12.1 Functions of several variables

Definition1

*A **function of two variables** is a rule that assigns a real number $f(x,y)$ to each ordered pair of real numbers (x,y) in the domain of the function.*
For a function f defined on the domain $D \subseteq \mathbb{R}^2$, we sometimes write $f: D \subseteq \mathbb{R}^2 \to \mathbb{R}$ to indicate that f maps points in two dimensions to real numbers.

*Likewise, a **function of three variables** is a rule that assigns a real number $f(x,y,z)$ to each ordered triple of real numbers (x,y,z) in the domain $D \subseteq \mathbb{R}^3$ of the function. We sometimes write $f: D \subseteq \mathbb{R}^3 \to \mathbb{R}$ to indicate that f maps points in three dimensions to real numbers.*

For instance, $f(x,y,z) = \dfrac{\cos(x+z)}{xy}$ and $g(x,y,z) = x^2 y - e^{xz}$ are both functions of the three variables x, y and z.

Example 1 (Finding the Domain of a Function of Two Variables)

Find and sketch the domain for
(a) $f(x,y) = x \ln y$.
(b) $g(x,y) = \dfrac{2x}{y - x^2}$.
Solution:
(a) For $f(x,y) = x \ln y$, recall that $\ln y$ is defined only for $y > 0$. The domain of f is then the set $D = \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$, that is, the half-plane lying above the x-axis (see Figure 1).

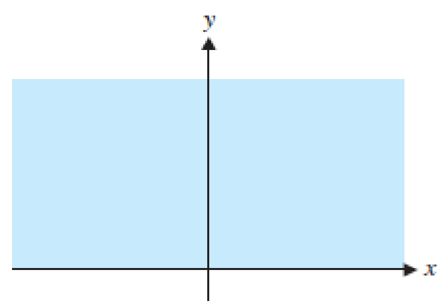

Figure1: the domain of $f(x,y) = x \ln y$

(b) $g(x,y)=\dfrac{2x}{y-x^2}$, note that g is defined unless there is a division by zero, which occurs when $y-x^2=0$. The domain of g is then $D=\left\{(x,y)\in\mathbb{R}^2 \mid y\neq x^2\right\}$, which is the entire xy-plane with the parabola $y=x^2$ removed (see Figure 2).

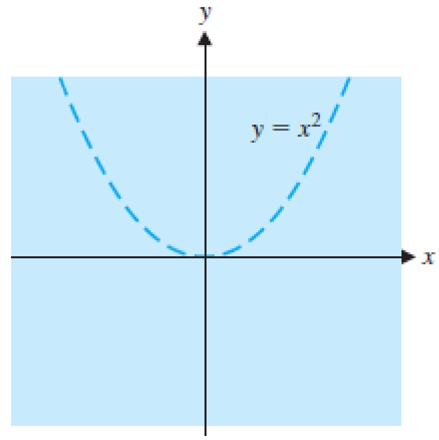

Figure2: the domain of $g(x,y)=\dfrac{2x}{y-x^2}$

Example 2(Finding the Domain of a Function of Three Variables)

Find and describe in graphical terms the domains of

(a) $f(x,y,z)=\dfrac{\cos(x+z)}{xy}$.

(b) $g(x,y,z)=\sqrt{1-x^2-y^2-z^2}$.

Solution

(a) For $f(x,y,z)=\dfrac{\cos(x+z)}{xy}$, there is a division by zero if $xy=0$, which occurs if $x=0$ or $y=0$. The domain is then $D=\left\{(x,y,z)\in\mathbb{R}^3 \mid x\neq 0 \,\&\, y\neq 0\right\}$, which is all of three-dimensional space, excluding the yz-plane ($x=0$) and the xz-plane ($y=0$).

(b) Notice that for $g(x,y,z)=\sqrt{1-x^2-y^2-z^2}$ to be defined, you'll need to have $1-x^2-y^2-z^2\geq 0$, or $x^2+y^2+z^2\leq 1$. The domain of g is then the unit sphere of radius 1 centered at the origin and its interior (see Figure 3).

