Compact Submanifolds in Euclidean Space *

Hanan Alohali
King Saud University
Mathematics department
P.O.Box 225899, Riyadh 11324
Saudi Arabia
hn_338@hotmail.com

Haila Alodan
King Saud University
Mathematics department
P.O.Box 22452, Riyadh 11495
Saudi Arabia
halodan1@ksu.edu.sa

Abstract: In this paper we study n-dimensional compact immersed submanifold M of a Euclidean space $(R^{n+p}, \langle , \rangle)$ with the immersion $\psi : M \to R^{n+p}$ under the restriction that the tangential component ψ^T of the position vector vector field ψ is a conformal vector field and find a characterization of a n-sphere in the Euclidean space R^{n+p}. We also find a condition under which the vector field ψ^T is a conformal vector field.

Key–Words: Ricci curvature, Conformal gradient vector field, Submanifolds.

1 Introduction

Given an immersed n-dimensional submanifold M of a Euclidean space $(R^{n+p}, \langle , \rangle)$, where $\langle , \rangle$ is the Euclidean metric. Of so many questions, one of the important questions is to find conditions under which the submanifold M lies on the hypersphere $S^{n+p-1}(c)$ of the Euclidean space R^{n+p} and this question has been studied in ([1]-[3]). Recall that a smooth vector field ξ on a Riemannian manifold (M, g) is said to be a conformal vector field if its flow consists of conformal transformations of the Riemannian manifold (M, g) and it is equivalent to the condition$\pounds_\xi g = 2\rho g$,where $\pounds_\xi$ is the Lie derivative with respect to the vector field ξ and ρ is a smooth function on M called the potential function of the conformal vector field ξ. Conformal vector fields have been used to characterize spheres among compact Riemannian manifolds (cf. [5]-[7]). If M is an n-dimensional immersed submanifold of the Euclidean space R^{n+p} with the immersion $\psi : M \to R^{n+p}$, then treating ψ as position vector field of points of M, we have$\psi = \psi^T + \psi^\perp$,where ψ^T is the tangential component of ψ to M and $\psi^\perp$ is the normal component of ψ. Thus it is a natural question to find conditions under which the vector field ψ^T is a conformal vector field on M as well as to study the geometry of the submanifold for which the vector field ψ^T is a conformal vector field. In this paper we answer this question as well as show that if ψ^T is a conformal vector field then under certain curvature conditions M either lies on a hypersphere $S^{n+p-1}(c)$ or is isometric to a sphere $S^n(c)$.

2 Preliminaries

Let M be an n-dimensional submanifold of the Euclidean space R^{n+p} with immersion $\psi : M \to R^{n+p}$. We denote by $\langle , \rangle$ and $\overline{\nabla}$ the euclidean metric and the Euclidean connection on $R^{n \div p}$, we also denote by the letter g and by ∇ the induced metric and the Riemannian connection on the submanifold M . Then we have the following equations for the submanifold M.

$$\begin{aligned} \overline{\nabla}_X Y &= \nabla_X Y + h(X, Y), \\ \overline{\nabla}_X N &= -A_N X + \nabla^\perp_X N \end{aligned} \tag{1}$$

$X, Y \in \mathfrak{X}(M)$, $N \in \Gamma(\upsilon)$, where $\mathfrak{X}(M)$ is the Lie algebra of smooth vector fields on M, $\Gamma(\upsilon)$ is the space of smooth sections of the normal bundle υ of M, h is the second fundamental form, A_N

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is the Weingarten map with respect to the normal $N \in \Gamma(v)$ which is related to the second fundamental form h by

$$g(A_N X, Y) = g(h(X,Y), N), \quad X, Y \in \mathfrak{X}(M)$$

and $\nabla^{\perp}$ is the connection in the normal bundle v. We also have the following equation

$$R(X,Y)Z = A_{h(Y,Z)}X - A_{h(X,Z)}Y. \quad (2)$$

where $R(X,Y)Z$, $X, Y, Z \in \mathfrak{X}(M)$ is the curvature tensor field of the submanifold M. The Ricci tensor field of M is given by

$$\begin{aligned} Ric(X,Y) &= ng(h(X,Y), H) \\ &- \sum_{i=1}^{n} g(h(X,e_i), h(Y,e_i)), \end{aligned} \quad (3)$$

