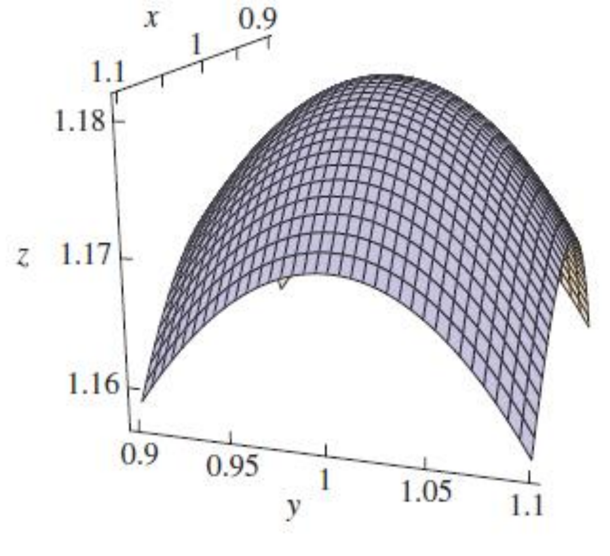
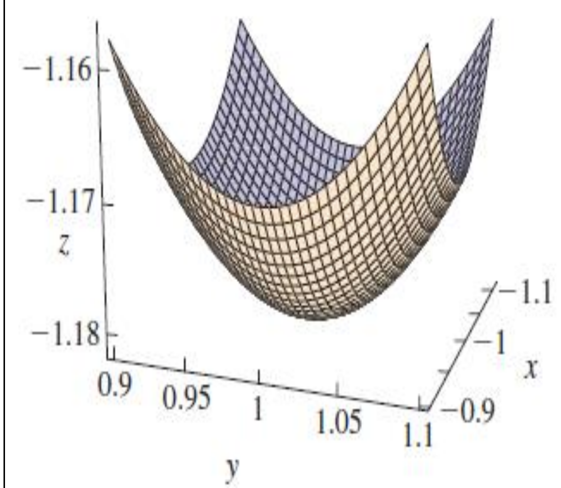
12.10 Extrema of functions of several variables

Definition1

*We call $f(a,b)$ a **local maximum** of f if there is an open disk R centered at (a,b), for which $f(a,b) \geq f(x,y)$ for all $(x,y) \in R$. Similarly, $f(a,b)$ is called a **local minimum** of f if there is an open disk R centered at (a,b), for which $f(a,b) \leq f(x,y)$ for all $(x,y) \in R$. In either case, $f(a,b)$ is called a **local extremum** of f.*



Local maximum at $(1,1)$ for $f(x,y) = x\,e^{-\frac{x^2}{2}-\frac{y^3}{3}+y}$



Local minimum at (-1.1) for $f(x,y) = x\,e^{-\frac{x^2}{2}-\frac{y^3}{3}+y}$

Definition2

*The point (a,b) is a **critical point** of the function $f(x,y)$ if (a,b) is in the domain of f and either $\frac{\partial f}{\partial x}(a,b) = \frac{\partial f}{\partial y}(a,b) = 0$ or one or both of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ do not exist at (a,b).*

Theorem1

If $f(x,y)$ has a local extremum at (a,b), then (a,b) must be a critical point of f.

Example1

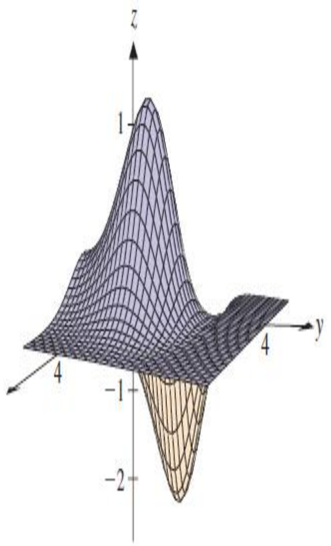
Find all critical points of $f(x,y) = x\,e^{-\frac{x^2}{2}-\frac{y^3}{3}+y}$.