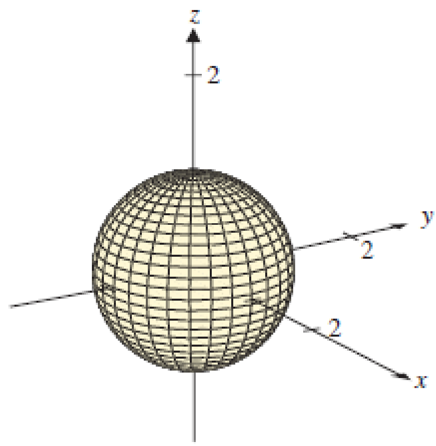

Figure3: the domain of $g(x, y, z) = \sqrt{1 - x^2 - y^2 - z^2}$

Definition 2

*The **graph** of the function f (x, y) is the graph of the equation z = f (x, y).*

Example 3 (Graphing Functions of Two Variables)

Graph (a) $f(x, y) = x^2 + y^2$ and (b) $g(x, y) = \sqrt{4 - x^2 + y^2}$.
Solution
(a) For $f(x, y) = x^2 + y^2$, you may recognize the surface $z = x^2 + y^2$ as a circular paraboloid. Notice that the traces in the planes $z = k > 0$ 0 are circles, while the traces in the planes $x = k$ and $y = k$ are parabolas. A graph is shown in Figure 4 .

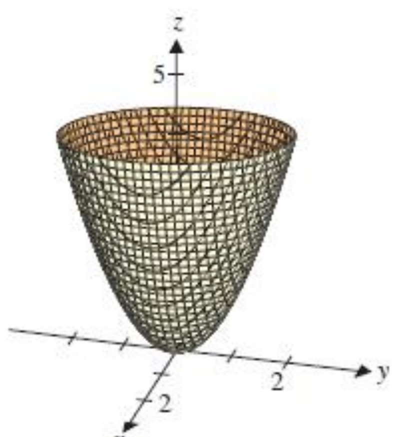

Figure 4: Graph of $z = x^2 + y^2$

(b) For $g(x,y)=\sqrt{4-x^2+y^2}$, note that the surface $z=\sqrt{4-x^2+y^2}$ is the top half of the surface $z^2=4-x^2+y^2$ or $x^2-y^2+z^2=4$. Here, observe that the traces in the planes $x=k$ and $z=k$ are hyperbolas, while the traces in the planes $y=k$ are circles. This gives us a hyperboloid of one sheet, wrapped around the y-axis. The graph of $z=g(x,y)$ is the top half of the hyperboloid, as shown in Figure 5.

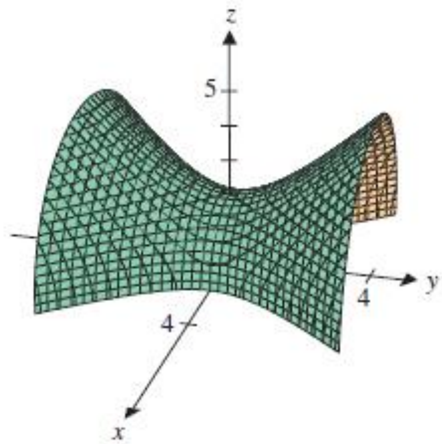

Figure 5: Graph of $z=\sqrt{4-x^2+y^2}$

Definition 3

*A **level curve** of the function $f(x,y)$ is the (two-dimensional) graph of the equation $f(x,y)=c$, for some constant c. (So, the level curve $f(x,y)=c$ is a two-dimensional graph of the trace of the surface $z=f(x,y)$ in the plane $z=c$.)*
*A **contour plot** of $f(x,y)$ is a graph of numerous level curves $f(x,y)=c$, for representative values of c.*

Example 4 (Sketching Contour Plots)

Sketch contour plots for (a) $f(x,y)=-x^2+y$ and (b) $g(x,y)=x^2+y^2$.
Solution
(a) First, note that the level curves of $f(x,y)$ are defined by $-x^2+y=c$,where c is a constant. Solving for y, you can identify the level curves as the parabolas $y=x^2+c$. A contour plot with $c=-4,-2,0,2$ and 4 is shown in Figure 6.

