ME 581 Advanced Fluid Mechanics **3(3+0)**

Fundamental equations: continuity, momentum, energy; Elements of potential flow theory; Laminar flow; Selected exact solutions; Computational solution; Thin shear-layer approximation; Simple shear layers and more complex flows

Detailed Course Content

Basic Concepts and Fundamentals: Definition and properties of Fluids, Fluid as continuum, Langragian and Eulerian description, Velocity and stress field, Fluid statics, Fluid Kinematics.

Governing Equations of Fluid Motion and Some exact solutions of Navier-Stokes Equations: Reynolds transport theorem, Integral and differential forms of governing equations: mass, momentum and energy conservation equations, Navier-Stokes equations, Euler's equation, Bernoulli's Equation, Couette flows, Poiseuille flows, fully developed flows in non-circular cross-sections, Unsteady flows, Creeping flows.

Potential Flows: Revisit of fluid kinematics, Stream and Velocity potential function, Circulation, Irrotational vortex, Basic plane potential flows: Uniform stream; Source and Sink; Vortex flow, Doublet, Superposition of basic plane potential flows, Flow past a circular cylinder, Magnus effect; Kutta-Joukowski lift theorem; Concept of lift and drag.

Laminar Boundary Layers: Boundary layer equations, Boundary layer thickness, Boundary layer on a flat plate, similarity solutions, Integral form of boundary layer equations, Approximate Methods, Flow separation, Entry flow into a duct.

Turbulent Flow: Concept of small-disturbance stability, Orr-Somerfield equation, Inviscid stability theory, Boundary layer stability, Thermal instability, Transition to turbulence. Introduction to turbulent boundary layer, Fluctuations and time-averaging, General equations of turbulent flow, Turbulent boundary layer equation, Flat plate turbulent boundary layer, Turbulent pipe flow, Prandtl mixing hypothesis, Turbulence modeling, Free turbulent flows

Introduction to Computational Fluid Dynamics (CFD): Boundary conditions, Basic discretization Finite difference method, Finite volume method and Finite element method.

Ch:2 Fundamental Equations of Flow Fluid

There are three basic equations of fluid Mechanics

conservation of mass
conservation of momentum
conservation of Energy

Following are the unknown are
Velocity, Thermodynamic pressure and Absolute Temperature
There are other four parameter
Density ρ, Enthalpy h, Viscosity μ and coefficient of thermal conductivity k. ρ, h, μ and k are functions of P, T.
$\rho = \rho(p\,T)$, $h = h(p,T)$; $\mu = \mu(P,T)$ and $k = k(p,T)$.

$h = C_p T$, $U = C_v T$

Conservation of mass → Continuity Equation

$$\frac{Dm}{Dt} = \frac{D}{Dt}(\rho V) = \rho\frac{DV}{Dt} + V\frac{D\rho}{Dt} = 0 \quad -(1)$$

$\frac{DV}{Dt}$ = Total Dilatation $\quad \frac{1}{V}\frac{DV}{Dt} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$

$$\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \operatorname{div} V = \nabla \cdot V$$

Equation (1) can be written as

$$\frac{D\rho}{Dt} + \frac{\rho}{V}\frac{DV}{Dt} = 0$$

$$\frac{D\rho}{Dt} + \rho\,\nabla \cdot V = 0 \quad -(2)$$

$$\frac{\partial \rho}{\partial T} + \operatorname{div} \rho V = \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho V) = 0 \quad -(3)$$

Therefore general continuity equation is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho V) = 0 \quad -(4)$$

If flow is incompressible (Density is constant)

$$\nabla V = 0 \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad -(5)$$

(This involves constant volume)

For 2D case - Steady 2D flow equation (4) gives

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \quad -(6)$$

If we define stream function ψ such that

$\rho u = \frac{\partial \psi}{\partial y} \quad \rho v = -\frac{\partial \psi}{\partial x}$ then equation 6 becomes

$$\frac{\partial^2 \psi}{\partial x dy} - \frac{\partial^2 \psi}{\partial y dx} = 0 \qquad \psi = \text{Second Order derivative}$$

$$\psi = \psi(x, y)$$

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = -\rho v dx + \rho u dy$$

$d\psi = \rho V . dA = dm$

Constant stream lines ψ = constant $d\psi = 0$ $dm = 0$

∴ Stream lines for which ψ is constant are lines across which no mass flow takes place.

Conservation of momentum

This is statement of Newton's second law of motion

$F = ma$ — ①

$\rho V \frac{DV}{Dt} = F = F_{body} + F_{surface}$

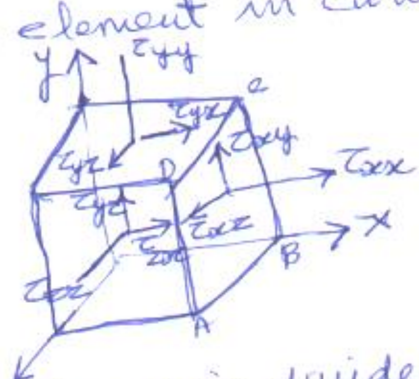
$\rho \frac{DV}{Dt} = f = f_{body} + f_{surface}$

f = applied force per unit volume

f_b = body force that acts on entire mass of system

f_s = surface force that act on surface of system

Consider fluid element in cartesian coordinates



Surface force is again divided into 2 components

① forces normal to surface τ_{xx}

② force on the plane of surface τ_{xy}

On the surface ABCD

The normal to surface is in positive direction of x axis The forces are

τ_{xx} Normal to face ABCD

τ_{xy} force acting on surface of ABCD is y direction

τ_{xz} force acting on surface of ABCD in z direction

The body force is only gravitational force. Ignore other force

$$f_{body} = \rho g$$

∴ The shear stress is a tensor

$$\tau_{ij} = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}$$

$$\therefore dF_x = \tau_{xx}\, dy\, dz + \tau_{yx}\, dx\, dz + \tau_{zx}\, dx\, dy$$

$$\therefore f_x = \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} \qquad \tau_{ij} = \tau_{ji}$$

$$\therefore f_s = \nabla \cdot \tau_{ij}$$

∴ Momentum equation becomes

$$\rho \frac{DV}{Dt} = \rho g + \nabla \cdot \tau_{ij}$$

This can be written as

$$\rho \frac{DV}{Dt} = \rho g - \nabla p + \frac{\partial}{\partial x_i}\left[\mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) + \delta_{ij} \lambda \nabla \cdot V\right]$$

λ = coefficient of bulk modulus
μ = coefficient of dynamic viscosity

δ_{ij} = Kronecker Delta $\delta_{ij} = 1 \quad i = j$
$\delta_{ij} = 0 \quad i \neq j$

If flow is incompressible $\nabla \cdot V = 0$

$$\rho \frac{DV}{Dt} = \rho g - \nabla P + \frac{\partial}{\partial x_j}\left[\mu \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right]$$

If the flow is incompressible with constant viscosity

$$\rho \frac{DV}{Dt} = \rho g - \nabla P + \mu \nabla^2 V \Rightarrow \frac{DV}{Dt} = g - \frac{\nabla P}{\rho} + \nu \nabla^2 V$$

This equation can be written in x, y, z coordinat

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = 0 - \frac{1}{\rho}\frac{\partial P}{dx} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = g - \frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = 0 - \frac{1}{\rho}\frac{\partial P}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