Solution

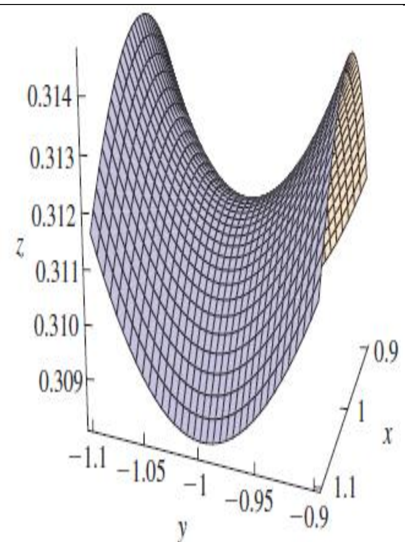
First, we compute the first partial derivatives:

$\frac{\partial f}{\partial x}(x,y) = (1-x^2)\,e^{-\frac{x^2}{2}-\frac{y^3}{3}+y}$ and $\frac{\partial f}{\partial y}(x,y) = x(1-y^2)\,e^{-\frac{x^2}{2}-\frac{y^3}{3}+y}$.

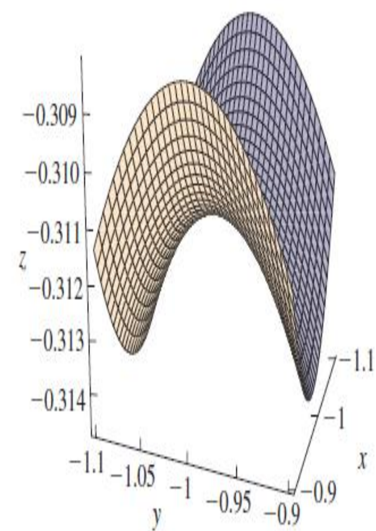
Since exponentials are always positive, we have $\frac{\partial f}{\partial x}(x,y)=0$ if and only if $1-x^2=0$, that is, when $x=\pm 1$. We have $\frac{\partial f}{\partial y}(x,y)=0$ if and only if $x\left(1-y^2\right)=0$, that is, when $x=0$ or $y=\pm 1$. So the set of critical points is $C_f=\{(-1,-1),(-1,1),(1,-1),(1,1)\}$.



$z=f(x,y)=x\,e^{-\frac{x^2}{2}-\frac{y^3}{3}+y}$



Saddle point at $(1,-1)$.



Saddle point at $(-1,-1)$.

Definition3

The point $P\left(a,b,f(a,b)\right)$ is a ***saddle point*** *of $z=f(x,y)$ if (a,b) is a critical point of f and if every open disk centered at (a,b) contains points (x,y) in the domain of f for which $f(x,y)<f(a,b)$ and points (x,y) in the domain of f for which $f(x,y)>f(a,b)$.*

<u>Theorem2 (Second Derivatives Test)</u>

Suppose that $f(x,y)$ has continuous second-order partial derivatives in some open disk containing the point (a,b) and that $f_x(a,b)=f_y(a,b)=0$. Define the discriminant D for the point (a,b) by $D(a,b)=f_{xx}(a,b)f_{yy}(a,b)-\left[f_{xy}(a,b)\right]^2$.

- **If $D(a,b)>0$ and $f_{xx}(a,b)>0$, then *f* has a local minimum at (a,b).**
- **If $D(a,b)>0$ and $f_{xx}(a,b)<0$, then *f* has a local maximum at (a,b).**
- **If $D(a,b)<0$, then *f* has a saddle point at (a,b).**
- **If $D(a,b)=0$, then no conclusion can be drawn.**