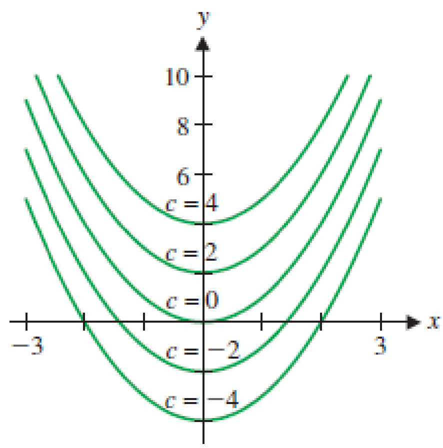

Figure 6: Contour plot $f(x,y) = -x^2 + y$

(b) The level curves for $g(x,y)$ are the circles $x^2 + y^2 = c$. In this case, note that there are level curves *only* for $c \geq 0$. A contour plot with $c = 1, 4, 7$ and 10 is shown in Figure7.

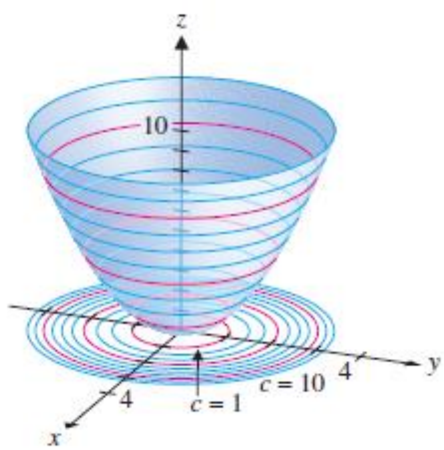

Figure 7: Contour plot $g(x,y) = x^2 + y^2$

12.2 Limits of Functions in Several Variables

Definition 1 (Formal Definition of Limit)
Let f *be defined on the interior of a circle centered at the point* (a,b)*, except possibly at* (a,b) *itself. We say that* $\lim_{(x,y)\to(a,b)} f(x,y) = L$ *if for every* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that* $|f(x,y) - L| < \varepsilon$ *whenever* $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$.
We illustrate the definition in Figure 1.

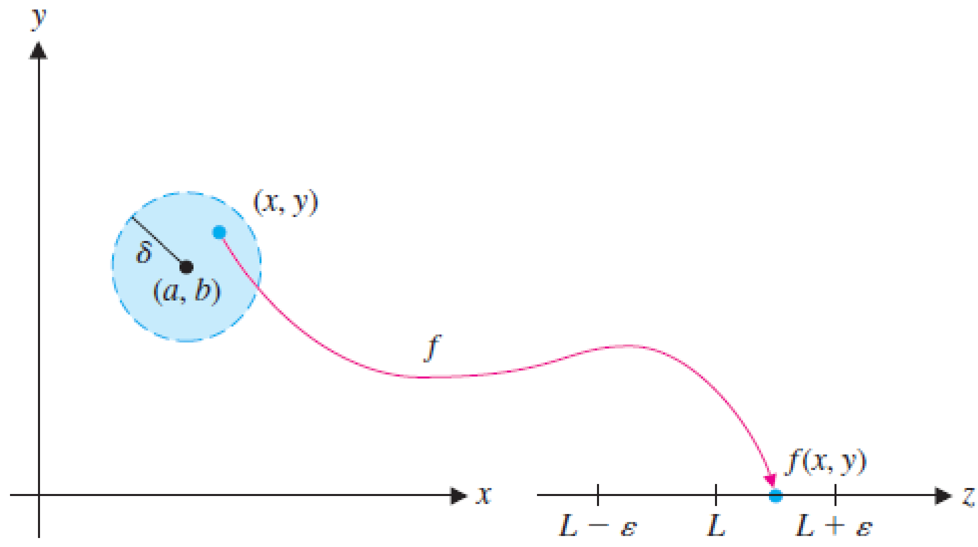

Figure 1: Limit of a Function of Two Variables

Remark 1

The definition of the limit of a function of three variables is completely analogous to the definition for a function of two variables. We say that $\lim_{(x,y,z)\to(a,b,c)} f(x,y,z) = L$, if we can make $f(x,y,z)$ as close as desired to L by making the point (x,y,z) sufficiently close to (a,b,c).

Example 1 (Finding a Simple Limit)

Evaluate $\lim_{(x,y)\to(2,1)} \dfrac{2x^2y+3xy}{5xy^2+3y}$.

