Complex numbers

A complex number z can be represented as a sum of real and imaginary part $z = x + yi$, where x and y are real numbers and $i = \sqrt{-1}\left(i^2 = -1\right)$.

- The complex number $x + yi$ can be represented by the order pair (x, y), and plotted in a plane (called the Argand plane) as shown in figure 1. In the Argand plane the horizontal axis is called the real axis and the vertical axis called the imaginary axis.

The **real part** of the complex number $x + yi$ is the real number x and the **imaginary part** is the real number y. Thus, the real part of $5 - 7i$ is 5 and the imaginary part is -7.

- $z = x + yi$ **is called a Cartesian complex number**
- Two complex numbers $z_1 = x_1 + i\,y_1$ and $z_2 = x_2 + i\,y_2$ are **equal** if $x_1 = x_2$ and $y_1 = y_2$.
- Complex numbers are **used** to solve polynomials (for example $x^2 = -1$ is a polynomial of 2 degree so it should have 2 roots, $x = \pm i$), used to solve differential equations (ODE and PDE). Also used for analyzing oscillation and waves (phase component).

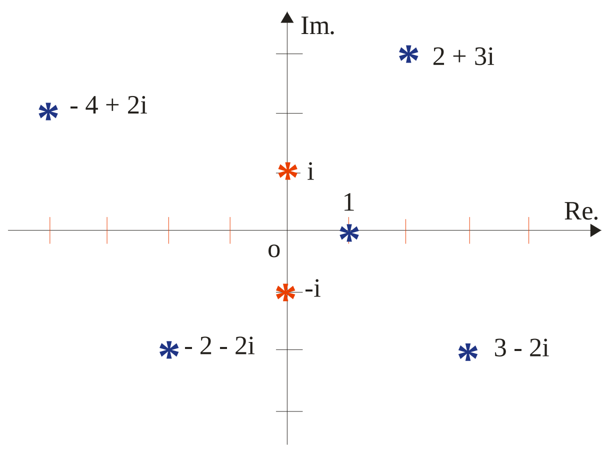


Figure 1. Complex numbers as points in the Argand plane.

- The **sum and difference** of two complex numbers are defined by adding or subtracting their real parts and their imaginary parts. (It is same as we do with the real numbers)

If $z_1 = x_1 + i\,y_1$ and $z_2 = x_2 + i\,y_2$, then $z_1 \pm z_2 = \left(x_1 \pm x_2\right) + i\left(y_1 \pm y_2\right)$

Example 1. If $z_1 = 1 - i$ **and** $z_2 = 4 + 7i$**,** $z_1 + z_2 = (1 - i) + (4 + 7i) = 5 + 6i$

Multiplication of complex numbers: Multiplication of complex numbers is achieved by assuming all quantities involved are real and then using $i^2 = -1$ simplify by separating real and imaginary parts. Multiplication is the most interesting operation in complex numbers. For $z_1 = x_1 + i\,y_1$ and $z_2 = x_2 + i\,y_2$

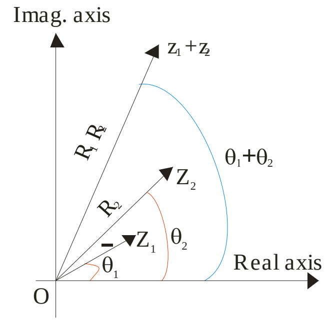
$$z_1 z_2 = \left(x_1 + i\,y_1\right)\left(x_2 + i\,y_2\right) = x_1 x_2 + x_1\left(i\,y_2\right) + \left(i\,y_1\right)x_2 + i^2 y_1 y_2 = x_1 x_2 + i\left(x_1 y_2 + y_1 x_2\right) - y_1 y_2$$

$$\Rightarrow z_1 z_2 = \left(x_1 x_2 - y_1 y_2\right) + i\left(x_1 y_2 + x_2 y_1\right)$$

- It is much simpler and easier to multiply and divide the complex numbers in polar coordinates system.

- $z_1 z_2 = R_1 R_2 e^{i\theta_1} e^{i\theta_2} = R_1 R_2 e^{i(\theta_1 + \theta_2)}$ $\qquad \dfrac{z_1}{z_2} = \dfrac{R_1 e^{i\theta_1}}{R_2 e^{i\theta_2}} = \dfrac{R_1}{R_2} e^{i(\theta_1 - \theta_2)}$

- In complex plane: A nice geometrical interpretation of complex number multiplication is shown in the following figure. Simple multiply the magnitudes $R_1 R_2$ and add the angles $\theta_1+\theta_2$.