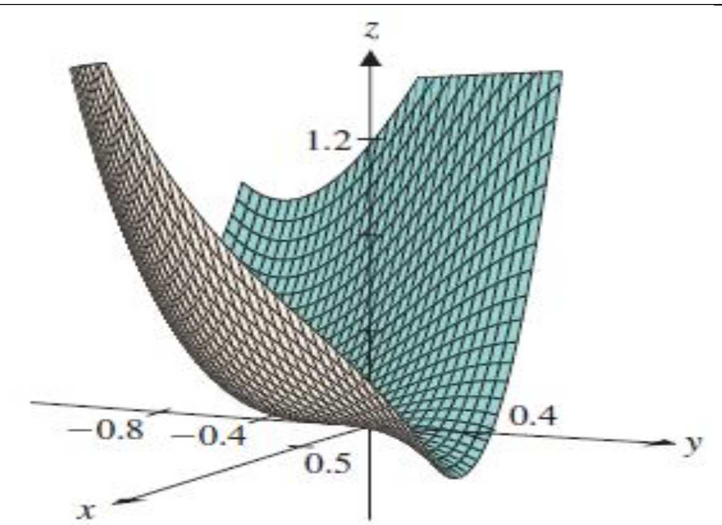
Example2 (Using the Discriminant to Find Local Extrema)

Locate and classify all critical points for $f(x,y) = 2x^2 - y^3 - 2xy$.

Solution

We first compute the first partial derivatives: $f_x = 4x - 2y$ and $f_y = -3y^2 - 2x$. Since both f_x and f_y are defined for all (x, y) , the critical points are solutions of the two equations: $f_x = 4x - 2y = 0$ and $f_y = -3y^2 - 2x = 0$. Solving the first equation for y , we get $y = 2x$. Substituting this into the second equation, we have $0 = -3(4x^2) - 2x = -12x^2 - 2x = -2x(6x+1)$, so that $x = 0$ or $x = \frac{-1}{6}$. The corresponding y -values are $y = 0$ and $y = \frac{-1}{3}$. The only two critical points are then $(0,0)$ and $\left(\frac{-1}{6}, \frac{-1}{3}\right)$. To classify these points, we first compute the second partial derivatives: $f_{xx} = 4, f_{yy} = -6y$ and $f_{xy} = -2$, and then test the discriminant. We have $D(0,0) = 4\times 0 - (-2)^2 = -4 < 0$ and $D\left(\frac{-1}{6}, \frac{-1}{3}\right) = 4\times(-6)\times\left(\frac{-1}{3}\right) - (-2)^2 = 4 > 0$.
From Theorem 2, we conclude that there is a <u>*saddle point*</u> *of* f *at* $(0,0)$, since $D(0,0) < 0$. Further, there is a <u>*local minimum at*</u> $\left(\frac{-1}{6}, \frac{-1}{3}\right)$ since $D\left(\frac{-1}{6}, \frac{-1}{3}\right) > 0$ and $f_{xx}\left(\frac{-1}{6}, \frac{-1}{3}\right) = 4 > 0$.

Point	$(0, 0)$	$\left(-\frac{1}{6}, -\frac{1}{3}\right)$
$f_{xx} = 4$	4	4
$f_{yy} = -6y$	0	2
$f_{xy} = -2$	−2	−2
$D(a, b)$	−4	4

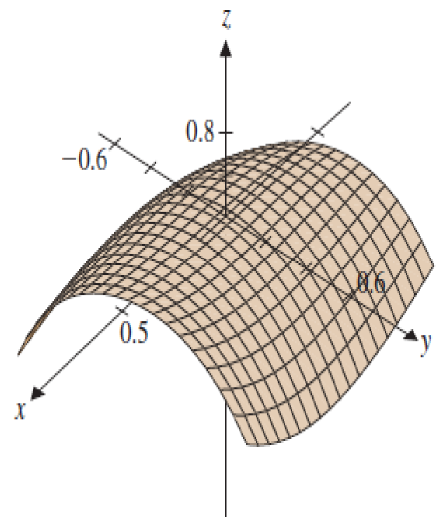


Example3 (Classifying Critical Points)