where $\{e_1, ..., e_n\}$ is a local orthonormal frame on M and

$$H = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i)$$

is the mean curvature vector field. The Ricci operator Q is a symmetric operator defined by

$$Ric(X,Y) = g(Q(X), Y), \qquad X, Y \in \mathfrak{X}(M).$$

If we express $\psi = \psi^T + \psi^{\perp}$, where $\psi^T \in \mathfrak{X}(M)$ is the tangential component and $\psi^{\perp} \in \Gamma(v)$ is the normal component of ψ, and if we denote by $B = A_{\psi^{\perp}}$ the Weingarten map with respect to the normal vector field $\psi^{\perp}$ then using the equation (1), we have

$$\begin{aligned} \nabla_x \psi^T &= X + BX, \\ \nabla_X^{\perp}\psi^{\perp} &= -h\left(X, \psi^T\right), \quad X, Y \in \mathfrak{X}(M) \end{aligned} \quad (4)$$

We use the mean curvature vector field H to define a smooth function $F : M \to R$ on the submanifold M by $F = \langle H, \psi^{\perp}\rangle$. Now for an n-dimensional compact submanifold $\psi : M \to R^{n+p}$, and a local orthonormal frame $\{e_1, ..., e_n\}$ on M, we have

$$\begin{aligned} div\psi^T &= \sum_{i=1}^{n}\left\langle \nabla_{e_i}\psi^T, e_i\right\rangle \\ &= n(1+F). \end{aligned}$$

$$\therefore div\psi^T = n(1+F) \quad (5)$$

We have the following Lemmas:

Lemma 2.1 [1] Let M be an n-dimensional compact submanifold of the Euclidean space R^{n+p}, then $\int_M (1+F)dv = 0$

Lemma 2.2 [1] Let M be an n-dimensional submanifold of R^{n+p} then the tensor field B satisfies

(i) $trB = nF$

(ii) $(\nabla B)(X,Y) - (\nabla B)(Y,X) = R(X,Y)\psi^T$

(iii) $\sum_{i=1}^{n}(\nabla B)(e_i, e_i) = n\nabla F + Q\left(\psi^T\right)$,

where $(\nabla B)(X,Y) = \nabla_X BY - B\nabla_X Y$ and $X, Y \in \chi(M)$.

Lemma 2.3 [1] Let $\psi : M \to R^{n+p}$ be an n-dimensional compact submanifold. Then a necessary and sufficient condition for $\psi(M) \subseteq S^{n+p-1}$ is that $\psi^T = 0$ and $F = -1$.

Definition 2.1 A smooth vector field ξ on a Riemannian manifold (M, g) is said to be a conformal vector field if there exists a smooth function ρ on M that satisfies $\pounds_\xi g = 2\rho g$, ρ called a potential function, where $\pounds_\xi g$ is the Lie derivative of g with respect to ξ. We say that ξ is non trivial conformal vector field if the potential function ρ is not a constant. A conformal vector field ξ is said to be gradient conformal vector field if $\xi = \nabla f$ for a smooth function f on M. Using Koszul's formula we immediately obtain the following for a vector field ξ on M

$$\begin{aligned} 2g(\nabla_X \xi, Y) &= (\pounds_\xi g)(X,Y) \\ &+ d\eta(X,Y) \qquad X, Y \in \mathfrak{X}(M) \end{aligned}$$

where η is the 1-form dual to ξ, that is $\eta(X) = g(X, \xi)$, $X \in \mathfrak{X}(M)$. Define a skew-symmetric tensor field φ of type (1,1) on M by

$$d\eta(X,Y) = 2g(\varphi X, Y), \quad X, Y \in \mathfrak{X}(M).$$

Then using the definition of a conformal vector field, we have

Lemma 2.4 [5] Let ξ be a conformal vector field on an n-dimensional Riemannian manifold $(M.g)$, with potential function ρ. Then

$$\nabla_X \xi = \rho X + \varphi X, \quad X \in \mathfrak{X}(M)$$

and

$$div\xi = n\rho$$

Lemma 2.5 [6] Let ξ be a conformal gradient vector field on a compact Riemannian manifold (M, g), then for $\rho = n^{-1}div\xi$,

$$\int_M \rho dv = 0$$

3. .Submanifolds with ψ^T as conformal vector field

Let M be an n-dimensional submanifold of the Euclidean space R^{n+p} , with immersion $\psi : M \to R^{n+p}$. In this section we study the geometry of the submanifold M for which the vector field ψ^T is a conformal vector field. First we prove the following lemmas.