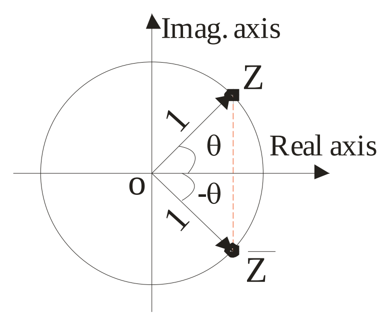


Example 2. If $z_1=3+2i$ and $z_2=4-5i$,

$$z_1 z_2=(3+2i)(4-5i)=12-15i+8i-10i^2=12-7i+10=22-7i$$

Complex conjugate: The **complex conjugate** of $z=x+yi$ is $\overline{z}=x-yi$. It is very important and is the mirror image of the number in the real axis.

- $z+\overline{z}=x+yi+x-yi=2x$, is a real number and
- $z\overline{z}=(x+i\,y)(x-i\,y)=x^2-ixy+ixy+y^2 \Rightarrow z\overline{z}=x^2+y^2$ is a real number and is equal to the length of z.



Division of complex numbers: For the quotient (division) of two complex numbers, to get rid of complex term from the denominator we multiply the numerator and denominator by the **complex conjugate** of the denominator.

For $z_1=x_1+i\,y_1$ and $z_2=x_2+i\,y_2$, then $\frac{z_1}{z_2}=\frac{x_1+i\,y_1}{x_2+i\,y_2}$, Multiplying and dividing by the complex conjugate of z_2, $\frac{z_1}{z_2}=\frac{x_1+i\,y_1}{x_2+i\,y_2}\times\frac{x_2-i\,y_2}{x_2-i\,y_2}=\frac{z_1\overline{z}_2}{|z_2|^2}$

Example 3. Express the number $\frac{-1+3i}{2+5i}$ in the form $x+yi$.

Solution. We multiply numerator and denominator by the complex conjugate of the denominator that is, $2-5i$.

$$\frac{-1+3i}{2+5i}=\frac{-1+3i}{2+5i}\times\frac{2-5i}{2-5i}=\frac{-1\times(2-5i)+3i(2-5i)}{2\times(2-5i)+5i(2-5i)}=\frac{-2+5i+6i-15i^2}{4-10i+10i-25i^2}=\frac{-2+11i-15(-1)}{4-25(-1)}=\frac{13+11i}{29}=\frac{13}{29}+\frac{11i}{29}$$

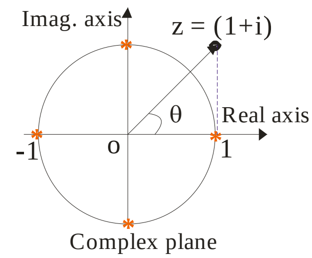
Example 4. Find the roots of the equation: $x^2+1=0$

$x^2=-1$, No real solution. Invent $i=\sqrt{-1}\left(i^2=-1\right)$.

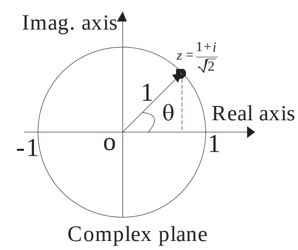
$$x^2=i^2 \Rightarrow x^2-i^2=0 \Rightarrow (x+i)(x-i)=0 \Rightarrow x+i=0 \; and \; x-i=0 \Rightarrow x=-i \; and \; x=i$$

Example 5. Find the roots of the equation: $x^4 = 1$, four degree equation have four solutions

$$x^4 - 1 = 0 \Rightarrow (x^2 - 1)(x^2 + 1) = 0 \Rightarrow (x+1)(x-1)(x+i)(x-i) = 0 \Rightarrow x = -1,\ x = +1\ \ x = -i\ \ x = i.$$



Example 6. The complex number $z = \frac{1+i}{\sqrt{2}} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$ is shown in the figure find



(i) $\bar{z} = ?\quad z = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \Rightarrow \bar{z} = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$

(ii) $z^2 = ?\quad z^2 = \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) \Rightarrow z^2 = \frac{1}{2} + \frac{i}{2} + \frac{i}{2} - \frac{1}{2} \Rightarrow z^2 = i$