Solution

First, note that this is the limit of a rational function (i.e., the quotient of two polynomials). Since the limit in the denominator is

$\lim_{(x,y)\to(2,1)} 5xy^2+3y = 13 \neq 0$, we have $\lim_{(x,y)\to(2,1)} \dfrac{2x^2y+3xy}{5xy^2+3y} = \dfrac{14}{13}$.

Remark 2

- We can show that the limit of any polynomial always exists and is found simply by substitution.
- We can show that the limit of any rational function at a point in its domain always exists and is found simply by substitution.

<u>Theorem 1</u>

If $f(x,y)$ approaches L_1 as (x,y) approaches (a,b) along a path P_1 and $f(x,y)$ approaches $L_2 \neq L_1$ as (x,y) approaches (a,b) along a path P_2, then $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

Remark 3

Unlike the case for functions of a single variable where we must consider left- and right-hand limits in two dimensions, instead of just two paths approaching a given point,

there are infinitely many (and you obviously can't check each one individually).
In practice, when you suspect that a limit does not exist, you should check the limit along the simplest paths first (Figure 2).

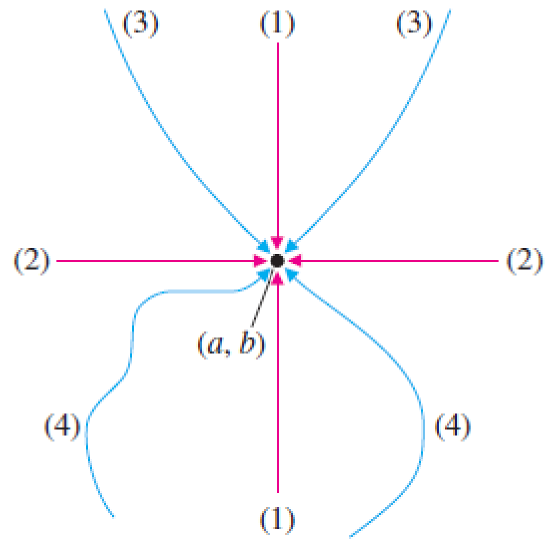

Figure 2: Various paths to (a,b)

Example 2 (A Limit That Does Not Exist)

Evaluate $\lim_{(x,y)\to(1,0)} \dfrac{y}{x+y-1}$.

Solution

First, we consider the vertical line path along the line $x=1$ and compute the limit as y approaches 0 0. If $(x,y)\to(1,0)$ along the line $x=1$, we have

$\lim_{y\to 0} \dfrac{y}{1+y-1} = 1.$

We next consider the path along the horizontal line $y=0$ and compute the limit as x approaches 1 . Here, we have $\lim_{x\to 1} \dfrac{0}{x+0-1} = 0$

Since the function approaches two different values along two different paths to the point (1, 0), the limit does not exist.

Example 3 (A Limit that is the same along two paths but Does Not Exist)

Evaluate $\lim_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$.

Solution

First, we consider the limit along the path $x=0$. We have $\lim_{y\to 0} \dfrac{0}{0^2+y^2} = 0$.

Similarly, for the path $y = 0$, we have $\lim_{x \to 0} \frac{0}{x^2 + 0^2} = 0$.

Be careful; just because the limits along the first two paths you try are the same does *not* mean that the limit exists. For a limit to exist, the limit must be the same along *all* paths through (0, 0) (not just along two). Here, we may simply need to look at more paths.

Notice that for the path $y = m\, x$ with $m \in \mathbb{R}^*$, we have $\lim_{x \to 0} \frac{mx^2}{x^2 + (mx)^2} = \frac{m}{1+m^2}$.

Since the limit along this path depends of m , the limit does not exist.

Example 4 (A Limit Problem Requiring a More Complicated Choice of Path)

Evaluate $\lim_{(x,y) \to (0,0)} \frac{xy^2}{x^2 + y^4}$.

Solution

Notice that for the path $x = m\, y^2$ with $m \in \mathbb{R}$ (pass through the origin point $(0,0)$), we have

$$\lim_{y \to 0} \frac{my^4}{(my^2)^2 + y^4} = \frac{m}{m^2+1}$$

Since the limit along this path depends of m , the limit does not exist (see Figure 3).

