Lecture 2.1: Elementary Row operations

2.1.1 Elementary Row operations:

Elementary row operations are steps for solving the linear system of equations:

I. Interchange two rows.
II. Multiply a row with non zero real number.
III. Add a multiple of one row to another row.

Note: *Elementary row operations produce same results when operated either on a system or on its augmented matrix form.*

2.1.2 Methods for solving System of Linear equations

1. **Gaussian Elimination Method**

2. **Gauss – Jordan Elimination Method**

2.1.3 Gaussian Elimination Method

STEP 1. by using elementary row operations

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & A_{12} & A_{13} & B_1 \\ 0 & 1 & A_{23} & B_2 \\ 0 & 0 & 1 & B_3 \end{bmatrix}$$

STEP 2. Find solution by back – substitutions.

Example:3. Solve the system of linear equations by Gaussian- elimination method

$$x_1 + x_2 + 2x_3 = 8$$
$$-x_1 - 2x_2 + 3x_3 = 1$$
$$3x_1 - 7x_2 + 4x_3 = 10$$

Solution: **Augmented matrix is**

$$[A \vdots b] = \begin{bmatrix} 1 & 1 & 2 & \mathbf{8} \\ -1 & -2 & 3 & \mathbf{1} \\ 3 & -7 & 4 & \mathbf{10} \end{bmatrix}$$

STEP 1.

$a_{11=1}$ is the leading entry, we want to reduce a_{21} and a_{31} to zero.
Add R_1 and R_2 and add-3 R_1 to R_3

$$\approx \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{bmatrix} \quad R_1+R_2, \quad -3R_1+R_3$$

To create a leading entry in the row R_2 multiply a_{22} by (-1),
and to reduce a_{32} and a_{22} multiple row R_2 by -10 and add to row R_3.

$$\approx \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{bmatrix} -R_2, \quad 10R_2+R_3$$

To create a leading entry in row R_3 divide a_{33} by (-52),

$$\approx \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{bmatrix} -R_3/52$$

Equivalent system of equations form is:

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 8 \\ x_2 - 5x_3 &= -9 \\ x_3 &= 2 \end{aligned}$$

STEP 2. **Back Substitution**

$$\begin{aligned} x_3 &= 2 \\ x_2 &= 5x_3 - 9 = 10 - 9 = 1 \\ x_1 &= -x_2 - 2x_3 + 8 = -1 - 4 + 8 = 3 \end{aligned}$$

Solution is $x_1 = 3$, $x_2 = 1$, $x_3 = 2$.

Lecture 2.2: Gauss Jordan method and Row Echelon Form

2.2.1 Gauss – Jordan Elimination Method

Matrix equation form

AX=b

$$[A:b] \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & B_1 \\ 0 & 1 & 0 & B_2 \\ 0 & 0 & 1 & B_3 \end{bmatrix}$$

By using elementary row operations we reduced the given system of equation with "1" as diagonal entries and all other entries of Matrix A are " 0 "

Example.1. Solve the system of linear equations by Gauss - Jordan elimination method

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 8 \\ -x_1 - 2x_2 + 3x_3 &= 1 \\ 3x_1 - 7x_2 + 4x_3 &= 10 \end{aligned}$$

Solution: Augmented matrix is

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{bmatrix}$$

$$\approx \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{bmatrix} \quad \mathbf{R_1+R_2, \ -3R_1+R_3}$$

$$\approx \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{bmatrix} \quad \mathbf{-R_2, \ 10R_2+R_3}$$

$$\approx \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad -R_3/52$$

$$\approx \begin{bmatrix} 1 & 1 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad -2R_3+R_1, \ 5R_3+R_2$$

$$\approx \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad -R_2+R_1$$

Equivalent system of equations form is:

$x_1 = 3$
$x_2 = 1$
$x_3 = 2$ is the solution of the system.

2.2.2 Row Echelon Form

A form of a matrix, which satisfies following conditions, is row echelon form

i. '1' (leading entry) must be in the beginning of each row,
ii. '1' must be on the right of the above leading entry,
iii. Below the leading entry all values must be zero,
iv. A row containing all zero values must be in the bottom.