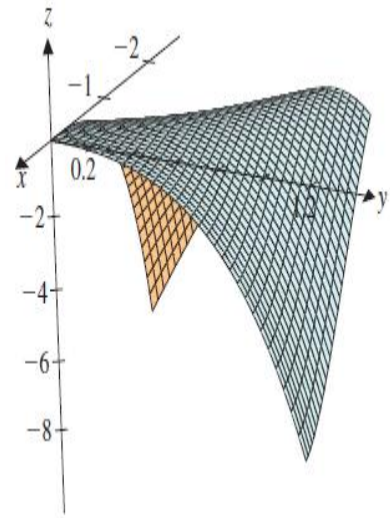
Locate and classify all critical points for $f(x,y)=x^3-2y^2-2y^4+3x^2y$.

Solution

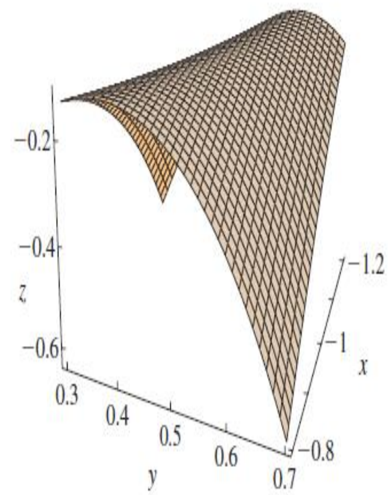
Here, we have $f_x=3x^2+6xy$ and $f_y=-4y-8y^3+3x^2$. Since both f_x and f_y exist for all (x,y), the critical points are solutions of the two equations: $f_x=3x^2+6xy=0$ and $f_y=-4y-8y^3+3x^2$=0. From the first equation, we have $0=3x^2+6xy=3x(x+2y)$, so that at a critical point, $x=0$ or $x=-2y$.
Substituting $x=0$ into the second equation, we have $0=-4y-8y^3=-4y(1+2y^2)$. The only (real) solution of this equation is $y=0$. This says that for $x=0$, we have only one critical point: $(0,0)$.
Substituting $x=-2y$ into the second equation, we have $0=-4y-8y^3+3(-2y)^2=-4y(1+2y^2-3y)=-4y(2y-1)(y-1)$. The solutions of this equation are $y=0, y=\frac{1}{2}$ and $y=1$, with corresponding critical points $(0,0),(-1,\frac{1}{2})$ and $(-2,1)$.
To classify the critical points, we compute the second partial derivatives, $f_{xx}=\frac{\partial}{\partial x}\left(3x^2+6xy\right)=6x+6y$ $\quad f_{yy}=\frac{\partial}{\partial y}\left(-4y-8y^3+3x^2\right)=-4-24y^2$, and $f_{xy}=\frac{\partial}{\partial y}\left(3x^2+6xy\right)=6x$, and evaluate the discriminant at each critical point. We have $D(0,0)=0$, $D\left(-1,\frac{1}{2}\right)=-6<0$ and $D(-2,1)=24>0$. From Theorem 2, we conclude that <u>f has a saddle point</u> at $\left(-1,\frac{1}{2}\right)$, since $D\left(-1,\frac{1}{2}\right)=-6<0$. Further, <u>f has a local maximum at</u> $(-2,1)$ since $D(-2,1)=24>0$ and $f_{xx}(-2,1)=-3<0$. Unfortunately, Theorem 2 gives us no information about the critical point $(0,0)$, since $D(0,0)=0$. However, notice that in the plane $y=0$ we have $f(x,y)=x^3$. In two dimensions, the curve $z=x^3$ has an inflection point at $x=0$. This shows that there is <u>no local extremum at</u> $(0,0)$.