(iii) $z + \bar{z} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} = \frac{2}{\sqrt{2}} \Rightarrow z + \bar{z} = \sqrt{2}\ \ (a\ real\ number)$

(iv) $z\bar{z} = \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) = \frac{1}{2} - \frac{i}{2} + \frac{i}{2} + \frac{1}{2} \Rightarrow z\bar{z} = 1$

Problem If $z = -1 + i2$

Find (i) $\bar{z}$ (ii) $z\bar{z}$ (iii) $z + \bar{z}$ (iv) Plot each result in the complex plane

Example 7. Find the roots of the equation: $x^2 + x + 1 = 0$

Using quadratic formula, $x = \frac{-1 \pm \sqrt{1^2 - 4}}{2} \Rightarrow x = \frac{-1 \pm \sqrt{-3}}{2} \Rightarrow x = \frac{-1 \pm i\sqrt{3}}{2}$

The roots are: $\left(\frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}\right)$. The solution of the example are complex conjugate of each other.

Example 8. Evaluate $(a)\,i^3,\ (b)\,i^4,\ (c)\,i^{23},\ (d)\,\frac{-4}{i^9}$

$$i^3 = i^2 \times i = (-1) \times i = -i, \text{ since } i^2 = -1$$

$$i^4 = i^2 \times i^2 = (-1) \times (-1) = 1$$

$$i^{23} = i \times i^{22} = i \times (i^2)^{11} = i \times (-1)^{11} = i \times (-1) = -i$$

$$i^9 = i \times i^8 = i \times (i^2)^4 = i \times (-1)^4 = i \times 1 = i$$

Hence

$$\frac{-4}{i^9} = \frac{-4}{i} = \frac{-4}{i} \times \frac{i}{i} = \frac{-4i}{i^2} = \frac{-4i}{-1} = 4i$$

Problem 1. Evaluate $(a)\,i^8,\ (b)\,-\frac{1}{i^7},\ (c)\,\frac{4}{2i^{13}}$ Answer: $(a)\,1,\ (b)\,-i,\ (c)\,-2i$

Problem 2. Evaluate in $a+ib$ form, given: $z_1 = 1+2i,\ z_2 = 4-3i,\ z_3 = -2+3i\ \ and\ \ z_4 = -5-i.$

$$(1)\frac{z_1 z_3}{z_1+z_3} \qquad (2)\, z_2+\frac{z_1}{z_4}+z_3 \qquad\qquad answer\left((1)\frac{3}{26}+\frac{41}{26}i \qquad (2)\frac{45}{26}-\frac{9}{26}i\right)$$

Problem 3. Show that: $\frac{-25}{2}\left(\frac{1+2i}{3+4i}-\frac{2-5i}{-i}\right)=57+24i$

Complex equations

If two complex numbers are equal, then their real parts are equal and their imaginary parts are equal. Hence if $a+ib=c+id, \quad then \quad a=c \quad and \quad b=d$

Example 9. Solve the complex equation $(x-2yi)+(y-3xi)=2+3i$

$(x-2yi)+(y-3xi)=2+3i \Rightarrow (x+y)+(-2y-3x)i=2+3i$, Equating real and imaginary parts gives

$$\Rightarrow x+y=2 \qquad (1)$$
$$-2y-3x=3 \qquad (2)$$

Solving equation (1) and (2), Multiplying equation (1) with 2 and then add both equations

$$2x+2y=4 \qquad (1)$$
$$-2y-3x=3 \qquad (2)$$

$\Rightarrow x=-7,$ Substituting x in equation (2) gives $y=9.$

Problem 4. Solve the complex equation

$$(x-2yi)-(y-xi)=2+i \qquad\qquad answer\,(x=3,\ y=1)$$

Hyperbolic functions

Pretty useful function especially in complex numbers.

$$\sinh(z)=\frac{e^z-e^{-z}}{2}, \quad \cosh(z)=\frac{e^z+e^{-z}}{2}, \quad \tanh(z)=\frac{\sinh(z)}{\cosh(z)}=\frac{e^z-e^{-z}}{e^z+e^{-z}} \Rightarrow \tanh(z)=\frac{e^z-e^{-z}}{e^z+e^{-z}}\times\frac{e^z}{e^z}=\frac{e^{2z}-1}{e^{2z}+1}$$

z is complex number.