Energy Equation

Energy equation is the statement of first law of Thermodynamics.

$$dE = dQ + dW$$

$$E = \rho\left(e + \frac{V^2}{2} - g\cdot r\right)$$

Q = heat added
W = work done on system
E = total energy
e = Internal energy
r = displacement
$\frac{V^2}{2}$ = kinetic energy

$$\therefore \frac{DE}{Dt} = \frac{DQ}{Dt} + \frac{DW}{Dt}$$

$$\frac{DE}{Dt} = \rho\left(\frac{De}{Dt} + V\frac{DV}{Dt} - g\cdot V\right)$$

$$\frac{DQ}{Dt} = \nabla.(k\nabla T) \qquad \rightarrow q = \frac{A\Delta T k}{L}$$

$$\frac{DW}{Dt} = \nabla.(V.\tau_{ij}) \qquad \rightarrow w_x = -(u\tau_{xx} + v\tau_{xy} + w\tau_{xz})$$

$$\nabla.(V.\tau_{ij}) = V.(\nabla.\tau_{ij}) + \tau_{ij}\frac{du_i}{dx_j}$$

$$V.\tau_{ij} = \rho\left(\frac{DV}{Dt} - g\right) \text{ from momentum equation}$$

$$\therefore \rho\frac{De}{Dt} = \nabla.(k\nabla T) + \tau_{ij}\frac{\partial u_i}{\partial x_j}$$

$$\tau_{ij}\frac{du_i}{\partial x_j} = \tau'_{ij}\frac{\partial u_i}{\partial x_j} - p\nabla V$$

$$p.\nabla.V = -\frac{p}{\rho}\frac{D\rho}{Dt} = \rho\frac{D\left(\frac{p}{\rho}\right)}{Dt} - \frac{Dp}{Dt}$$

$$h = e + \frac{p}{\rho}$$

$$\boxed{\rho\frac{Dh}{Dt} = \frac{DP}{Dt} + \nabla.(k\nabla T) + \tau_{ij}\frac{du_i}{\partial x_j}}$$

Boundary conditions

① At fluid solid interface $V_{fluid} = V_{solid}$

$T_{fluid} = T_{solid}$

$\left(k \frac{dT}{\partial n}\right)_{fluid} = q$ solid - liquid

② Interface of solid and gas

$V_{n\ liquid} = \frac{\partial n}{\partial t}$

$P_{liq} = P_{atm}$

$\left(\frac{\partial V}{\partial n}\right)_{liquid} = \left(\frac{\partial T}{\partial n}\right)_{liquid} = 0$

③ At inlet and Exit P, V and T should be known

Mathematical character of equation

$$A \frac{\partial^2 \phi}{dx^2} + B \frac{\partial^2 \phi}{\partial x \partial y} + C \frac{\partial^2 \phi}{\partial y^2} = D$$

$B^2 - 4AC < 0$ equation is elliptic – Boundary value Problem

$B^2 - 4AC = 0$ equation is parabolic – mixed initial value, Boundary value

$B^2 - 4AC > 0$ equation is hyperbolic → Initial Value

Example

Laplace Equation $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$ Elliptical

Heat conduction equation $\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial \phi}{\partial y} = 0$ Parabolic

Wave equation $\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2} = 0$ hyperbolic

Reynolds transport theorem

Reynolds transport theorem (also known as the Leibniz-Reynolds transport theorem), or in short **Reynolds theorem**, is a three-dimensional generalization of the Leibniz integral rule. This theorem is used to compute derivatives of integrated quantities.

Reynolds transport theorem can be simply stated as - What was already there plus what goes in minus what comes out is equal to what is there. Reynolds theorem is used in formulating the basic conservation laws of continuum mechanics, particularly fluid dynamics and large-deformation solid mechanics. These conservation laws (law of conservation of mass, law of conservation of linear momentum, and law of conservation of energy) are adopted from classical mechanics and thermodynamics where the system approach is normally followed. In fluid mechanics, it is often more convenient to work with control volumes as it is difficult to identify and follow a system of fluid particles. Thus, there is a need to relate the system equations and corresponding control volume equations. The link between the two is given by the Reynolds transport theorem. The theorem is named after Osborne Reynolds (1842–1912).

Imagine a system and a coinciding control volume with a control surface. Reynolds transport theorem states that the rate of change of an extensive property N, for the system is equal to the time rate of change of N within the control volume and the net rate of flux of the property N through the control surface. For an example, the law of conservation of mass states that rate of change of the property, mass, is equal to the sum of the rate of accumulation of mass within a control volume and the net rate of flow of mass across the control surface.

The differential forms of these equations with additional assumption of Newton's viscosity law are commonly known as the Navier-Stokes equations.

General form

The Reynolds transport theorem refers to any extensive property, N, of the fluid in a particular control volume. It is expressed in terms of a substantive derivative on the left-hand side.

$$\frac{DN_{sys}}{Dt} = \int_{c.v.} \frac{\partial}{\partial t}(\rho\eta)dV + \int_{c.s.} \rho\eta\vec{v}_b \cdot \hat{n}dA + \int_{c.s.} \rho\eta\vec{v}_r \cdot \hat{n}dA,$$

where η is the intensive property related to extensive property N, (N per unit mass), t is time, $c.v.$ refers to the control volume, $c.s.$ refers to the control surface, ρ is the fluid density, V is the volume, υ_b is the velocity of the boundary of the control volume (the control surface), υ_r is the velocity of the fluid with respect to the control surface, n is the outward pointing normal vector on the control surface, and A is the area.

Mass formulation

Also called the continuity equation, the control volume form of the conservation of mass is found by substituting mass in for N. This means that η is equal to 1.

$$0 = \int_{c.v.} \frac{\partial \rho}{\partial t} dV + \int_{c.s.} \rho\vec{v}_b \cdot \hat{n}\, dA + \int_{c.s.} \rho\vec{v}_r \cdot \hat{n}\, dA$$

All variables are defined as in the general formulation. M is equal to the mass of the control volume. Applying the Conservation of mass principle, the left hand side reduces to 0 since mass of a system cannot change in time. In a steady flow system, the first term on the right hand side of the equation will be equal to 0, i.e. the mass of the control volume does not change, implying that the mass flow rate into the control volume is equal to the mass flow rate out of the control volume.

Momentum formulation

The momentum equation is found by substituting momentum in for *N*. From this, η is found to be velocity. From Newton's second law, we have the time rate of change of momentum (now the left hand side of the equation) is equal to the net force. Thus,

$$\sum \vec{F} = \int_{c.v.} \frac{\partial}{\partial t}(\rho\vec{v})dV + \int_{c.s.} \rho\vec{v}(\vec{v}_b \cdot \hat{n})dA + \int_{c.s.} \rho\vec{v}(\vec{v}_r \cdot \hat{n})dA,$$

where *F* is force, υ is the velocity of fluid in a coordinate system attached to the control surface, and all other variables are defined as in the general formulation. Note that the integral form of the momentum equation is a vector equation.