The surface near $(0,0)$



The surface near $(-2,1)$



The surface near $\left(-1,\frac{1}{2}\right)$

Definition4

*We call $f(a,b)$ the **absolute maximum** of f on the region R if $f(a,b) \geq f(x,y)$ for all $(x,y) \in R$. Similarly, $f(a,b)$ is called the **absolute minimum** of f on R if $f(a,b) \leq f(x,y)$ for all $(x,y) \in R$. In either case, $f(a,b)$ is called an **absolute extremum** of f.*

<u>Theorem 3 (Extreme Value Theorem)</u>

Suppose that $f(x,y)$ is continuous on the closed and bounded region $R \subset \mathbb{R}^2$. Then f has both an absolute maximum and an absolute minimum on R. Further, an absolute extremum may only occur at a critical point in R or at a point on the boundary of R.

12.11 Constrained Optimization and Lagrange Multipliers

In this section, we develop a technique for finding the maximum or minimum of a function, given one or more constraints on the function's domain.

<u>Theorem1</u>

Suppose that $f(x,y,z)$ and $g(x,y,z)$ are functions with continuous first partial derivatives and $\nabla g(x,y,z) \neq 0$ on the surface $g(x,y,z)=0$. Suppose that either the minimum (or the maximum) value of $f(x,y,z)$ subject to the constraint $g(x,y,z)=0$ occurs at (x_0,y_0,z_0). Then $\nabla f(x_0,y_0,z_0)=\lambda \nabla g(x_0,y_0,z_0)$, for some constant λ (called a Lagrange multiplier).

Remark1

- Note that Theorem 1 says that if $f(x,y,z)$ has an extremum at a point (x_0,y_0,z_0) on the surface $g(x,y,z)=0$, we will have for $(x,y,z)=(x_0,y_0,z_0)$,

$$\begin{cases} f_x(x,y,z)=\lambda g_x(x,y,z) \\ f_y(x,y,z)=\lambda g_y(x,y,z) \\ f_z(x,y,z)=\lambda g_z(x,y,z) \\ g(x,y,z)=0 \end{cases}$$

  Finding such extrema then boils down to solving these four equations for the four unknowns x,y,z and λ .

- Notice that the Lagrange multiplier method we have just developed can also be applied to functions of two variables, by ignoring the third variable in Theorem1. That is, if $f(x,y)$ and $g(x,y)$ have continuous first partial derivatives and $f(x_0,y_0)$ is an extremum of f , subject to the constraint $g(x,y)=0$, then we must have $\nabla f(x_0,y_0)=\lambda\nabla g(x_0,y_0)$,for some constant λ .In this case, we end up with the three equations $f_x(x,y)=\lambda g_x(x,y)$, $f_y(x,y)=\lambda g_y(x,y)$ and $g(x,y)=0$, for the three unknowns x,y and λ.

Example 1 (Finding a Minimum Distance)

Use Lagrange multipliers to find the point on the line $y=3-2x$ that is closest to the origin.

Solution

For $f(x,y)=x^2+y^2$, we have $\nabla f(x,y)= \ <2x,2y>$ and for $g(x,y)=2x+y-3$, we have $\nabla g(x,y)= \ <2,1>$. The vector equation $\nabla f(x,y)=\lambda\nabla g(x,y)$ becomes $<2x\ ,2y>=\lambda \ <2,1>$ from which it follows that $2x=2\lambda$ and $2y=\lambda$.
The second equation gives us $\lambda=2y$. The first equation then gives us $x=\lambda=2y$.
Substituting $x=2y$ into the constraint equation $y=3-2x$, we have $5y=3$.
The solution is $y=\frac{3}{5}$, giving us $x=2y=\frac{6}{5}$. The closest point is then $\left(\frac{6}{5},\frac{3}{5}\right)$.

Example 2 (Optimization with an Inequality Constraint)

Suppose that the temperature of a metal plate is given by $T(x,y)=x^2+2x+y^2$, for points (x,y) on the elliptical plate defined by $x^2+4y^2\le 24$. Find the maximum and minimum temperatures on the plate.

Solution
The plate corresponds to the shaded region R shown in Figure 1.