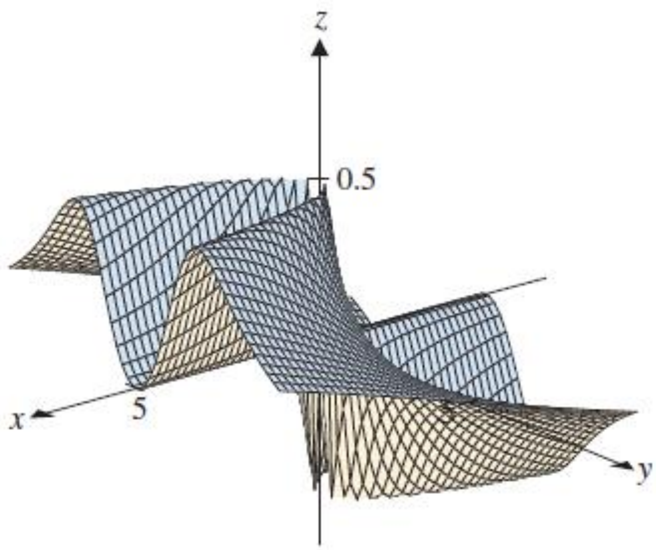

Figure 3: the surface of $z = \frac{xy^2}{x^2 + y^4}$ for $-5 \le x \le 5$ and $-5 \le y \le 5$

Theorem 2

Suppose that $|f(x,y) - L| \le g(x,y)$ *for all* (x,y) *in the interior of some circle centered at* (a,b), *except possibly at* (a,b) .

If $\lim_{(x,y) \to (a,b)} g(x,y) = 0$, *then* $\lim_{(x,y) \to (a,b)} f(x,y) = L$.

Example 5 (Proving That a Limit Exists)

Evaluate $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^2+y^2}$.

Solution

As we did in earlier examples, we start by looking at the limit along several paths through $(0,0)$.

Along the path $x=0$, we have $\lim\limits_{(0,y)\to(0,0)} \dfrac{0^2y}{0^2+y^2}=0$.

Similarly, along the path $y=0$, we have $\lim\limits_{(x,0)\to(0,0)} \dfrac{x^2.0}{x^2+0^2}=0$.

Further, along the path $y=mx$ (with m a real number), we have

$$\lim_{(x,mx)\to(0,0)} \frac{x^2mx}{x^2+(mx)^2}=\lim_{x\to 0}\frac{mx}{1+m^2}=0.$$

We know that if the limit exists, it must equal 0. After simplifying the expression, there remained an extra power of x in the numerator forcing the limit to 0. To show that the limit equals 0, consider $\left|f(x,y)-0\right|=\left|\dfrac{x^2y}{x^2+y^2}\right|$.

Notice that without the y^2 term in the denominator, we could cancel the x^2 terms.

Since $x^2+y^2\geq x^2$, we have that for $x\neq 0$, $\left|f(x,y)-0\right|=\left|\dfrac{x^2y}{x^2+y^2}\right|\leq\left|\dfrac{x^2y}{x^2}\right|\leq|y|$.

Since $\lim\limits_{(x,y)\to(0,0)}|y|=0$, Theorem 2 gives us $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^2+y^2}=0$, also.

12.3 Continuity of functions in two or three variables

Definition 1

Suppose that $f(x,y)$ is defined in the interior of a circle centered at the point (a,b).

*We say that f is **continuous** at (a,b) if $\lim\limits_{(x,y)\to(a,b)} f(x,y)=f(a,b)$.*

*If $f(x,y)$ is not continuous at (a,b), then we call (a,b) a **discontinuity** of f.*

*We say that a function $f(x,y)$ is **continuous on a region** R if it is continuous at each point in R.*

Remark 1

- The definition of the continuity of a function of three variables is completely analogous to the definition for a function of two variables:

Suppose that $f(x,y,z)$ is defined in the interior of a sphere centered at (a,b,c). We say that *f* is **continuous** at (a,b,c) if $\lim\limits_{(x,y,z)\to(a,b,c)} f(x,y,z)=f(a,b,c)$

If $f(x,y,z)$ is not continuous at (a,b,c), then we call (a,b,c) a **discontinuity** of f.

- Notice that because we define continuity in terms of limits, we immediately have the following results, which follow directly from the corresponding results for limits. If $f(x,y)$ and $g(x,y)$ are continuous at (a,b), then $f+g$, $f-g$ and $f \cdot g$ are all continuous at (a,b). Further, $\frac{f}{g}$ is continuous at (a,b), if, in addition, $g(a,b) \neq 0$.