Energy formulation

The energy equation is found by substituting energy in for *N*. From this, η is found to be energy per unit mass.

$$\dot{Q} - \sum \dot{W} = \int_{c.v.} \frac{\partial}{\partial t}\left[\rho\left(\frac{v^2}{2} + gz + \tilde{u}\right)\right] dV + \int_{c.s.} \left[\frac{v^2}{2} + gz + \tilde{u} + \frac{p}{\rho}\right] \rho\vec{v}_b \cdot \hat{n} dA + \int_{c.s.} \left[\frac{v^2}{2} +\right.$$

where *Q* is the heat transfer into the control volume, *W* is the work done by the system, *g* is the acceleration due to gravity, *z* is the vertical distance from an arbitrary datum, $\tilde{u}$is the specific internal energy of the fluid, *p* is the pressure and all other variables are defined as in the general formulation.

Note that these equations make no consideration for chemical reactions or potential energy associated with electromagnetic fields.

Formulation used in solid mechanics

Suppose $\Omega(t)$ is a region in Euclidean space with boundary $\partial\Omega(t)$, and let $\mathbf{n}(\mathbf{x}, t)$be the outward unit normal to the boundary at time *t*. Let $\mathbf{x}(t)$be the positions of points in the region, $\mathbf{v}(\mathbf{x}, t)$the velocity field in the region, and let $\mathbf{f}(\mathbf{x}, t)$be a vector field in the region (it may also be a scalar field). Reynolds transport theorem states that[1]

$$\frac{d}{dt}\left(\int_{\Omega(t)} \mathbf{f}\ \mathrm{dV}\right) = \int_{\Omega(t)} \frac{\partial \mathbf{f}}{\partial t}\ \mathrm{dV} + \int_{\partial\Omega(t)} (\mathbf{v} \cdot \mathbf{n})\mathbf{f}\ \mathrm{dA}\ .$$

Axisymmetric Flow

We now turn to inviscid, incompressible, axisymmetric potential flow. Using cylindrical coordinates, (r, θ, z), where $r = 0$ is the axis of the axisymmetric flow and (u_r, u_θ, u_z) are the velocities in those (r, θ, z) directions the continuity equation (see equation (Bce11)) is

$$\frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{\partial(u_z)}{\partial z} = 0 \quad \text{(Bgfa1)}$$

and this allows the definition of another stream function, ψ, known as Stokes' stream function (different from the stream function used in planar flow) defined as

$$u_z = \frac{1}{r}\frac{\partial\psi}{\partial r} \quad ; \quad u_r = -\frac{1}{r}\frac{\partial\psi}{\partial z} \quad \text{(Bgfa2)}$$

and whose definition automatically assures that the continuity equation (Bgfa1) is satisfied.

For future reference we also note that the vorticity components in incompressible axisymmetric flow (see equations (Bba27) to (Bba29) with $\partial/\partial\theta$ terms set to zero) are

$$\omega_r = -\frac{\partial u_\theta}{\partial z} \quad \text{(Bgfa3)}$$

$$\omega_\theta = \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \quad \text{(Bgfa4)}$$

$$\omega_z = \frac{1}{r}\frac{\partial(ru_\theta)}{\partial r} \quad \text{(Bgfa5)}$$

and in the absence of swirl $(u_\theta = 0)$ only the θ component remains:

$$\omega_r = \omega_z = 0 \quad \text{and} \quad \omega_\theta = \omega = \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \quad \text{(Bgfa6)}$$

Deleting the viscous terms and absorbing the force field terms into the pressure, the equations of motion (equations (Bhg1) to (Bhg3)) yield

$$\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial\theta} + u_z\frac{\partial u_r}{\partial z} - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \quad \text{(Bgfa7)}$$

$$\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial\theta} + u_z\frac{\partial u_\theta}{\partial z} + \frac{u_\theta u_r}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial\theta} \quad \text{(Bgfa8)}$$

$$\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial\theta} + u_z\frac{\partial u_z}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} \quad \text{(Bgfa9)}$$

Note that the last terms on the left hand sides of equations (Bfga7) and (Bfga8), namely u_θ^2/r and $u_\theta u_r/r$, are due to the centripetal and Corioli's components of acceleration. Setting $\partial/\partial\theta$ terms equal to zero for axisymmetric flow, equations (Bgfa7) to (Bgfa9) become

$$\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + u_z\frac{\partial u_r}{\partial z} - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \quad \text{(Bgfa10)}$$

EXAMPLE 3.1

Problem Statement *Flow in a diverging channel.* Consider water flowing in the diverging channel shown in Figure 3.4. The velocity V_1 at the inlet with cross-sectional area 0.2 m^2 is 1 m/s. What are the mass and volumetric flow rates through the channel? If the cross-sectional area at the exit is 0.35 m^2, find the velocity at the exit of the channel.

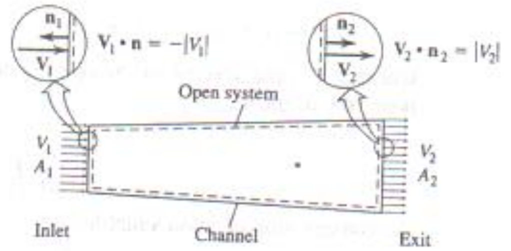


FIGURE 3.4 Flow in a diverging channel with uniform conditions at the inlet and exit.

Governing Equations By selecting the section of the channel shown in Figure 3.4 as a control volume, we have an open system (mass crosses the boundary). The mass flow rate $\dot{m}$ through an area A_r is defined as

$$\dot{m} = \int_{A_r} \rho(\mathbf{V}\cdot\mathbf{n})\,dA_r,$$

and the volumetric flow rate $\dot{Q}$ as

$$\dot{Q} = \int_{A_r} (\mathbf{V}\cdot\mathbf{n})\,dA_r,$$

where ρ is the density of the fluid and $\mathbf{V}\cdot\mathbf{n} = V_n$ is the velocity normal to the area A_r.

The conservation of mass for an open system is

$$\frac{d}{dt}\int_{\mathcal{V}_{os}} \rho d\mathcal{V} + \int_{A_r} \rho(\mathbf{V}\cdot\mathbf{n})\,dA_r \equiv 0,$$

where $\mathcal{V}_{os}$ is the volume of the open system.

Basic Assumptions It is assumed that the flow is incompressible (constant density ρ), steady, and uniform across the channel.

SOLUTION By assuming uniform conditions at the inlet and exit, the mass flow rate in kg/s simplifies as

$$\dot{m} = \int_{A_r} \rho|(\mathbf{V}\cdot\mathbf{n})|dA_r = \rho V_n A_r,$$

and the volumetric flow rate in m^3/s as

$$\dot{Q} = \int_{A_r} |(\mathbf{V}\cdot\mathbf{n})|\,dA_r = V_n A_r.$$

Using the numerical values, the mass and volumetric flow rates are readily calculated as

$$\dot{m} = \rho V_{n_1} A_1 = 998\ kg/m^3 \times 1\ m/s \times 0.2\ m^2 = 199.6\ kg/s.$$

and

$$\dot{Q} = 1\ m/s \times 0.2\ m^2 = 0.2\ m^3/s.$$

Similarly, since we have uniform conditions at the inlet and exit, conservation of mass reduces to

$$\rho V_{n_1} A_1 = \rho V_{n_2} A_2.$$

The velocity at the exit then is

$$V_{n_2} = \frac{\rho V_1 A_1}{\rho A_2} = \frac{1\ m/s \times 0.2\ m^2}{0.35\ m^2} = 0.57\ m/s.$$

This problem shows that the velocity of an incompressible fluid in a diverging channel decreases, whereas in a converging channel it increases.

