Lecture 4.1: Solving Linear system by Inverse Matrix

Let a given linear system of equations is

$$AX = B$$

Find A^{-1}

Multiply with A^{-1} from left

$$A^{-1}AX = A^{-1}B$$

$$I\,X = A^{-1}B$$

$$X = A^{-1}B \text{ is a solution.}$$

Note: To find A^{-1} we use *Elementary Matrix method*.

Example1.

Write the system of equations in a matrix form, find A^{-1}, use A^{-1} to solve the system

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 4 \\ 2x_1 + 2x_2 + x_3 &= -1 \\ 2x_1 + 3x_2 + x_3 &= 3 \end{aligned}$$

Solution: 1. Matrix Form is:

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & 2 & 1 \\ 2 & 3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} \quad \text{is in form of } AX = B$$

2. Find A^{-1} by using Elementary Matrix method

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 & 1 & 0 \\ 2 & 3 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 & 1 & 0 \\ 2 & 3 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & -4 & -1 & -2 & 1 & 0 \\ 0 & -3 & -1 & -2 & 0 & 1 \end{array}\right] \quad \mathrm{R}_2 - 2R_1, R_3 - 2R_1$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & -4 & -1 & -2 & 1 & 0 \\ 0 & 0 & -1 & -2 & -3 & 4 \end{array}\right] \quad 4R_3 - 3R_2$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & -1 & -3 & 4 \\ 0 & -4 & 0 & 0 & 4 & -4 \\ 0 & 0 & -1 & -2 & -3 & 4 \end{array}\right] \quad R_1 + R_2, R_2 - R_3$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & -1 & -3 & 4 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & -2 & -3 & 4 \end{array}\right] \quad -\tfrac{1}{4} R_2, -R_3$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & -1 & -3 & 4 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & -2 & -3 & 4 \end{array}\right] \quad -\tfrac{1}{4} R_2, -R_3$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & -2 & -3 & 4 \end{array}\right] \quad -3R_2 + R_3$$

$$\equiv \left[\mathrm{I} \middle| \mathrm{A}^{-1}\right]$$

$$\mathrm{A}^{-1} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 2 & 3 & -4 \end{bmatrix}$$

$$\mathrm{X} = \mathrm{A}^{-1}\mathrm{B} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 2 & 3 & -4 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ -7 \end{bmatrix}$$

Solution set is $x_1 = -1, x_2 = 4, x_3 = -7.$

Lecture 4.2 Determinant

4.1 Determinant of a matrix

The determinant is a useful value that can be computed from the elements of a square matrix. The **determinant** of a matrix A is denoted det(A), det A, or | A|.

4.2 Evaluation of determinant of Matrix

1.The determinant of a (1×1) matrix $A = [\ a\ ]$ is just $\det A = a$.

2. The determinant of 2x2 matrix is defined as

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$|A| = \det A = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - cb$$

Example:1. Find determinant of matrix

$$A = \begin{bmatrix} 4 & 5 \\ 3 & 6 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 4 & 5 \\ 3 & 6 \end{bmatrix}$$

$$\begin{aligned} \det A &= 4x6 - 3x5 \\ &= 24 - 15 \\ &= 9 \end{aligned}$$

4.3 The determinant of 3x3 matrix is defined as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\det A = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Example:2. Find determinant of matrix

$$A = \begin{bmatrix} 2 & 4 & 5 \\ 3 & 6 & 8 \\ 4 & 5 & 9 \end{bmatrix}$$

Solution:

Expanding along the top row and noting alternating signs $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$

$$\det A = +2x\begin{vmatrix} 6 & 8 \\ 5 & 9 \end{vmatrix} - 4x\begin{vmatrix} 3 & 8 \\ 4 & 9 \end{vmatrix} + 5x\begin{vmatrix} 3 & 6 \\ 4 & 5 \end{vmatrix}$$

$$= 2x(54-40) - 4x(27-32) + 5x(15-24)$$

$$= 2x(14) - 4x(-5) + 5x(-9)$$

$$= 28 + 20 - 45 = 48 - 45 = 3$$

Note: we can write determinant of a matrix as

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ or } \begin{vmatrix} a & b \\ c & d \end{vmatrix} \text{ or } \det A \text{ or } |A|$$