- $\cosh^2 z-\sinh^2 z=\left(\frac{e^z+e^{-z}}{2}\right)^2-\left(\frac{e^z-e^{-z}}{2}\right)^2=\frac{1}{4}\left(e^{2z}+2e^ze^{-z}+e^{2z}\right)-\frac{1}{4}\left(e^{2z}-2e^ze^{-z}+e^{2z}\right)$

$$=\frac{1}{4}\left(e^{2z}+2+e^{2z}\right)-\frac{1}{4}\left(e^{2z}-2+e^{2z}\right)=\frac{1}{4}\left(e^{2z}+2+e^{2z}-e^{2z}+2-e^{2z}\right)=\frac{1}{4}(4)=1$$

$$\therefore \cosh^2 z-\sinh^2 z=1$$

Trignometric functions

Pretty useful function especially in complex numbers.

$$\sin(z)=\frac{e^{iz}-e^{-iz}}{2i}, \quad \cos(z)=\frac{e^{iz}+e^{-iz}}{2}, \qquad\qquad \cos^2 z+\sin^2 z=1$$

- Sin and cos is between +1 and -1, but it is not true for complex z .

$|\sin(10i)|=\left|\frac{e^{-10}-e^{10}}{2i}\right|\geq 10000$ a huge number

- Hyperbolic, trignometir and exponancial functions are interrelated.

Identities:

- $i\sin(z)=\sinh(iz)$ $\qquad\qquad$ $\cos(z)=\cosh(iz)$
- $\sin(iz)=i\sinh(z)$ $\qquad\qquad$ $\cos(iz)=\cosh(z)$
- $e^{z}=\cosh(z)+\sinh(z)$ $\qquad\qquad$ $e^{z}=\cos(iz)-i\sin(iz)$
- $e^{iz}=\cos z+i\sin z,\ e^{-iz}=\cos z-i\sin z,\quad replace\ z\ with\ iz\quad e^{-i(iz)}=\cos(iz)-i\sin(iz)$

$\Rightarrow e^{z}=\cos(iz)-i\sin(iz)\Rightarrow e^{-z}=\cos(iz)+i\sin(iz)$

$\Rightarrow e^{z}+e^{-z}=2\cos(iz)\Rightarrow \cos(iz)=\dfrac{e^{z}+e^{-z}}{2}\Rightarrow \cos(iz)=\cosh z$

- $e^{iz}=\cos z+i\sin z,\ e^{-iz}=\cos z-i\sin z,\quad replace\ z\ with\ iz\quad e^{-i(iz)}=\cos(iz)-i\sin(iz)$

$\Rightarrow e^{z}=\cos(iz)-i\sin(iz)\Rightarrow e^{z}=\cos(iz)-i\sin(iz)\Rightarrow e^{z}=\cosh z-i\sinh z$

The Polar form of a complex number

Let a complex number Z be $x+yi$ as shown in the Argand diagram of figure 2. Let distance OZ be r and the angle OZ makes with the positive real axis be θ.

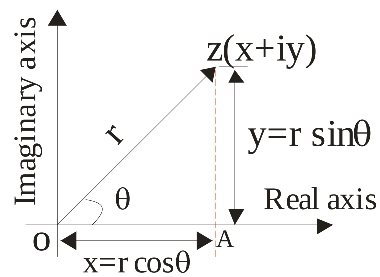


Figure 2.

From trigonometry,

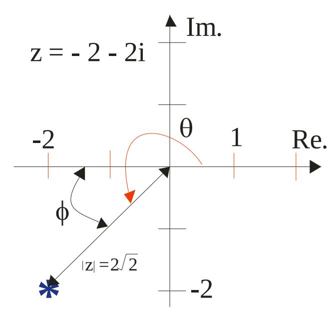
$x=r\cos\theta\ and\ y=r\sin\theta,\quad r=\sqrt{x^{2}+y^{2}}\quad \theta=\tan^{-1}\left(\dfrac{y}{x}\right)$

Where r is called the **modulus** (or magnitude) of *Z* and is written as mod *Z* or |*Z*|. r is determined using Pythagoras' theorem on triangle OAZ and θ is called the **argument** (or amplitude) of *Z* and is written as arg *Z*. Whenever changing from Cartesian form to polar form, or vice-versa, a sketch is invaluable for determining the quadrant in which the complex number occurs.

Hence $z=x+yi=r\cos\theta+ir\sin\theta\Rightarrow Z=r(\cos\theta+i\sin\theta)$**,** where r is the distance and θ is direction.