EXAMPLE 3.2

Problem Statement *Ground water infiltration.* Consider the hydraulic dam shown in Figure 3.5, releasing water at a rate of 200 m^3/s into a dry river bed 50 m wide. The ground absorbs water at a rate of 0.01 m/s (velocity of the water entering the ground). Find the extent of the river that will be affected by the released water.

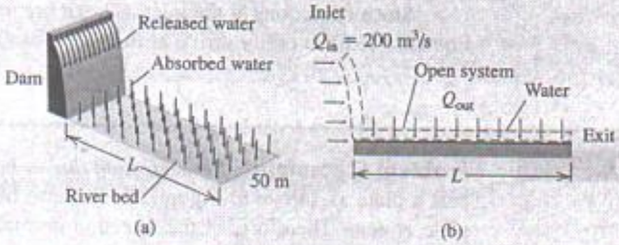


FIGURE 3.5 (a) Water released from a dam infiltrates the ground at a given velocity rate; (b) the selected open system considers both the released and the infiltrating water.

Governing Equations By considering the water as an open system, the conservation of mass is expressed using

$$\frac{d}{dt}\int_{V_{os}} \rho dV + \int_{A_r} \rho(\mathbf{V}\cdot\mathbf{n})\, dA_r = 0.$$

Basic Assumptions Assume steady state and incompressible flow, with uniform conditions at the locations the water enters and leaves the boundary of the selected open system.

SOLUTION In this problem, mass crosses the boundary of the open system at the locations shown in Figure 3.5 as inlet and exit. For steady flow $\frac{d}{dt}\int_{V_{os}} \rho\, dV = 0$, the continuity equation reduces to

$$\int_{A_r} (\mathbf{V}\cdot\mathbf{n})\, dA_r = \int_{in} (\mathbf{V}\cdot\mathbf{n})\, dA_i + \int_{out} (\mathbf{V}\cdot\mathbf{n})\, dA_o = 0.$$

At the inlet we have $\mathbf{V} = V_i\mathbf{i}$ and $\mathbf{n} = -\mathbf{i}$, where V_i is the magnitude of the velocity of the released water at the dam. Therefore,

$$\int_{in} (\mathbf{V}\cdot\mathbf{n})\,dA_r = \int_{in} (V_i\mathbf{i})\cdot(-\mathbf{i})\,dA_i = -V_i \int_{in} dA_i = -V_i A_i = -\dot{Q},$$

where $\dot{Q}$ is the volumetric rate of water release. At the boundary, where water leaves the open system and enters the ground, we have $\mathbf{V} = -V_e\mathbf{j}$ and $\mathbf{n} = -\mathbf{j}$, where V_e is the velocity of the water being absorbed by the ground. Therefore,

$$\int_{out} (\mathbf{V}\cdot\mathbf{n})\,dA_e = \int_{out} (-V_e\mathbf{j})\cdot(-\mathbf{j})\,dA_e = V_e \int_{in} dA_e = V_e A_e,$$

where A_e is the affected area. From the geometry of the problem, $A_e = LW$, where W is the width of the river and L the length of this area.

By adding the terms at the inlet and exit, the conservation of mass yields

$$-\dot{Q} + V_e A_e = -\dot{Q} + V_e LW = 0.$$

Therefore,

$$L = \frac{\dot{Q}}{V_e W} = \frac{200\ m^3/s}{0.01\ m/s \times 50\ m} = 400\ m.$$

Since conditions at the inlet and exit are uniform, we could have alternatively used Equation (3.8) to easily arrive at the same result (left as an exercise for the reader).

EXAMPLE 3.3

Problem Statement *Flow ejection due to boundary layer effects.* Consider the flow past a plate as shown in Figure 3.6. Because of friction, the velocity of the fluid at the plate is zero. Therefore, in the direction normal to the incoming flow direction (i.e., in the y-direction) the velocity changes from zero at the plate to the value of the freestream velocity (u_0) away from it. The layer across which the velocity varies is known as the *boundary layer*. The thickness of this layer (δ) increases with the distance x from the front end of the plate in the direction of flow—that is, $\delta = \delta(x)$.

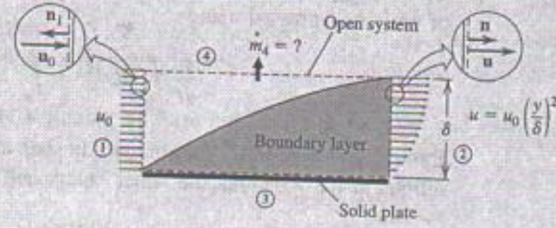


FIGURE 3.6 Due to the zero velocity at the plate, the velocity profile changes from a uniform profile at the inlet u_0 into a nearly parabolic profile $u = u_0(y/\delta)^2$ at the exit at a distance L away from the inlet. As a result, excess fluid is deflected away from the plate.

In this example, consider a uniform flow u_0 approaching a flat plate that changes into a parabolic profile $u = u_0\left(\frac{y}{\delta}\right)^2$ at a distance L away from the front edge of the plate where $\delta = \delta(x = L)$. For the conditions shown in the figure, find the mass flow rate across each boundary of the selected open system.

Governing Equations The conservation of mass for the selected open system is given by

$$\frac{d}{dt}\int_{\mathcal{V}_{os}} \rho \, d\mathcal{V} + \int_{A_r} \rho(\mathbf{V}\cdot\mathbf{n})\, dA_r = 0.$$

Basic Assumptions The flow is assumed to be steady, two-dimensional and incompressible.

SOLUTION By assuming steady flow $\frac{d}{dt}\int_{\mathcal{V}_{os}} \rho d\mathcal{V} = 0$ and by considering all four boundaries of the open system, the conservation of mass reduces to

$$\underbrace{\int_1 \rho(\mathbf{V}\cdot\mathbf{n}) W dS}_{I_1} + \underbrace{\int_2 \rho(\mathbf{V}\cdot\mathbf{n}) W dS}_{I_2} + \underbrace{\int_3 \rho(\mathbf{V}\cdot\mathbf{n}) W dS}_{I_3} + \underbrace{\int_4 \rho(\mathbf{V}\cdot\mathbf{n}) W \, dS}_{I_4} = 0.$$

At each boundary numbered as shown in Figure 3.6, we have the following conditions:

$$\mathbf{V}_1 = u_o\mathbf{i}, \quad \mathbf{n}_1 = -\mathbf{i}, \quad dS_1 = dy, \quad \mathbf{V}_2 = u_o\left(\frac{y}{\delta}\right)^2\mathbf{i}, \quad \mathbf{n}_2 = \mathbf{i}, \quad dS_2 = dy,$$
$$\mathbf{V}_3 = 0, \quad \mathbf{n}_3 = \mathbf{j}, \quad dS_3 = dx, \quad \mathbf{V}_4 = ?, \quad \mathbf{n}_4 = -\mathbf{j}, \quad dS_4 = dx,$$