<u>**Example:3.**</u>

Find the determinant of $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

Solution:

$$\det A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$\det A = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = +1\times\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\times\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\times\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= 1(45-48) - 2(36-42) + 3(32-35) = -3+12-9 = 0$$

Example4. Find determinant of matrix of order 4x4

$$A = \begin{bmatrix} 0 & 1 & 2 & 5 \\ 2 & -1 & 2 & 3 \\ 3 & 2 & 1 & 5 \\ 1 & 0 & 4 & 0 \end{bmatrix}$$

Solution:

Two entries in 4th row are zero, so determinant is calculated by opening from 4th row.

$$\begin{aligned} \det A &= a_{41}c_{41} + a_{42}c_{42} + a_{43}c_{43} + a_{44}c_{44} \\ &= (1)c_{41} + (0)c_{42} + (4)c_{43} + (0)c_{44} \\ &= c_{41} + (4)c_{43} \end{aligned}$$

$$\det A = c_{41} + (4)c_{43} = -\begin{vmatrix} 1 & 2 & 5 \\ -1 & 2 & 3 \\ 2 & 1 & 5 \end{vmatrix} - 4\begin{vmatrix} 0 & 1 & 5 \\ 2 & -1 & 2 \\ 3 & 2 & 1 \end{vmatrix}$$

Finding values of cofactors c_{41} and c_{43}

$$\begin{aligned} \det A &= -(4) - 4(34) \\ &= -4 - 136 \\ &= -140 \end{aligned}$$

Example:5.

Solving matrix equation

Find all values of λ for which det (A) = 0 for matrix

$$A = \begin{bmatrix} \lambda - 4 & 0 & 0 \\ 0 & \lambda & 2 \\ 0 & 3 & \lambda - 1 \end{bmatrix}$$

Solution: Two entries of 1st row are zero, we open it from first row

$$\det\ A = (\lambda-4)\begin{vmatrix} \lambda & 2 \\ 3 & \lambda-1 \end{vmatrix}$$

$$=(\lambda-4)\left[\lambda(\lambda-1)-6\right]$$

$$=(\lambda-4)\left[\lambda^2-\lambda-6\right]$$

$$=(\lambda-4)(\lambda-3)(\lambda+2)$$

We need to find the value of λ, when det A = 0

$$\Rightarrow\ (\lambda-4)(\lambda-3)(\lambda+2) = 0$$

$\Rightarrow$ λ=4, λ=3 and λ= -2.

Lecture 4.3: Determinant of triangular matrices

Upper triangular matrix

In upper triangular matrix all the entries below the diagonal are zero.

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix}$$

Lower triangular matrix

In lower triangular matrix all the entries above the diagonal are zero.

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 8 & 4 & 0 \\ 4 & 7 & 3 \end{bmatrix}$$

Note: Determinant of triangular matrix is product of diagonal elements.

Det A = (1)(4)(5) = 20

Det B = (1)(4)(3) = 12

Example:6. The determinant of Triangular matrix

$$A = \begin{bmatrix} 2 & 4 & 5 & 3 \\ 0 & 5 & 3 & -1 \\ 0 & 0 & 3 & 9 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

$$\det A = (2)(4)(5)(3) = 120$$

Diagonal Matrices

Diagonal matrix is matrix whose off diagonal elements are zero.

Example:7. Determinant of Diagonal matrix

Find determinant of matrix $B = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Solution:

$$\det B = (5)(4)(3) = 60$$

Example:8.

Evaluate $\det C = \begin{vmatrix} 3 & 0 & 0 & 0 & 0 \\ -4 & 2 & 0 & 0 & 0 \\ 67 & e & 4 & 0 & 0 \\ 0 & 1 & -47 & 2 & 0 \\ \pi & -3 & 6 & -\sqrt{2} & -1 \end{vmatrix}$

Matrix C is lower triangular $\Rightarrow$ $\det C = 3\times2\times4\times2\times(-1) = -48$

Example:9.

Evaluate $\det D = \begin{vmatrix} 2 & -1 & 1 & 1 \\ -3 & 2 & -4 & -3 \\ 4 & 2 & 7 & 4 \\ 2 & 3 & 11 & 2 \end{vmatrix}$

Columns 1 and 4 of matrix D are identical $\Rightarrow$ $\det D = 0$.