$(x+yi)^{2}=r^{2}(\cos\theta+i\sin\theta)^{2}=r^{2}\left(\cos^{2}\theta-\sin^{2}\theta+i(2\sin\theta\cos\theta)\right)=r^{2}(\cos 2\theta+i\sin 2\theta)$

- $x+yi=r(\cos\theta+i\sin\theta)$
- $(x+yi)^{2}=r^{2}(\cos 2\theta+i\sin 2\theta)$
- **Example 10.** Express the number $z=-2-2i$ in the polar (trigonometric) form



$r=\sqrt{(-2)^{2}+(-2)^{2}}=\sqrt{8}=2\sqrt{2}$

$\tan\phi=\dfrac{-2}{-2}=1\Rightarrow \phi=\tan^{-1}(1)\Rightarrow \phi=45^{o},\qquad \theta=180^{o}+\phi\Rightarrow \theta=225^{o}$

$$\therefore z=2\sqrt{2}\left(\cos 225^{o}+i\sin 225^{o}\right)$$

Multiplication of complex numbers (polar coordinates)

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then their product

$$z_1 z_2 = r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = r_1 r_2(\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + i^2\sin\theta_2\sin\theta_1)$$

$$\Rightarrow z_1 z_2 = r_1 r_2(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)) = r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$$

We have very easy calculation to find the products of two complex numbers in polar coordinates. We simply multiply the moduli and add the arguments.

Division of complex numbers (polar coordinates)

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then their quotient

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \times \frac{(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 - i\sin\theta_2)}$$

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1\cos\theta_2 - i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 - i^2\sin\theta_1\sin\theta_2)}{r_2(\cos^2\theta_2 - i\sin\theta_2\cos\theta_2 + i\sin\theta_2\cos\theta_2 - i^2\sin^2\theta_2)}$$

$$\Rightarrow \frac{z_1}{z_2} = \frac{r_1(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2))}{r_2(\cos^2\theta_2 + \sin^2\theta_2)} = \frac{r_1}{r_2}(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2))$$

We have very easy calculation to find the quotient of two complex numbers in polar coordinates. We simply find quotient of the moduli and the difference of the arguments.

- Using Euler (Leonhard Euler Swiss Mathematician 1707 - 1783) formula (Euler identity) (One of the most beautiful formula in mathematics and is the most important formula in complex analysis (is the heart of complex numbers)) $\cos\theta + i\sin\theta = e^{i\theta}$, $r^2(\cos\theta + i\sin\theta)^2 = r^2 e^{i2\theta}$
- $e^{i\frac{\pi}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$
- $e^{i\pi} = \cos\pi + i\sin\pi = -1$
- $i^i = \left(e^{i\frac{\pi}{2}}\right)^i = e^{-\frac{\pi}{2}}$ *is a real number*

$Z = r(\cos\theta + i\sin\theta)$ is usually abbreviated to $Z = r\angle\theta$ which is known as the **polar form** of a complex number.

Problem 5. For the complex number $z = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$ find $(i)\ r\ \ (ii)\ \theta(45^o)\ \ (iii)\ e^{i\theta}\ \ (iv)\ z^2$

Powers of complex numbers De Moivre's theorem

Abraham De Moivre (French, 1667 - 1754) in his later years he began to sleep more and more. It is reported that he predicted the day of his own death. After observing his sleep time increase each day by an additional 15 minutes he calculated the arithmetic progression until he will sleep forever. His calculations were correct. He died November 7, 1754.

Arithmetic progression (sequence): is a sequence of numbers such that the the difference between the consecutive terms is constant. For example 7, 11, 15, 19,………is a arithmetic progression with common difference of 4.

De Moivre's theorem:

If $z = r(\cos\theta + i\sin\theta)$ is a complex number and n is a positive integer then,

$$z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$

Proof of De Moivre's theorem:

To prove De Moivre's theorem, we use simple proof of induction. For a complex number $z = r(\cos\theta + i\sin\theta)$, we can easily show that by repeated multiplication for $n = 0,1,2,3,4,\cdots\cdots$

$$z^n = \left[r(\cos\theta + i\sin\theta)\right]^n = r^n(\cos\theta + i\sin\theta)^n = r^n(\cos n\theta + i\sin n\theta)$$

$z^n = r^n(\cos\theta + i\sin\theta)^n$

$Z^0 = r^0(\cos\theta + i\sin\theta)^0 = 1,$ for $n = 0$ (1)