Each of the boundary integrals is evaluated as follows:

$$\int_{0_1}^{\delta} \rho(\mathbf{V}\cdot\mathbf{n}) W dy = \int_{0_1}^{\delta} \rho(u_o\mathbf{i})\cdot(-\mathbf{i}) W dy = -\int_{0_1}^{\delta} \rho u_o W dy = -\rho u_o \delta W.$$

Similarly,

$$\int_{0_2}^{\delta} \rho(\mathbf{V}\cdot\mathbf{n}) W dy = \int_{0_2}^{\delta} \rho u_o \left(\frac{y}{\delta}\right)^2 W dy = \rho u_o \left[\frac{y^3}{3\delta^2}\right]_0^{\delta} W = \rho u_o \frac{\delta}{3} W.$$

Since the velocity along the wall is zero,

$$\int_3 \rho(\mathbf{V}\cdot\mathbf{n}) W dx = 0.$$

In order to conserve mass,

$$\underbrace{-\rho u_o \delta W}_{I_1} + \underbrace{\rho u_o \frac{\delta}{3} W}_{I_2} + \underbrace{0}_{I_3} + \underbrace{\dot{m}_4}_{I_4} = 0.$$

The mass flow rate at edge of the boundary layer then is

$$\dot{m}_4 = \rho u_o \delta W - \rho u_o \frac{\delta}{3} W \Rightarrow \dot{m}_4 = \frac{2}{3}\rho u_o \delta W.$$

Therefore, because of the fluid friction induced slowdown in the flow, two-thirds of the incoming mass flow rate is deflected away from the surface. ∎

EXAMPLE 3.4

Problem Statement *Entrance flow.* Consider the entrance flow in the pipe as shown in Figure 3.7. Because of fluid-solid friction at the pipe wall, after a certain distance downstream of the inlet, the velocity develops into a parabolic profile. For the conditions shown, find the maximum velocity in the pipe.

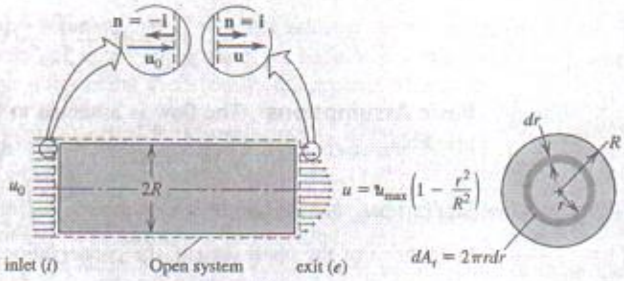


FIGURE 3.7 Uniform flow velocity entering a pipe changes into a parabolic flow. As result, the maximum velocity in the pipe increases and ultimately achieves a constant value.

Governing Equations The conservation of mass for the selected open system is given by

$$\frac{d}{dt}\int_{\mathcal{V}_{os}} \rho d\mathcal{V} + \int_{A_r} \rho(\mathbf{V}\cdot\mathbf{n})\, dA_r = 0.$$

Basic Assumptions This two-dimensional flow is assumed to be steady and incompressible.

SOLUTION For steady flow $\frac{d}{dt}\int \rho d\mathcal{V} = 0$, the conservation of mass simplifies to

$$\int_{A_r} \rho(\mathbf{V}\cdot\mathbf{n})\, dA_r = 0.$$

The surface integrals vanish everywhere except at the inlet (subscript i) and exit (subscript e) of the pipe where mass crosses the boundary of the selected system. At the inlet, we have $\mathbf{V}_i = u_o\mathbf{i}$ and $\mathbf{n}_i = -\mathbf{i}$. At the exit, we have $\mathbf{V}_e = u_{max}\left(1-\frac{r^2}{R^2}\right)\mathbf{i}$ and $\mathbf{n}_e = \mathbf{i}$. For the cylindrical geometry, an infinitesimal cross-sectional area is expressed as $dA_r = 2\pi r\, dr$. Therefore,

$$\int_{inlet} \rho(\mathbf{V}\cdot\mathbf{n})dA_r + \int_{exit} \rho(\mathbf{V}\cdot\mathbf{n})dA_r = 0 \Rightarrow$$

$$\int_{inlet} \rho \underbrace{(u_o\mathbf{i})}_{\mathbf{V}_i}\cdot\underbrace{(-\mathbf{i})}_{\mathbf{n}_i}\underbrace{2\pi r\, dr}_{dA_i} + \int_{exit} \rho \underbrace{\left(u_{max}\left[1-\left(\frac{r}{R}\right)^2\right]\mathbf{i}\right)}_{\mathbf{V}_e}\cdot\underbrace{(\mathbf{i})}_{\mathbf{n}_e}\underbrace{2\pi r dr}_{dA_e} = 0,$$

then,

$$-\rho u_o\pi\int_0^R 2r\, dr + \rho u_{max} 2\pi\int_0^R \left[r - \frac{r^3}{R^2}\right] dr = 0$$

$$-\rho u_o\pi R^2 + 2\pi\rho u_{max}\left[\frac{r^2}{2} - \frac{r^4}{4R^2}\right]_0^R = 0$$

$$-\rho u_o \pi R^2 + 2\pi \rho u_{max}\left[\frac{R^2}{2} - \frac{R^4}{4R^2}\right] = -\rho u_o \pi R^2 + 2\pi \rho u_{max}\frac{R^2}{4} = 0$$

$$\Rightarrow u_{max} = 2u_o.$$

The velocity profile

$$u = u_0\left(1 - \frac{r^2}{R^2}\right)$$

is characteristic of a type of flow known as laminar pipe flow and is encountered well downstream from the inlet.

EXAMPLE 3.5

Problem Statement *Mixing.* Consider a medical device used to mix and deliver to a patient two drugs with roughly the same density 1050 kg/m^3 (Figure 3.8). For the conditions shown in the figure, find the maximum velocity after the two drug streams mix.

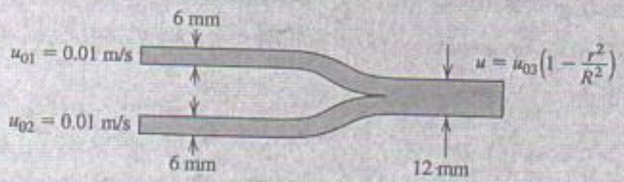


FIGURE 3.8 Mass conservation during mixing of two drug streams.

Governing Equations The conservation of mass for the selected open system is given by

$$\frac{d}{dt}\int_{\mathcal{V}_{os}} \rho\, d\mathcal{V} + \int_{A_r} \rho(\mathbf{V}\cdot\mathbf{n})\, dA_r = 0.$$

Basic Assumptions The flow is assumed to be steady and incompressible. Also, the velocity profile at the exit of the delivery tube is assumed to be parabolic of the form $u = u_0\left(1 - \frac{r^2}{R^2}\right)$.