Example 1(Determining Where a Function of Two Variables Is Continuous)

Find all points where the given function is continuous:

(a) $f(x,y) = \frac{x}{x^2 - y}$.

(b) $g(x,y) = \begin{cases} \frac{x^4}{x\left(x^2+y^2\right)}, (x,y) \neq (0,0) \\ 0, \quad (x,y) = (0,0) \end{cases}$.

Solution

- For (a), notice that $f(x,y)$ is a quotient of two polynomials (i.e., a rational function) and so, it is continuous at any point where we don't divide by 0. Since division by zero occurs only when $y = x^2$, we have that f is continuous at all points (x,y) with $y \neq x^2$.

- For (b), the function g is also a quotient of polynomials, except at the origin. Notice that there is a division by 0 whenever $x = 0$. We must consider the point $(0,0)$ separately, however, since the function is not defined by the rational expression there. We can verify that $\lim_{(x,y)\to(0,0)} g(x,y) = 0 = g(0,0)$ using the following string of inequalities. Notice that for $(x,y) \neq (0,0)$,

$$|g(x,y)| = \left|\frac{x^4}{x\left(x^2+y^2\right)}\right| \leq \left|\frac{x^4}{x\left(x^2\right)}\right| = |x|$$

and $|x| \to 0$ as $(x,y) \to (0,0)$. We deduce that $\lim_{(x,y)\to(0,0)} g(x,y) = 0 = g(0,0)$, so that g is continuous at $(0,0)$. Putting this all together, we get that g is continuous at the origin and also at all points (x,y) with $x \neq 0$.

Theorem 1

Suppose that $f(x,y)$ is continuous at (a,b) and $g(x)$ is continuous at the point $f(a,b)$. Then $h(x,y) = g \circ f(x,y) = g(f(x,y))$ is continuous at (a,b).

Example 2 (Determining Where a Composition of Functions Is Continuous)

Determine where $f(x,y) = e^{x^2 y}$ is continuous?

Solution

Notice that $f(x,y) = g(h(x,y))$, where $g(t) = e^t$ and $h(x,y) = x^2 y$. Since g is continuous for all values of t and h is a polynomial in x and y (and hence continuous on $\mathbb{R}^2$) , it follows from Theorem 1 that f is continuous on $\mathbb{R}^2$.

Example 3

Determine where $h(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$ is continuous ?

Solution

- The function $f(x,y) = \frac{y}{x}$ is a rational function and therefore continuous except on the line $x = 0$.

- The function $g(t) = \tan^{-1} t$ is continuous everywhere.

It follows from Theorem 1 that h is continuous on $\mathbb{R}^2 \setminus \{x = 0\}$ (see Figure1).

