c)\begin{vmatrix}1 & 1 & 1\\a & b & c\\1 & 1 & 1\end{vmatrix}$$ Taking common from R_1

$$= 0.$$ R_1 and R_3 are equal

Example14.

Find the values of x for which the matrix does not have inverse

$$A = \begin{bmatrix}x+2 & 2x+3 & 3x+4\\2x+3 & 3x+4 & 4x+5\\3x+5 & 5x+8 & 10x+17\end{bmatrix}$$

Solution:

$$\det A=\begin{vmatrix} x+2 & 2x+3 & 3x+4 \\ 2x+3 & 3x+4 & 4x+5 \\ 3x+5 & 5X+8 & 10x+17 \end{vmatrix}$$

By the row operations R_3-R_2**, and** R_3-R_1

$$=\begin{vmatrix} x+2 & 2x+3 & 3x+4 \\ x+1 & x+1 & x+1 \\ x+2 & 2x+4 & 6x+12 \end{vmatrix}$$

Taking common $x+1$ **from Row 1 and** $x+2$ **from Row 2**

$$=(x+1)(x+2)\begin{vmatrix} x+2 & 2x+3 & 3x+4 \\ 1 & 1 & 1 \\ 1 & 2 & 6 \end{vmatrix}$$

By subtracting column 1 from column 2 and column 3

$$=(x+1)(x+2)\begin{vmatrix} x+2 & x+1 & 2x+2 \\ 1 & 0 & 0 \\ 1 & 1 & 5 \end{vmatrix}$$

Opening from Row2

$=(x+1)(x+2)(-3(x+1))$

$\det A=0 \Rightarrow -3(x+1)(x+2)(x+1)=0$

is zero $\Rightarrow x=-1 \text{ or } x=-2$

Note: We can apply the operation in columns we perform operations on rows.

<u>Example 15</u>.

Use determinants to find which real value(s) of *c* make this matrix invertible:

$$A=\begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & c \\ 2 & c & 1 \end{bmatrix}$$

Solution:

$$A=\begin{vmatrix} 1 & 2 & -1 \\ 0 & -1 & c \\ 2 & c & 1 \end{vmatrix} = 0+(-1)\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} - c\begin{vmatrix} 1 & 2 \\ 2 & c \end{vmatrix}$$

$= -(1 + 2) - c(c - 4) = -(c^2 - 4c + 3) = -(c - 1)(c - 3)$

$\det A = 0 \Rightarrow c = 1$ or $c = 3$

Therefore the matrix is invertible for all real values of c except $c = 1$ or $c = 3$.

Finding determinant by using Elementary row operations, reducing it to upper triangular matrix form

Example 16. Evaluate

$$\det A = \begin{vmatrix} 1 & -1 & 5 & 5 \\ 3 & 1 & 2 & 4 \\ -1 & -3 & 8 & 0 \\ 1 & 1 & 2 & -1 \end{vmatrix}.$$

Solution: Use elementary row operations to carry the matrix to upper triangular form:

$$\begin{vmatrix} 1 & -1 & 5 & 5 \\ 3 & 1 & 2 & 4 \\ -1 & -3 & 8 & 0 \\ 1 & 1 & 2 & -1 \end{vmatrix} \xrightarrow[\substack{R_3 + R_1 \\ R_4 - R_1}]{R_2 - 3R_1} \begin{vmatrix} 1 & -1 & 5 & 5 \\ 0 & 4 & -13 & -11 \\ 0 & -4 & 13 & 5 \\ 0 & 2 & -3 & -6 \end{vmatrix}$$

$$\xrightarrow[\substack{R_3 + R_2 \\ R_4 - \frac{1}{2}R_2}]{} \begin{vmatrix} 1 & -1 & 5 & 5 \\ 0 & 4 & -13 & -11 \\ 0 & 0 & 0 & -6 \\ 0 & 0 & \frac{7}{2} & -\frac{1}{2} \end{vmatrix} \xrightarrow[R_3 \leftrightarrow R_4]{} -\begin{vmatrix} 1 & -1 & 5 & 5 \\ 0 & 4 & -13 & -11 \\ 0 & 0 & \frac{7}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & -6 \end{vmatrix}$$

$$\Rightarrow \det A = -1 \times 4 \times \tfrac{7}{2} \times (-6) = +84.$$