SOLUTION For steady flow then $\frac{d}{dt}\int_{\mathcal{V}_{os}} \rho d\mathcal{V} = 0$, we have

$$\int_{A_r} \rho_i(\mathbf{V}_i\cdot\mathbf{n}_i)\, dA_r = 0,$$

where i denotes places where mass crosses the system boundary. At these locations we have the following conditions:

$$\mathbf{V}_1 = u_{o1}\mathbf{i} \quad \mathbf{n}_1 = -\mathbf{i}$$

$$\mathbf{V}_2 = u_{o2}\mathbf{i} \quad \mathbf{n}_2 = -\mathbf{i}$$

$$\mathbf{V}_3 = u_{o3}\left(1 - \frac{r^2}{R^2}\right)\mathbf{i}, \quad \mathbf{n}_3 = \mathbf{i}.$$

For the cylindrical geometry, an infinitesimal cross-sectional area is expressed as $dA_r = 2\pi r dr$. Therefore,

$$\int_1 \rho \mathbf{V}_1 \cdot \mathbf{n}_1 dA_1 + \int_2 \rho \mathbf{V}_2 \cdot \mathbf{n}_2 dA_2 + \int_3 \rho \mathbf{V}_3 \cdot \mathbf{n}_3 dA_3 = 0 \Rightarrow$$

$$\int_1 \rho \underbrace{(u_{o1}\mathbf{i})}_{\mathbf{V}_1} \cdot \underbrace{(-\mathbf{i})}_{\mathbf{n}_1} \underbrace{2\pi r dr}_{dA_1} + \int_2 \rho \underbrace{(u_{o2}\mathbf{i})}_{\mathbf{V}_2} \cdot \underbrace{(-\mathbf{i})}_{\mathbf{n}_2} \underbrace{2\pi r dr}_{dA_1} +$$

$$\int_3 \underbrace{\rho\, u_{o3} \left[1 - \left(\frac{r}{R}\right)^2\right] (\mathbf{i})}_{\mathbf{V}_3} \cdot \underbrace{(\mathbf{i})}_{\mathbf{n}_3} \underbrace{2\pi r dr}_{dA_3} = 0,$$

by eliminating $2\rho\pi$ and by performing the scalar product between the velocity and the outward normal vectors, the preceding expression simplifies to

$$-u_{o1} \int_0^{R_1} r dr - u_{o2} \int_0^{R_2} r dr + u_{o3} \int_0^{R_3} \left[r - \frac{r^3}{R^2} \right] dr = 0$$

$$-u_{o1} \frac{R_1^2}{2} - u_{o2} \frac{R_2^2}{2} + u_{o3} \left[\frac{r^2}{2} - \frac{r^4}{4R^2} \right]_0^{R_3} = 0$$

$$-u_{o1} \frac{R_1^2}{2} - u_{o2} \frac{R_2^2}{2} + u_{o3} \frac{R_3^2}{4} = 0$$

$$u_{o3} = \frac{2u_{o1} R_1^2 + 2u_{o2} R_2^2}{R_3^2}.$$

By substituting the numerical values,

$$u_{o3} = \frac{2 \times 0.01\ m/s \times 0.003^2\ m^2 + 2 \times 0.01\ m/s \times 0.003^2\ m^2}{0.006^2\ m^2} = 0.01\ m/s.$$

EXAMPLE 3.6

Problem Statement *Reservoir filling.* Consider an initially empty reservoir with volume 2 m^3 connected to a supply line through a valve as shown in Figure 3.9. Find the time it takes to fully fill the reservoir when the valve opens.

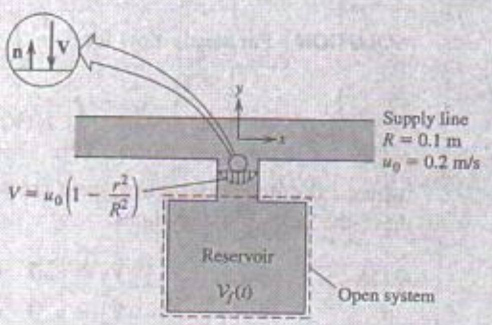


FIGURE 3.9 A tank with fixed volume is filled with water from a supply line.

Governing Equations The conservation of mass for the selected open system is given by

$$\frac{d}{dt}\int_{\mathcal{V}} \rho d\mathcal{V} + \int_1 \rho(\mathbf{V}\cdot\mathbf{n})dA_r = 0.$$

In this problem, however, only part of the selected volume is filled with fluid. Therefore, at any instant t, the density ρ is defined only over the volume $\mathcal{V}_f(t)$ that the fluid occupies. Therefore, the unsteady term reduces to

$$\frac{d}{dt}\int_{\mathcal{V}} \rho d\mathcal{V} = \frac{d}{dt}\int_{\mathcal{V}_f} \rho d\mathcal{V}_f.$$

Basic Assumptions The flow at the supply line is assumed to be constant and incompressible.

SOLUTION At the inlet of the reservoir we have the following conditions: $\mathbf{V}_i = u_o\left(1-\frac{r^2}{R^2}\right)(-\mathbf{j})$ and $\mathbf{n}_i = \mathbf{j}$. Substituting the appropriate terms in the conservation of mass

$$\frac{d}{dt}\int_{\mathcal{V}_f} \rho d\mathcal{V}_f + \int_1 \rho \underbrace{u_o\left[1-\frac{y^2}{R^2}\right](-\mathbf{j})}_{\mathbf{V}_i}\cdot \underbrace{(\mathbf{j})}_{\mathbf{n}_i}\, \underbrace{2\pi r dr}_{dA_i} = 0,$$

$$\rho\frac{d}{dt}\int_{\mathcal{V}_f} d\mathcal{V}_f - 2\pi\rho u_o \int_1 \left[1-\frac{r^2}{R^2}\right] r dr = 0,$$

then,

$$\frac{d\mathcal{V}_f}{dt} = 2\pi u_o \left[\frac{r^2}{2} - \frac{r^4}{4R^2}\right]_0^R$$

$$\frac{d\mathcal{V}_f}{dt} = \pi u_o \frac{R^2}{2} \Rightarrow d\mathcal{V}_f = \pi u_o \frac{R^2}{2} dt$$

By integration,

$$\int_0^{\mathcal{V}_t} d\mathcal{V}_f = \pi u_o \frac{R^2}{2}\int_0^{t_{fill}} dt.$$

The upper limit of the integration is set to $\mathcal{V}_f = \mathcal{V}_t$, where $\mathcal{V}_t$ is the volume of the tank (since the entire tank is finally filled with fluid). The final result then is

$$\mathcal{V}_t = \pi u_o \frac{R^2}{2} t_{fill}.$$

The reservoir will fill in

$$t_{fill} = \frac{2\mathcal{V}_t}{\pi u_o R^2} = \frac{2\times 2\ m^3}{\pi \times 0.2\ m/s \times 0.1^2\ m^2} = 636.6\ s.$$

3.4 Conservation of Linear Momentum

If in Equation (3.6) we substitute $\alpha = \mathbf{V}$ and $B = \sum \mathbf{F}$, the conservation of linear momentum for an open system that is *stationary or moving with constant velocity (i.e., for an inertial coordinate system)*, emerges as

$$\underbrace{\frac{d}{dt}\int_{\mathcal{V}} \rho \mathbf{V} d\mathcal{V}}_{\text{Rate of change of momentum}} + \underbrace{\int_{A_r} \rho \mathbf{V}(\mathbf{V}\cdot\mathbf{n}) dA_r}_{\text{Momentum flux at boundary}} \equiv \underbrace{\sum \mathbf{F}}_{\text{Net Force}}. \tag{3.9}$$

Again, $\sum \mathbf{F}$ is a vector representing the resultant of the applied forces. The velocity vector $\mathbf{V}$ and its time derivative are expressed *relative to the selected coordinate system.*

If flow conditions at locations where mass crosses the boundary are uniform, the momentum equation simplifies to

$$\frac{d}{dt}\int_{\mathcal{V}} \rho \mathbf{V} d\mathcal{V} \pm \sum_{k=1}^{k=N} (\dot{m}\mathbf{V})_k \equiv \sum \mathbf{F}, \tag{3.10}$$

where N is the number of segments where mass crosses the boundary. Again, the proper sign is determined from the product $(\mathbf{V}\cdot\mathbf{n})$; the negative sign $(-)$ is associated with inlets, and the positive sign $(+)$ with exits.