Examples:

(i) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (iii) $\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

2.2.3 Reduced Row Echelon Form

A form of a matrix, which satisfies following conditions, is row echelon form

i. '1' (leading entry) must be in the beginning of each row,
ii. '1' must be on the right of the above leading entry,
iii. All entries in the column containing leading entry must be zero,
iv. A row containing all zero values must be in the bottom.

Examples

(i) $\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$, (ii) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, (iii) $\begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Note: 1. Gaussian Elimination method is reducing the given Augmented matrix to Row echelon form and backward substitution.
Note: 2. Gauss- Jordan Elimination method is reducing the given Augmented matrix to Reduced Row echelon form.

Example:2. Use Gauss – Jordan method to solve the system of linear system

$$\begin{aligned} x - y + 2z - w &= -1 \\ 2x + y - 2z - 2w &= -2 \\ -x + 2y - 4z + w &= 1 \\ 3x \qquad\qquad - 3w &= -3 \end{aligned}$$

Solution: Gauss-Jordan method is same as to reduce the augmented matrix to reduced row echelon from.
Augmented matrix is

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 2 & 1 & -2 & -2 & -2 \\ -1 & 2 & -4 & 1 & 1 \\ 3 & 0 & 0 & -3 & -3 \end{bmatrix}$$

There is a leading entry '1' in the first row, making all other entries in the first column zero

$$\approx \begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{bmatrix} \quad (-2R_1+R_2)/3\ ,\ R_1+R_3\ ,\ -3R_1+R_4$$

Lecture 2.3 Conditions on Solutions and Homogenous System

2.3.1 No solution

Example: 1. Solve the system of linear equations

$$\begin{aligned} x - 2y + z - 4u &= 1 \\ x + 3y + 7z + 2u &= 2 \\ x - 12y - 11z - 16u &= 5 \end{aligned}$$

Solution:

Augmented matrix is:

$$\begin{bmatrix} 1 & -2 & 1 & -4 & 1 \\ 1 & 3 & 7 & 2 & 2 \\ 1 & -12 & -11 & -16 & 5 \end{bmatrix}$$

Reducing it to row echelon form (using Gaussian - elimination method)

$$\approx \begin{bmatrix} 1 & -2 & 1 & -4 & 1 \\ 0 & 5 & 6 & 6 & 1 \\ 0 & -10 & -12 & -12 & 4 \end{bmatrix} \qquad R_2 - R_1,\ \ R_3 - R_1$$

$$\approx\begin{bmatrix}1 & -2 & 1 & -4 & 1\\ 0 & 5 & 6 & 6 & 1\\ 0 & 0 & 0 & 0 & 6\end{bmatrix} \qquad -R_3+2R_2$$

Last equation is

$$0x+0y+0z+0u=6, \text{ which can never be satisfied, but}$$
$$0\neq 6$$

hence there is no solution for the given system of linear equations.

2.3.2 Conditions on Solutions

Example:2. For which values of 'a' will be following system

$$\begin{aligned} x+2y-3z &= 4\\ 3x-y+5z &= 2\\ 4x+y+(a^2-14)z &= a+2\end{aligned}$$

(i) infinitely many solutions?
(ii) No solution?
(iii) Exactly one solution?

Hence find the solution(s) if exists.

Solution:

Augmented matrix is

$$\begin{bmatrix}1 & 2 & -3 & 4\\ 3 & -1 & 5 & 2\\ 4 & 1 & a^2-14 & a+2\end{bmatrix}$$

Reducing it to row echelon form

$$\approx\begin{bmatrix}1 & 2 & -3 & 4\\ 0 & -7 & -14 & -10\\ 0 & -7 & a^2-2 & a-14\end{bmatrix} \qquad R_2-3R_1,\ R_3-4R_1$$

$$\approx\begin{bmatrix}1 & 2 & -3 & 4\\ 0 & 1 & -2 & \frac{10}{7}\\ 0 & 0 & a^2-16 & a-4\end{bmatrix} \qquad -\tfrac{1}{7}R_2,\ R_3-R_2$$

writing in the equation form,

$$\begin{aligned} x+2y-3z &= 4 \quad \to 1 \\ y-2z &= \tfrac{10}{7} \quad \to 2 \\ (a^2-16)z &= a-4 \quad \to 3 \end{aligned}$$

or equation 3 can be written as

$$(a+4)\,(a-4)z = a-4$$

CASE I .

$$a=4 \quad \Rightarrow \; 0z=0$$

$$\begin{aligned} x+2y-3z &= 4 \\ y-2z &= \tfrac{10}{7} \end{aligned}$$

as number of equations are less than number of unknowns, hence the system has infinite many solutions.