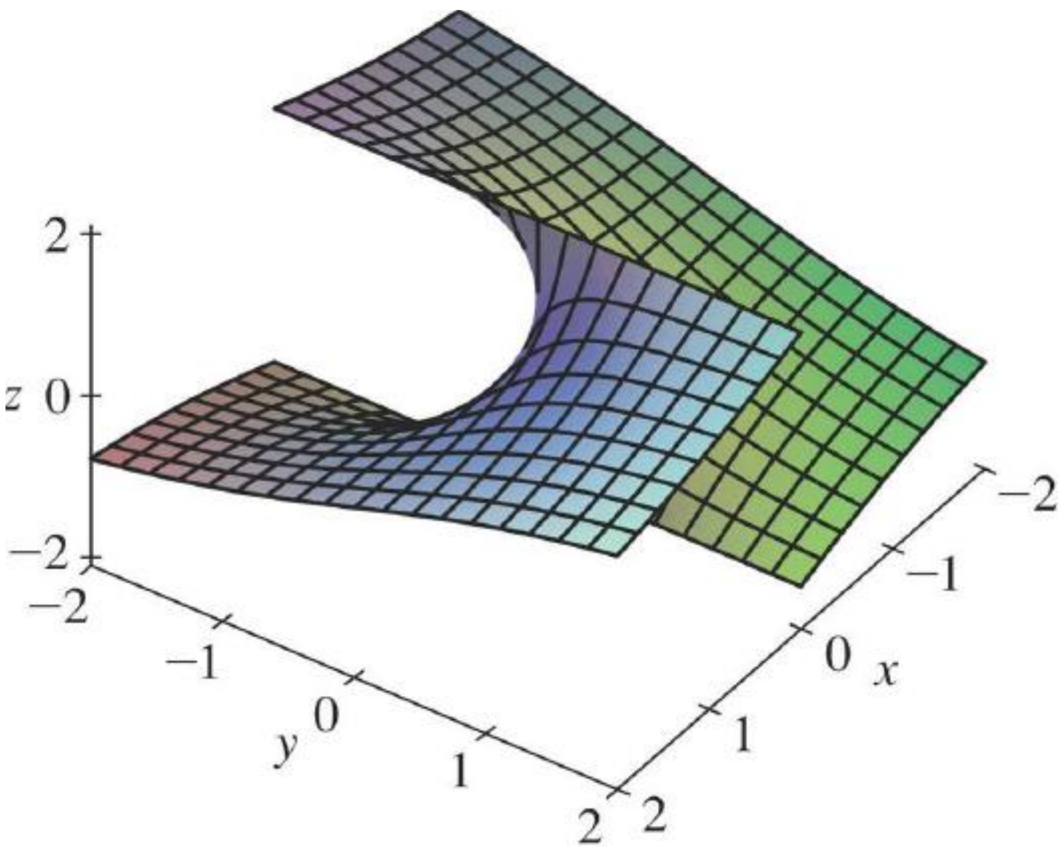

Figure1: the figure shows the break in the graph of $h(x,y)=\tan^{-1}\left(\frac{y}{x}\right)$ above the y-axis

Example 4

Determine where $f(x,y)=\begin{cases}\frac{3x^2y}{x^2+y^2}, (x,y)\neq(0,0)\\ 0, \quad (x,y)=(0,0)\end{cases}$ is continuous?

Solution

We know f is continuous for $(x,y)\neq(0,0)$. Since it is equal to a rational function there.

Also we have $\lim_{(x,y)\to(0,0)}\frac{3x^2y}{x^2+y^2}=0=f(0,0)$. Thus f is continuous at $(0,0)$.

So f is continuous on $\mathbb{R}^2$ (see Figure 2).