In steady flows the equation simplifies to

$$\sum_{k_e=1}^{k=N_e} (\dot{m}_k \mathbf{V}_k)_{exit} - \sum_{k_i=1}^{k=N_i} (\dot{m}_k \mathbf{V}_k)_{inlet} \equiv \sum \mathbf{F}, \tag{3.11}$$

where subscripts (i) and (e) denote, respectively, inlets and exits. In cases of steady flow with a single inlet and a single exit we have

$$(\dot{m}\mathbf{V})_{exit} - (\dot{m}\mathbf{V})_{inlet} \equiv \sum \mathbf{F}.$$

But according to the continuity equation $\dot{m}_{in} = \dot{m}_{out} = \dot{m}$. Therefore,

$$\dot{m}(\mathbf{V}_{exit} - \mathbf{V}_{inlet}) \equiv \sum \mathbf{F}. \tag{3.12}$$

For the conditions stated above, Equation (3.12) is a useful alternative form of the momentum equation.

EXAMPLE 3.7

Problem Statement *Force on a solid surface generated by an impinging jet of fluid.* During severe flooding, the manhole cover of the drainage system is lifted as shown in Figure 3.10. If the diameter D of the cover is 85 cm and the thickness 6 cm, find the volume of water per second out of the manhole. Consider a cast-iron cover whose density is 8200 kg/m^3.

Governing Equations For the analysis, we have a choice for the selection of the control volume; we can either select the jet and the cover as a single control volume, or select the jet and the cover as separate control volumes. Irrespective of the choice of control volume, however, the analysis should yield the same answer. Selecting the cover and water as shown in Figure 3.10 as the control volume, the conservation of momentum

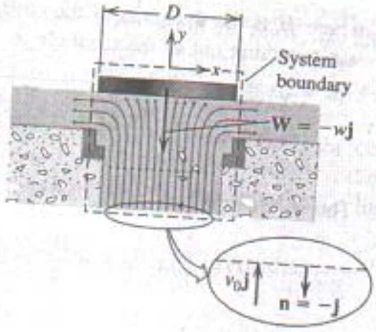


FIGURE 3.10 Force on a solid surface generated by a jet of fluid.

is expressed as

$$\frac{d}{dt}\int_{\mathcal{V}_{os}} \mathbf{V}\rho d\mathcal{V} + \int_{A_r} \rho\mathbf{V}(\mathbf{V}\cdot\mathbf{n})dA_r \equiv \sum \mathbf{F}.$$

Basic Assumptions The flow of water is assumed to be steady and uniform at those locations where it crosses the boundary of the selected control volume.

SOLUTION For steady flow $\frac{d}{dt}\int_{\mathcal{V}_{os}} \mathbf{V}\rho d\mathcal{V} = 0$, the momentum equation reduces to

$$\int_{A_r} \rho\mathbf{V}(\mathbf{V}\cdot\mathbf{n})dA_r \equiv \sum \mathbf{F}.$$

Because of symmetry, it is obvious that there is no net force in the horizontal direction. Hence, we need to consider only the y-component of the momentum equation

$$\int_{A_r} \rho v(\mathbf{V}\cdot\mathbf{n})dA_r \equiv \sum F_y.$$

The surface integral is evaluated at those locations where mass crosses the boundary and where v is not zero. This is true only at the location of the water jet (noted as inlet). The velocity vector at this location is $\mathbf{V} = v_o\,\mathbf{j}$, with v_o being the velocity of the water, and the outward unit normal vector is $\mathbf{n} = -\,\mathbf{j}$. The only force acting on the selected system is the weight of the cover $\mathbf{W} = -\rho_c g \mathcal{V}_c\,\mathbf{j}$, where ρ_c and $\mathcal{V}_c$ are, respectively, the density and volume of the cover and g is the gravitational acceleration.

Substituting the various terms into the momentum equation,

$$\int_{A_r} \rho(v_o\mathbf{j})(v_o\mathbf{j})\cdot(-\mathbf{j})dA_r = -\rho_c\mathcal{V}_c g\mathbf{j},$$

upon simplification and by using $\mathcal{V}_c = A_r H$, we get

$$-\rho v_o^2 A_r \mathbf{j} = -\rho_c \mathcal{V}_c g\mathbf{j} \Rightarrow v_o = \sqrt{g\frac{\rho_c}{\rho}\frac{A_r H}{A_r}} = \sqrt{g\frac{\rho_c}{\rho}H},$$

where H is the thickness of the cover. In terms of numerical values, the velocity of the water coming out of the manhole is

$$v_o = \sqrt{9.81\ m/s^2 \times \frac{8200\ kg/m^3}{998\ kg/m^3} \times 0.06\ m/s} = 2.2\ m/s.$$

The volumetric flow rate $\dot{Q}$ is

$$\dot{Q} = v_o A_r = v_o \pi \frac{D^2}{4} = 2.2\ m/s \times 3.14 \times \frac{0.85^2\ m^2}{4} = 1.25\ m^3/s.$$

EXAMPLE 3.8

Problem Statement *Force induced by flow in a 90° elbow.* Consider flow of water in a 90° elbow in a constant 0.06 m diameter pipe as shown in Figure 3.11. If the velocity u_o in the pipe is 0.15 m/s, find the required force to keep the elbow in place.

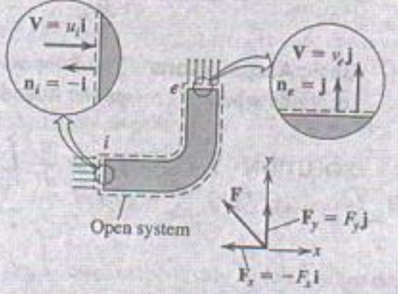


FIGURE 3.11 Force induced by flow in a 90° elbow.

Governing Equations By selecting a segment of the pipe as shown in the figure, we have an open system for which the conservation of momentum is expressed as

$$\frac{d}{dt}\int_{\mathcal{V}_{os}} \mathbf{V}\rho d\mathcal{V} + \int_{A_r} \rho\mathbf{V}(\mathbf{V}\cdot\mathbf{n})dA_r = \sum \mathbf{F},$$

and the conservation of mass as

$$\frac{d}{dt}\int_{\mathcal{V}_{os}} \rho d\mathcal{V} + \int_{A_r} \rho\mathbf{V}\cdot\mathbf{n}dA_r = 0.$$

Basic Assumptions The flow is assumed to be steady, one-dimensional, incompressible, and with negligible gravity effects. We also neglect the pressure difference between the two ends of the pipe. Note that this is an oversimplification of the real flow, since a pressure difference across the pipe is necessary in order to induce and sustain the flow.