CASE II

$a=-4 \quad \Rightarrow \quad 0\text{z} = \text{-8}$, but $0 \neq \text{-8}$, hence, there is no solution.

CASE III

$$\text{a} \neq 4, \text{a} \neq -4,$$

the system will have unique solution when $a \neq 4$ and $a \neq \text{-}4$

Example:3. What conditions must a, b, and c satisfy in order for the system of equations

$$\begin{aligned} x+y+2z &= a \\ x \qquad + z &= b \\ 2x+y+3z &= c \end{aligned}$$

to be consistent.

Solution: The augmented matrix is

$$\begin{bmatrix} 1 & 1 & 2 & a \\ 1 & 0 & 1 & b \\ 2 & 1 & 3 & c \end{bmatrix}$$ reducing it to reduced row echelon form

$$\approx \begin{bmatrix} 1 & 1 & 2 & a \\ 0 & -1 & -1 & b-a \\ 0 & -1 & -1 & c-2a \end{bmatrix} \quad R_2\text{-}R_1,\ R_3\text{-}2R_1$$

$$\approx \begin{bmatrix} 1 & 1 & 2 & a \\ 0 & -1 & -1 & b-a \\ 0 & 0 & 0 & c-a-b \end{bmatrix} \quad R_3\text{-}R_1$$

The system will be consistent if only if $c - a - b = 0$

Or $c = a + b$

Thus the required condition for system to be consistent is

$c = a + b.$

2.3.3 Homogeneous system

A system of equations of the form

AX = 0.,

That is with all constants b's taken as zero, is called homogeneous system.

Note:

1. The homogeneous system has solutions, $x_1 = x_2 = x_3 = ... = x_n = 0$ called **trivial solution.**

2. The homogeneous system has infinitely many non- trivial solutions in addition to the trivial solutions.

3. The homogeneous system will have a non- trivial solution if and only if A is a singular matrix $\Rightarrow$ $|A| = 0.$

Example:4. Solve the homogeneous system of linear equations

$$\begin{aligned} 2x+2y+4z&=0 \\ w \qquad - y-3z&=0 \\ 2w+3x+ \quad y+ z&=0 \\ -2w+ x+3y-2z&=0 \end{aligned}$$

Solution: The augmented matrix is

$$\begin{bmatrix} 0 & 2 & 2 & 4 & 0 \\ 1 & 0 & -1 & -3 & 0 \\ 2 & 3 & 1 & 1 & 0 \\ -2 & 1 & 3 & -2 & 0 \end{bmatrix} \Leftrightarrow \overset{R_1 \leftrightarrow R_2}{\begin{bmatrix} 1 & 0 & -1 & -3 & 0 \\ 0 & 2 & 2 & 4 & 0 \\ 2 & 3 & 1 & 1 & 0 \\ -2 & 1 & 3 & -2 & 0 \end{bmatrix}} \Leftrightarrow$$

$R_2/2$, $R_3 - 2R_1$, R_4+R_1 $\qquad$ R_3-$3R_2$, $-R_2+R_4$

$$\begin{bmatrix} 1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 3 & 3 & 7 & 0 \\ 0 & 1 & 1 & -8 & 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -10 & 0 \end{bmatrix} \Leftrightarrow$$

R_1+3R_3, R_2-$2R_3$, R_4-$10R_3$

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

System form is;

$$\begin{aligned} w+x \qquad &=0 \\ x+y \quad &=0 \\ z&=0 \end{aligned}$$

leading entries are w, x, and z , free entry is y

let $y=t$

$x=-y=-t$

$w=-x=t$

$z=0$

solution is $w=t, x=-t, y=t, z=0,$ where $t \in R, \quad t \neq 0.$

so there are inifitly many solutions.