Lemma 3.1 Let M be an n-dimensional submanifold of the Euclidean space R^{n+p}, with immersion $\psi : M \to R^{n+p}$ and $f = \frac{1}{2}\left\|\psi^{\perp}\right\|^2$. If the gradient ∇f of the smooth function f is a conformal vector field, then

$$Ric\left(\psi^T, \psi^T\right) + n\psi^T (F) + n\rho + nF + \|B\|^2 = 0,$$

where ρ is the potential function of ∇f.

Proof. As ∇f is conformal vector field with potential function say ρ, we have$\pounds_{\nabla f} g = 2\rho g$. Put $B = A_{\psi^\perp}$ and let η be the 1-form dual to ∇f, then $\eta(X) = g(X, \nabla f) = X(f)$, and define skew symmetric tensor field

rphi $of type (1, 1) on M$ $by d\eta(X, Y) = 2g(\varphi X, Y)$, $X, Y \in \mathfrak{X}(M)$. Then by lemma2.4 $\nabla_X (\nabla f) = \rho X + \varphi X$ and $div(\nabla f) = n\rho$, and η is closed 1-form, which gives ∇f is also closed and as $d\eta(X, Y) = 2g(\varphi X, Y) = 0$, $X, Y \in \mathfrak{X}(M)$, we get that $\varphi = 0$. Thus

$$\nabla_X (\nabla f) = \rho X, and \quad \Delta f = n\rho, \qquad (6)$$

where Δ is the Laplacian operator. Now for $X \in \mathfrak{X}(M)$, we have

$$\begin{aligned} g(\nabla f, X) &= X(f) = X\left(\frac{1}{2}\left\|\psi^{\perp}\right\|^2\right) \\ &= -\left(A_{\psi^\perp}\psi^T, X\right) \end{aligned}$$

which gives

$$\nabla f = -A_{\psi^\perp}\psi^T = -B\psi^T.$$

Put $\xi = \psi^T$ and thus $\nabla f = -B\xi$ gives,

$$\nabla_X (\nabla f) = -B\xi = -\left[(\nabla B)(X, \xi) + B\nabla_X \xi\right],$$

which on using equation (4) gives

$$\nabla_X (\nabla f) = -(\nabla B)(X, \xi) - BX - B^2 X \quad (7)$$

Now using the lemma 2.2 and the fact that B is a symmetric operator, we have

$$\begin{aligned} \sum_{i=1}^n g\left((\nabla B)(e_i, \xi), e_i\right. &= g\left(\sum_{i=1}^n (\nabla B)(e_i, e_i), \xi\right) \\ &= n\xi(F) + Ric(\xi, \xi) \qquad (8) \end{aligned}$$

Also, using the equations (6) and (7), we have

$$\sum_{i=1}^n g\left((\nabla B)(e_i, \xi), e_i\right) = -n\rho - trB - \|B\|^2 \qquad (9)$$

Thus using $trB = nF$ and the equations (8) and (9), we have

$$Ric(\xi, \xi) + n\xi(F) + n\rho + nF + \|B\|^2 = 0,$$

which proves the Lemma.

Lemma 3.2 *Let $\psi : M \to R^{n+p}$ be an n-dimensional compact submanifold. Then*

$$\int_M \left\{Ric\left(\psi^T, \psi^T\right) - n^2 (1 + F)^2 + \|B\|^2 - n dv = 0.\right.$$

Proof. Consider a local orthonormal frame $\{e_1, ..., e_n\}$, and put $\xi = \psi^T$, then using lemma 2.2 and equation (5) to compute $div(B\xi)$, we have

$$\begin{aligned} div(F\xi) &= \xi(F) + F div\xi \\ &= g(\nabla F, \xi) + nF(1 + F) \end{aligned}$$