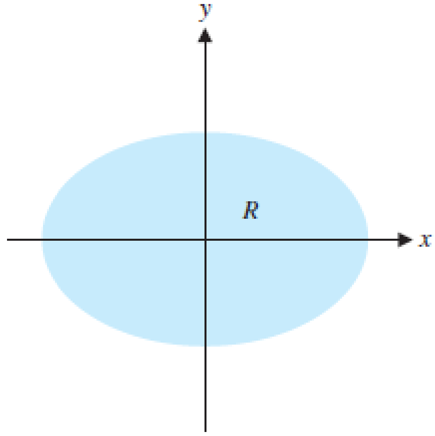


Figure 1: A metal plate.

We first look for critical points of $T(x,y)$ inside the region R. We have $\nabla T(x,y) = <2x+2, 2y> = <0,0>$ if $(x,y)=(-1,0)$, which is in R. At this point, $T(-1,0)=-1$. We next look for the extrema of $T(x,y)$ on the ellipse $x^2+4y^2=24$. We first rewrite the constraint equation as $g(x,y)=x^2+4y^2-24=0$. From Theorem 1, any extrema on the ellipse will satisfy the Lagrange multiplier equation: $\nabla T(x,y)=\lambda\nabla g(x,y)$ or $<2x+2, 2y> = \lambda <2x, 8y> = <2\lambda x, 8\lambda y>$.
This occurs when $2x+2=2\lambda x$ and $2y=8\lambda y$.

Notice that the second equation holds when $y=0$ or $\lambda=\frac{1}{4}$.

If $y=0$, the constraint $x^2+4y^2=24$ gives $x=\pm\sqrt{24}$.

If $\lambda=\frac{1}{4}$, the first equation becomes $2x+2=\frac{1}{2}x$ so that $x=-\frac{4}{3}$. The constraint $x^2+4y^2=24$ now gives $y=\pm\frac{\sqrt{50}}{3}$.

Finally, we compare the function values at all of these points (the one interior critical point and the candidates for boundary extrema):
and $T(-1,0)=-1$, $T(\sqrt{24},0)=24+\sqrt{24}\approx 33.8$, $T(-\sqrt{24},0)=24-2\sqrt{24}\approx 14.2$
$T\left(-\frac{4}{3},\frac{\sqrt{50}}{3}\right)=\frac{14}{3}\approx 4.7$, $T\left(-\frac{4}{3},-\frac{\sqrt{50}}{3}\right)=\frac{14}{3}\approx 4.7$.

From this list, it's easy to identify the minimum value of -1 at the point $(-1,0)$ and the maximum value of $24+2\sqrt{24}$ at the point $(\sqrt{24},0)$.

We close this section by considering the case of finding the minimum or maximum value of a differentiable function $f(x,y,z)$ subject to two constraints $g(x,y,z)=0$ and $h(x,y,z)=0$, where g and h are also differentiable (see Figure 2 below).

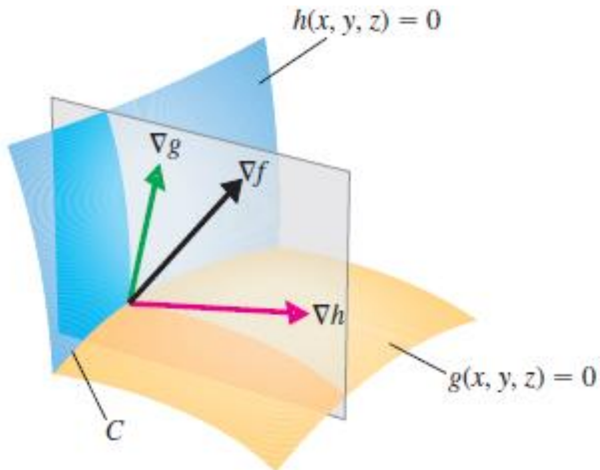


Figure 2: Constraint surfaces and the plane determined by the normal vectors ∇g and ∇h.