$Z^1 = r^1(\cos\theta + i\sin\theta)^1 = r(\cos\theta + i\sin\theta)$ For $n = 1$ (2)

Let us assume that it is true for $n = k$

$\left[r(\cos\theta + i\sin\theta)\right]^k = r^k(\cos k\theta + i\sin k\theta)$

We must show that it is true for $n = k+1$, that is

$\left[r(\cos\theta + i\sin\theta)\right]^{k+1} = r^{k+1}(\cos(k+1)\theta + i\sin(k+1)\theta)$

As $\left[r(\cos\theta + i\sin\theta)\right]^k = r^k(\cos k\theta + i\sin k\theta)$

Multiply both sides by $r(\cos\theta + i\sin\theta)$

$\left[r(\cos\theta + i\sin\theta)\right]^k\left[r(\cos\theta + i\sin\theta)\right] = \left[r^k(\cos k\theta + i\sin k\theta)\right]\left[r(\cos\theta + i\sin\theta)\right]$

$\Rightarrow \left[r(\cos\theta + i\sin\theta)\right]^{k+1} = r^k r(\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$

$\Rightarrow \left[r(\cos\theta + i\sin\theta)\right]^{k+1} = r^{k+1}(\cos k\theta\cos\theta + i\cos k\theta\sin\theta + i\sin k\theta\cos\theta + i^2\sin k\theta\sin\theta)$

$\Rightarrow \left[r(\cos\theta + i\sin\theta)\right]^{k+1} = r^{k+1}(\cos k\theta\cos\theta + i\cos k\theta\sin\theta + i\sin k\theta\cos\theta + i^2\sin k\theta\sin\theta)$

$\Rightarrow \left[r(\cos\theta + i\sin\theta)\right]^{k+1} = r^{k+1}(\cos k\theta\cos\theta - \sin k\theta\sin\theta + i(\sin k\theta\cos\theta + \cos k\theta\sin\theta))$

By applying the use of trigonometric formulas for the sum of angles for sine and cosine, we get

$$\begin{bmatrix} \sin(x+y) = \sin x\cos y + \cos x\sin y & \\ \cos(x+y) = \cos x\cos y - \sin x\sin y, & x = k\theta, \quad y = \theta \end{bmatrix}$$

$\Rightarrow \left[r(\cos\theta + i\sin\theta)\right]^{k+1} = r^{k+1}(\cos(k\theta+\theta) + i\sin(k\theta+\theta)) = r^{k+1}(\cos(k+1)\theta + i\sin(k+1)\theta)$

So it is true for all positive integers.

Example 11. Using De Moivre's theorem we can easily compute the power of a complex number such as $z = 2 + 2i.$

We can write in polar form: $z = r(\cos\theta + i\sin\theta)$, with $r = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$ and

$\tan\phi = \frac{2}{2} = 1 \Rightarrow \phi = \tan^{-1}(1) \Rightarrow \phi = 45^o$, where z lies in the first quadrant.

$$z = 2\sqrt{2}(\cos 45^o + i\sin 45^o)$$

Then

$z^6 = (2+2i)^6 = \left[2\sqrt{2}(\cos 45^o + i\sin 45^o)\right]^6 = (2\sqrt{2})^6(\cos 6\times 45^o + i\sin 6\times 45^o)$

$= 512(\cos 270^o + i\sin 270^o) = -512i$

Problem 6. Using De Moivre's theorem find the power of a complex number.

(i) $2(\sqrt{3} + i)^5$ (ii) $\left[3(\cos 150^o + i\sin 150^o)\right]^4$ Answer: $(i) - 32\sqrt{3} + 32i$ $(ii) - 40.5 + -40.5\sqrt{3}i$

Example 12. Find the solution of the equation $x^5 - 243 = 0$, and represent the solution graphically.