We find after simple calculations that,

$$\begin{aligned} div(B\xi) &= \sum_{i=1}^n g\left(\nabla_{e_i} B\xi, e_i\right) \\ &= ng(\nabla F, \xi) + Ric(\xi, \xi) \\ &+ nF + \|B\|^2, \end{aligned}$$

and

$$g(\nabla F,\xi)=div(F\xi)-nF^2-nF,$$

which gives

$$ng(\nabla F,\xi)=ndiv(F\xi)-n^2F^2-n^2F.$$

Thus

$$\begin{aligned} div(B\xi) &= ndiv(F\xi)-n^2F^2-n^2F \\ &+ Ric(\xi,\xi)+nF+\|B\|^2 \end{aligned}$$

and we have

$$\begin{aligned} div(B\xi-nF\xi) &= Ric(\xi,\xi)-n^2F^2 \\ &- n^2F+nF+\|B\|^2, \end{aligned}$$

which on integration gives

$$\int_M \left\{Ric(\xi,\xi)-n^2\left(F^2-1\right)+\|B\|^2-n\right\}dv=0 \quad (10)$$

Also using Lemma 2.1, we have

$$\int_M (1+F)^2\,dv=\int_M \left(F^2-1\right)dv,$$

thus using the above equation in the equation (10), we get

$$\int_M \left\{Ric(\xi,\xi)-n^2(1+F)^2+\|B\|^2-n\right\}dv=0.$$

Theorem 3.1 *Let $\psi : M \to R^{n+p}$ be an n-dimensional compact submanifold with the tangential component ψ^T a nonzero conformal vector field with potential function ρ. If the Ricci tensor on M satisfies*

(i) $Ric\left(\nabla\rho+c\psi^T,\nabla\rho+c\psi^T\right)>0.$

(ii) $Ric(\nabla\rho,\nabla\rho)\le(n-1)c\|\nabla\rho\|^2$ *for a constant c.*

Then M is isometric to a sphere $S^n(c)$.

Proof. *For $\psi=\psi^T+\psi^\perp$ let $\xi=\psi^T$ be a conformal vector field with potential function ρ. If we define $f=\frac{1}{2}\|\psi\|^2$, then it is straight forward to show that $\xi=\nabla f$. Thus ξ is a gradient conformal vector field and the 1-form η dual to ξ is $\eta=df$, and consequently $d\eta=d^2f=0$ and that $\varphi=0$ Thus the Lemma 2.4 gives $\nabla_X\xi=\rho X$. We have*

$$\nabla_X\xi=\nabla_X\psi^T=BX+X=\rho X,$$

which gives $B=(\rho-1)I$ and $div\xi=n\rho$. However as $\xi=\nabla f$, we have $\Delta f=n\rho$. We have$\nabla_X\xi=\nabla_X\psi^T=BX+X=\rho X$,which gives $B=(\rho-1)I$ and $div\xi=n\rho$. However as $\xi=\nabla f$, we have $\Delta f=n\rho$. Thus we have

$$\therefore (\nabla B)(X,Y)=X(\rho)Y$$

which together with Lemma 2.2 gives

$$X(\rho)Y-Y(\rho)X=R(X,Y)\nabla f \quad (11)$$

We get

$$\begin{aligned} Ric(\xi,X) &= g(Q(\xi),X) \\ &= \sum_{i=1}^{n} R(e_i,X;\xi,e_i) \\ &= g(X,\nabla\rho)-nX(\rho) \end{aligned}$$

consequently, we have

$$Q(\xi)=-(n-1)\nabla\rho \quad (12)$$

Also we compute,

$$Ric(\xi,\xi)=-(n-1)div(\rho\xi)+n(n-1)\rho^2 \quad (13)$$

Also, the equation (12) gives

$$\begin{aligned} Ric(\xi,\nabla\rho) &= g(-(n-1)\nabla\rho,\nabla\rho) \\ &= -(n-1)\|\nabla\rho\|^2 \quad (14) \end{aligned}$$

Let λ_1 be the first non zero eigenvalue of the Laplacian operator on M. Then the Lemma2.5 together with the minimum principle implies

$$\int_M \|\nabla\rho\|^2\,dv\ge\lambda_{1M}\rho^2dv \quad (15)$$

Using $c=\frac{\lambda_1}{n}$ and the equations (13), (14) and (15), we get$\int_M Ric(\nabla\rho+c\xi,\rho+c\xi)\,dv=$
$\int_M\{Ric(\nabla\rho,\nabla\rho)+n(n-1)c^2\rho^2$
$-2(n-1)c\|\nabla\rho\|^2\,dv$
$\le\int_M\left\{Ric(\nabla\rho,\nabla\rho)-(n-1)c\|\nabla\rho\|^2\right\}dv,$
Using the conditions in the statement, and the