The method of Lagrange multipliers for the case of two constraints then consists of finding the point (x, y, z) and the Lagrange multipliers λ and μ (for a total of five unknowns) satisfying the five equations defined by:

$$\begin{cases} f_x(x,y,z) = \lambda\, g_x(x,y,z) + \mu\, h_x(x,y,z) \\ f_y(x,y,z) = \lambda\, g_y(x,y,z) + \mu\, h_y(x,y,z) \\ f_z(x,y,z) = \lambda\, g_z(x,y,z) + \mu\, h_z(x,y,z) \\ \quad g(x,y,z) = 0 \quad \& \quad h(x,y,z) = 0 \end{cases}.$$

Example 3 (Optimization with Two Constraints)
The plane $x + y + z = 12$ intersects the paraboloid $z = x^2 + y^2$ in an ellipse. Find the point on the ellipse that is closest to the origin.

Solution
We illustrate the intersection of the plane with the paraboloid in Figure 3.

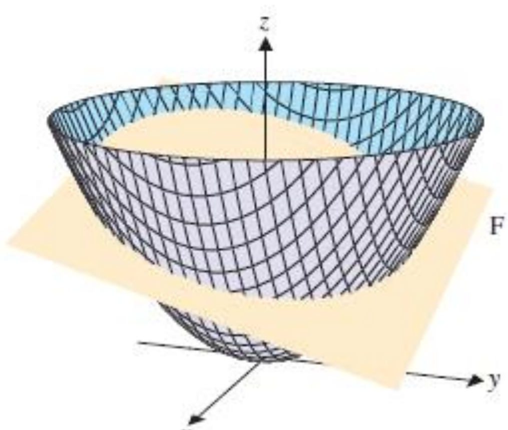


Figure 3: Intersection of a paraboloid and a plane.

Observe that minimizing the distance to the origin is equivalent to minimizing $f(x,y,z)=x^2+y^2+z^2$ [the *square* of the distance from the point (x,y,z) to the origin]. Further, the constraints may be written as $g(x,y,z)=x+y+z-12=0$ and $h(x,y,z)=x^2+y^2-z=0$. At any extremum, we must have that

$$\nabla f(x,y,z)=\lambda\nabla g(x,y,z)+\mu\nabla h(x,y,z) \text{ or}$$
$$<2x,2y,2z>=\lambda<1,1,1>+\mu<2x,2y,-1> .$$

Together with the constraint equations, we now have the system of equations:

$$\begin{cases} 2x=\lambda+2\mu x \quad (1) \\ 2y=\lambda+2\mu y \quad (2) \\ 2z=\lambda-\mu \quad (3) \\ x+y+z-12=0 \quad (4) \quad \& \quad x^2+y^2-z=0 \quad (5) \end{cases}$$

From (1), we have $\lambda=2x(1-\mu)$, while from (2), we have $\lambda=2y(1-\mu)$.
Setting these two expressions for λ equal gives us $2x(1-\mu)=2y(1-\mu)$,
from which it follows that either $\mu=1$ (in which case $\lambda=0$) or $x=y$. However, if $\mu=1$ and $\lambda=0$, we have from (3) that $z=-12$, which contradicts (5).
Consequently, the only possibility is to have $x=y$, from which it follows from (5) that $z=2x^2$. Substituting this into (4) gives us:
$0=x+y+z-12=x+x+2x^2-12=2x^2+2x-12=2(x+3)(x-2)$, so that $x=-3$ or $x=2$. Since $y=x$ and $z=2x^2$, we have that $(2,2,8)$ and $(-3,-3,18)$ are the only candidates for extrema. Finally, since $f(2,2,8)=72$ and $f(-3,-3,18)=342$, the closest point on the intersection of the two surfaces to the origin is $(2,2,8)$. By the same reasoning, observe that the farthest point on the intersection of the two surfaces from the origin is $(-3,-3,18)$.