$x^5 - 243 = 0 \Rightarrow x^5 = 243 \Rightarrow x = (243)^{\frac{1}{5}}$,

As $243 = 243(\cos 0 + i\sin 0) = 243(\cos(0+2\pi k) + i\sin(0+2\pi k))$, $where\ k = 0,1,2,3\cdots\cdots n-1$

Then, $x = \left[243(\cos(2\pi k) + i\sin(2\pi k))\right]^{\frac{1}{5}} = (243)^{\frac{1}{5}}\left(\cos\frac{2\pi k}{5} + i\sin\frac{2\pi k}{5}\right) = 3\left(\cos\frac{2\pi k}{5} + i\sin\frac{2\pi k}{5}\right)$

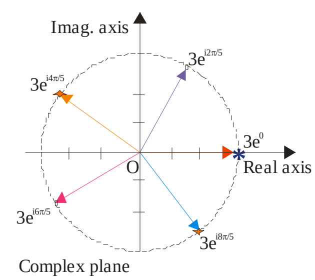
$\Rightarrow x = 3e^{\frac{i2\pi k}{5}}, where\ k = 0,1,2,3,4.$

$for\ k=0,\ x=3,$ $\qquad for\ k=1,\ x = 3e^{\frac{i2\pi}{5}} = 3\left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right) = 3\left(\cos 72^0 + i\sin 72^0\right),$

$for\ k=2,\ x = 3e^{\frac{i4\pi}{5}} = 3\left(\cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}\right) = 3\left(\cos 144^0 + i\sin 144^0\right),$

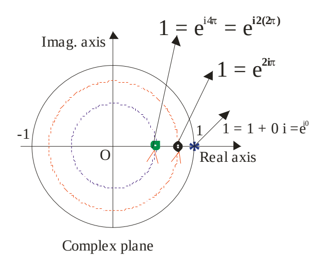
$for\ k=3,\ x = 3e^{\frac{i6\pi}{5}} = 3\left(\cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}\right) = 3\left(\cos 216^0 + i\sin 216^0\right),$

$for\ k=4,\ x = 3e^{\frac{i8\pi}{5}} = 3\left(\cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}\right) = 3\left(\cos 288^0 + i\sin 288^0\right).$



n^{th} roots of unity

$z^n = 1 \Rightarrow z = (1)^{\frac{1}{n}} = \sqrt[n]{1}$



$1 = e^{i0},\qquad 1 = e^{i2\pi},\qquad 1 = e^{i2(2\pi)},\qquad 1 = e^{ik(2\pi)},\qquad where\ k = 0,1,2,3\cdots\cdots n-1$

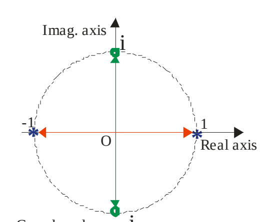
$z^n = 1 \Rightarrow z = (1)^{\frac{1}{n}} = \sqrt[n]{1} \Rightarrow z = \left(e^{ik2\pi}\right)^{\frac{1}{n}} \Rightarrow z = e^{\frac{ik2\pi}{n}},\qquad where\ k = 0,1,2,3\cdots\cdots n-1$

Example 13. Find 4th solution of the equation $z^4 = 1$, plot these solutions on the complex plane. $n = 4$

$z = \left(e^{ik2\pi}\right)^{\frac{1}{4}} \Rightarrow z = e^{\frac{ik\pi}{2}}\ where\ k = 0,1,2,3$

$for\ k=0,\ z = e^0 = 1,\quad for\ k=1,\ z = e^{\frac{i\pi}{2}} = i,\quad for\ k=2,\ z = e^{i\pi} = -1,\quad for\ k=3,\ z = e^{\frac{i3\pi}{2}} = -i$

$\{1,\ i,\ -1,\ -i\}$



Example 14. Find 8th solution of the equation $z^8 = 1$, plot these solutions on the complex plane. $n = 8$

$$z = e^{\frac{ik\pi}{n}}, \quad where\ k = 0,1,2,3\cdots\cdots n-1$$

$$with\ n = 8, \quad z = e^{\frac{ik\pi}{4}}\ where\ k = 0,1,2,3,4,5,6,7$$

$$for\ k = 0,\ z = e^0 = 1, \qquad for\ k = 1,\ z = e^{\frac{i\pi}{4}} = \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$$

$$for\ k = 2,\ z = e^{\frac{i\pi}{2}} = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = i, \qquad for\ k = 3,\ z = e^{\frac{i3\pi}{4}} = \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$$

$$for\ k = 4,\ z = e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1, \qquad for\ k = 5,\ z = e^{\frac{i5\pi}{4}} = \cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right) = -\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$$

$$for\ k = 6,\ z = e^{\frac{i3\pi}{2}} = \cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right) = -i, \qquad for\ k = 7,\ z = e^{\frac{i7\pi}{4}} = \cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right) = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$$