above inequality, we conclude that $\nabla \rho + c\xi = 0$, *which gives*

$$\nabla_X (\nabla \rho) = -c\rho X, \quad X \in \mathfrak{X}(M).$$

Thus if ρ *is non-constant, then the above equitation gives that* M *is isometric to an* n*-sphere by Obata's theorem (cf. [8]). If* ρ *is constant then*$\nabla \rho = 0 \implies \xi = 0$,*which gives a contradiction as* ξ *is a nonzero conformal vector field.*

In the following result, we consider a conformal vector field given by the normal component $\psi^{\perp}$ *and it is interesting to know that in this case we get the criterion for the submanifold to lie on the hypersphere in the Euclidean space.*

Theorem 3.2 *Let* $\psi : M \to R^{n+p}$ *be an* n*-dimensional compact submanifold with mean curvature* H. *Suppose* $\nabla^{\perp}_{\psi^T} H = 0$ *and* ∇f *is a conformal vector field, where* $f = \frac{1}{2}\left\|\psi^{\perp}\right\|^2$. *Then* $h\left(\psi^T, \psi^T\right) = 0$ *if and only if* $\psi(M) \subseteq S^{n+p-1}(c)$ *for some constant* $c > 0$.

Proof. Suppose $h\left(\psi^T, \psi^T\right) = 0$. *Then on taking* $\xi = \psi^T$,

$$\begin{aligned} \xi(F) = &= \xi g\left(H, \psi^{\perp}\right) \\ &= -g(H, h(\xi, \xi)) = 0 \end{aligned}$$

that is, $\xi(F) = 0$, *which together with Lemma 3.1 gives,*$Ric(\xi, \xi) + n\xi(F) + n\rho + nF + \|B\|^2 = 0$. *Integrating the above equation, we get*$\int_M \left\{Ric(\xi, \xi) + nF + \|B\|^2\right\} dv = \int_M \left\{Ric(\xi, \xi) + \|B\|^2 - n\right\} dv = 0$,*where we used the Lemma 2.1. Now by lemma 3.2 in the above equation, we get*$\int_M -n^2(1+F)^2 dv = 0$,*that is,* $F = -1$.*Thus by Lemma 2.3,* $\psi(M) \subseteq S^{n+p-1}(c)$ *for some constant* $c > 0$. *Conversely, if* $\psi(M) \subseteq S^{n+p-1}(c)$, $c > 0$ *then by lemma 2.3* $F = -1$ *and* $\psi^T = 0$, *thus* $h(\xi, \xi) = 0$.

In the next result, we find the condition under which the vector field ψ^T *becomes a conformal vector field on* M.

Theorem 3.3 *Let* $\psi : M \to R^{n+p}$ *be an* n*-dimensional compact submanifold, with* $\lambda = \inf \frac{1}{n-1} Ric$.*If* $\left\|\psi^T\right\|^2 \geq \frac{n}{\lambda}(1+F)^2$, *then* ψ^T *is a conformal vector field on* M.

Proof. Taking $\xi = \psi^T$, *in the Lemma 3.2 ,we get*

$$\int_M \left\{Ric(\xi, \xi) - n^2(1+F)^2 + \|B\|^2 - n\right\} dv = 0,$$

which gives $\int_M \left\{\left(Ric(\xi, \xi) - \lambda(n-1)\|\xi\|^2\right) + \left(\|B\|^2 - nF^2\right) + \left((n-1)\left(\lambda\|\xi\|^2 - n(1+F)^2\right)\right)\right\} dv = 0$. *Using* $Ric(\xi, \xi) \geq (n-1)\lambda\|\xi\|^2$, *and the Schwarz inequality* $\|B\|^2 \geq nF^2$ *and the condition in the statement* $\lambda\|\xi\|^2 \geq n(1+F)^2$ *in the above equation, we get the equality* $\|B\|^2 = nF^2$, *which holds if and only if* $B = FI$. *Thus*$\nabla_X \xi = BX + X = FX + X = (1+F)X = \rho X$,*that is* $\pounds_{\xi} g = 2\rho g$,*which proves that* $\xi = \psi^T$ *is a conformal vector field.*

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