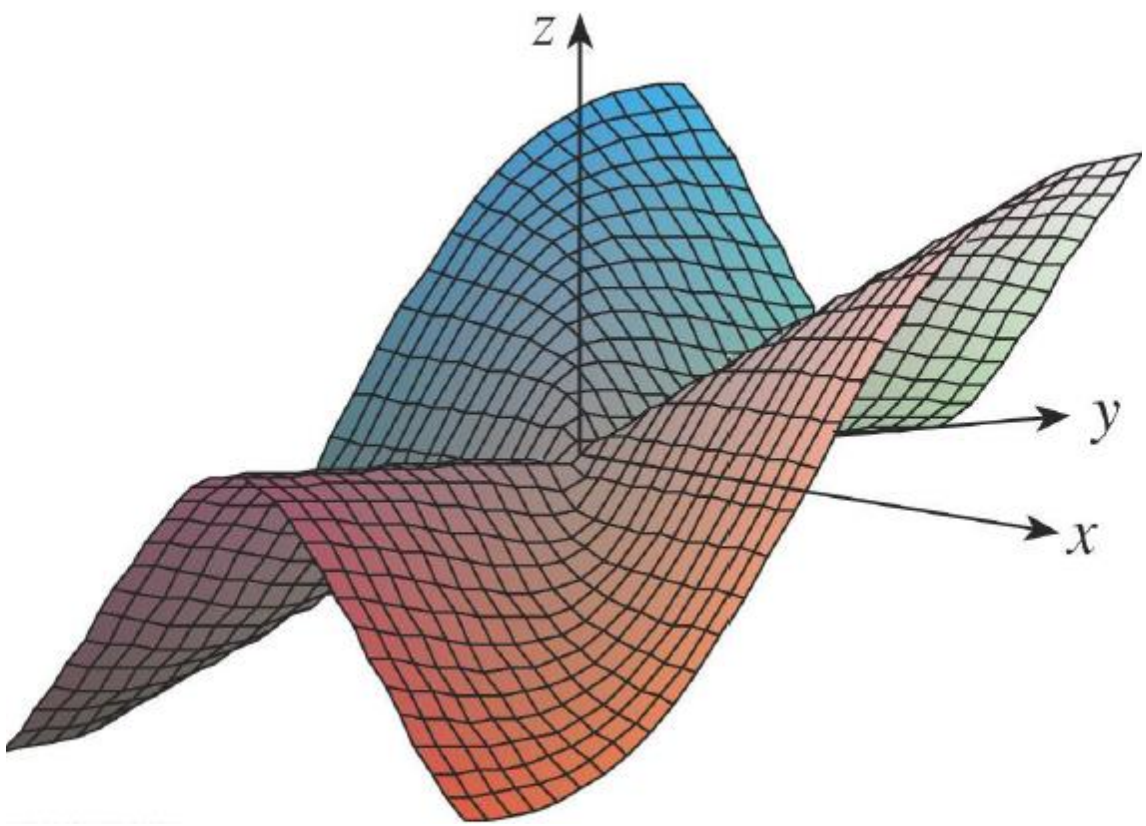

Figure2: Graph of $z = f(x,y) = \begin{cases} \dfrac{3x^2y}{x^2+y^2}, (x,y) \neq (0,0) \\ 0, \quad (x,y) = (0,0) \end{cases}$.

Example 5

Determine where $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, (x,y) \neq (0,0) \\ 0, \quad (x,y) = (0,0) \end{cases}$ is continuous?

Solution

We know f is continuous for $(x,y) \neq (0,0)$. Since it is equal to a rational function there. Also we have $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$ does not exist. Thus f is not continuous at $(0,0)$. So f is continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$ (see Figure3).

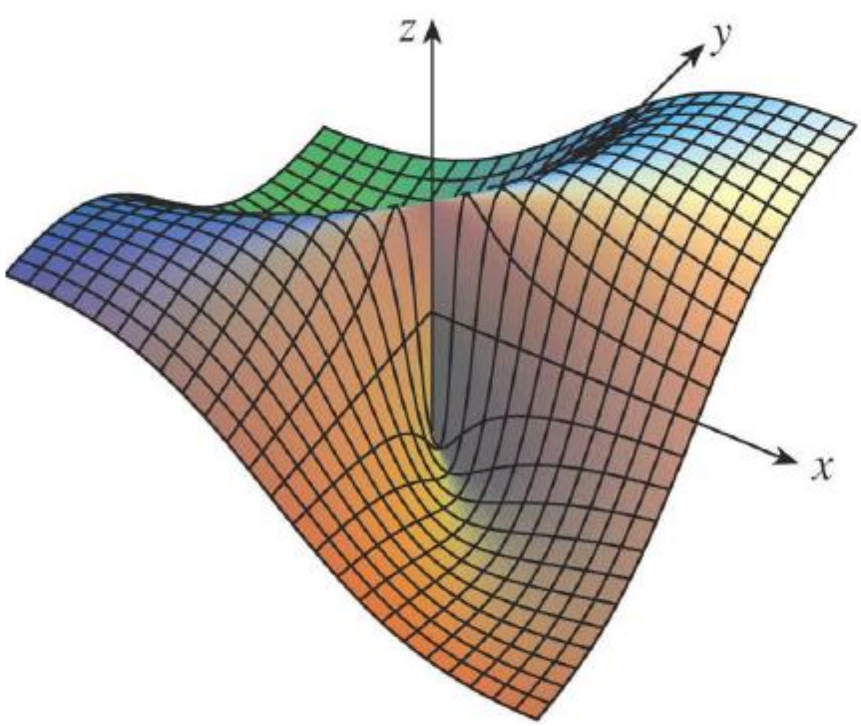

Figure3: Graph of $z = f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, (x,y) \neq (0,0) \\ 0, \quad (x,y) = (0,0) \end{cases}$.

Example 6 (Continuity for a Function of Three Variables)

Find all points where $f(x,y,z) = \ln\left(9 - x^2 - y^2 - z^2\right)$ is continuous.

Solution

Notice that $f(x,y,z)$ is defined only for $9 - x^2 - y^2 - z^2 > 0$. On this domain, f is a composition of continuous functions, which is also continuous. So, f is continuous for $x^2 + y^2 + z^2 < 9$, which you should recognize as the interior of the sphere of radius 3 centered at (0, 0, 0).