SOLUTION The surface integrals in the conservation laws are evaluated at the two locations where mass crosses the boundary denoted as *inlet* (i) and *exit* (e). At the *inlet*, $\mathbf{V}_i = u_i\mathbf{i}$ and $\mathbf{n}_i = -\mathbf{i}$. At the *exit*, $\mathbf{V}_e = v_e\mathbf{j}$ and $\mathbf{n}_e = \mathbf{j}$. The cross-sectional areas at both ends are equal; $A_i = A_e = A_r$.

After simplification, the conservation of mass for the open system shown in Figure 3.3 yields

$$\rho u_i A_r = \rho u_e A_r \Rightarrow u_i = v_e = u_o = 0.15\ m/s.$$

The only relevant surface force acting on the system is the contact force **F** required to support the elbow. Assuming that the force acts in the direction shown in the figure, it is expressed as $\mathbf{F} = -F_x\,\mathbf{i} + F_y\,\mathbf{j}$. The momentum equation then becomes

$$\int_{inlet} \rho \mathbf{V}(\mathbf{V}\cdot\mathbf{n})dA_r + \int_{exit} \rho \mathbf{V}(\mathbf{V}\cdot\mathbf{n})dA_r = -F_x\mathbf{i} + F_y\mathbf{j}.$$

Substituting the appropriate expressions for the velocity and unit normal vectors, we get

$$\int_{inlet} \rho \mathbf{V}(\mathbf{V}\cdot\mathbf{n})dA_r = \int_{inlet} \rho u_o\mathbf{i}(u_o\mathbf{i})\cdot(-\mathbf{i})dA_r = -\rho u_o^2 A_r\mathbf{i};$$

similarly,

$$\int_{exit} \rho \mathbf{V}(\mathbf{V}\cdot\mathbf{n})dA_r = \int_{exit} \rho u_o\mathbf{j}(u_o\mathbf{j})\cdot(\mathbf{j})dA_r = \rho u_o^2 A_r\mathbf{j}.$$

The momentum equation then takes the form

$$-\rho u_o^2 A_r\mathbf{i} + \rho u_o^2 A_r\mathbf{j} = -F_x\mathbf{i} + F_y\mathbf{j}.$$

The same result could have been obtained by using Equation (3.12) and by setting $\mathbf{V}_1 = u_o\mathbf{i}$ and $\mathbf{V}_2 = u_o\mathbf{j}$.

By substituting the numerical values

$$F_x = \rho u_o^2 A_r = 998\ kg/m^3 \times 0.15^2\ m^2/s^2 \times \pi \times \frac{0.06^2}{4}\ m^2 = 0.0635\ N.$$

and

$$F_y = \rho u_o^2 A_r = 998\ kg/m^3 \times 0.15^2\ m^2/s^2 \times \pi \times \frac{0.06^2}{4}\ m^2 = 0.0635\ N.$$

Since F_x and F_y are positive quantities, the force directions are as assumed. ■

EXAMPLE 3.9

Problem Statement *Force induced by boundary layer effects.* Find the horizontal force induced along the plate due to the development of the boundary layer (Figure 3.6). Since the boundary layer develops as a result of fluid-solid friction, the sought-for force is the friction force at the plate surface that resists the flow.

Governing Equations For the selected open system, the conservation of momentum is expressed as

$$\frac{d}{dt}\int_{\mathcal{V}_{os}} \mathbf{V}\rho d\mathcal{V} + \int_{A_r} \rho\mathbf{V}(\mathbf{V}\cdot\mathbf{n})dA_r \equiv \sum \mathbf{F},$$

and the conservation of mass as

$$\frac{d}{dt}\int_{\mathcal{V}_{os}} \rho d\mathcal{V} + \int_{A_r} \rho(\mathbf{V}\cdot\mathbf{n})dA_r \equiv 0.$$

Basic Assumptions The flow is assumed to be steady, two-dimensional, incompressible, and with negligible gravity effects.

SOLUTION For steady flow $\frac{d}{dt}\int_{\mathcal{V}_{os}} \mathbf{V}\rho d\mathcal{V} = 0$, and by considering the four boundaries of the open system, the momentum equation in the x-direction (since we are interested only in the force in this direction) reduces to

$$\int_1 \rho u_1(\mathbf{V}\cdot\mathbf{n})_1 W dS_1 + \int_2 \rho u_2(\mathbf{V}\cdot\mathbf{n})_2 W dS_2 + \int_3 \rho u_3(\mathbf{V}\cdot\mathbf{n})_3 W dS_3 + \int_4 \rho u_4(\mathbf{V}\cdot\mathbf{n})_4 W dS_4 = F_x.$$

Along the boundaries, we have the following conditions:

$$\mathbf{V}_1 = u_o\mathbf{i},\quad \mathbf{n}_1 = -\mathbf{i},\quad dS_1 = dy,\quad \mathbf{V}_2 = u_o\frac{y^2}{\delta^2}\mathbf{i},\quad \mathbf{n}_2 = \mathbf{i},\quad dS_2 = dy,$$
$$\mathbf{V}_3 = 0\mathbf{j},\quad \mathbf{n}_3 = -\mathbf{j},\quad dS_3 = dx,\quad \mathbf{V}_4 = ?,\quad \mathbf{n}_4 = \mathbf{j},\quad dS_4 = dx,$$

By substitution and by considering the proper sign according to $(\mathbf{V}\cdot\mathbf{n})$, we get

$$F_x = -\int_{0_1}^{\delta} \rho u_o^2\, W dy + \int_{0_2}^{\delta} \rho u_o^2 \left(\frac{y^2}{\delta^2}\right)^2 W dy + 0 + \int_{0_4}^{L} \rho u_o(\mathbf{V}\cdot\mathbf{n}) W dx,$$

where L is the length of the plate. By integration,

$$F_x = -\rho u_o^2 \delta W + \rho u_o^2 \left[\frac{y^5}{5\delta^4}\right]_0^{\delta} W + \int_{0_4}^{L} \rho u_o(\mathbf{V}\cdot\mathbf{n})_4 W dx.$$

For mass conservation, we can make use of the result obtained in Example 3.3, where $\dot{m}_4 = \frac{2}{3}\rho u_o \delta W$. The last term in the momentum equation simplifies to

$$\int_{0_4}^{L} \rho u_o(\mathbf{V}\cdot\mathbf{n})_4 W dx = u_o \underbrace{\int_{0_4}^{L} \rho(\mathbf{V}\cdot\mathbf{n})_4 W dx}_{\dot{m}} = u_o\dot{m}_4 = \frac{2}{3}\rho u_o^2 \delta W.$$

Finally,

$$F_x = -\rho u_o^2 \delta W + \rho u_o^2 \frac{1}{5}\delta W + \frac{2}{3}\rho u_o^2 \delta W = -\frac{2}{15}\rho u_o^2 \delta W.$$

This is the friction force the plate exerts on the fluid. By reaction, the force on the plate has the same magnitude but acts in the opposite direction

$$\Rightarrow F_{plate} = \frac{2}{15}\rho u_o^2 \delta W.$$

■

3.4.1 Conservation of Linear Momentum for a Non-Inertial Coordinate System

The conservation of momentum relative to a *non-inertial* coordinate system (i.e., relative to a coordinate system that is accelerating with respect to a fixed coordinate system) is