INTEGRAL CALCULUS (MATH 106)

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Chapter 6: Plane Curves and Polar Coordinates

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The student is expected to be able to:

1. Know the definition of parametric equations
2. Calculate the slope of the tangent line to parametric curve.
3. Calculate arc length of a parametric equations.
4. Calculate the surface area generated by revolving a parametric curve.
5. Know what is the polar coordinates.
6. Know the equation and graph of some of polar curves.
7. Know how to calculate the slope of the tangent line with polar coordinates.
8. Know how to calculate the area inside polar curves and between polar curves.
9. Know how to calculate the arc length of a polar curve.
10. Know how to calculate Surface Area Generated By Revolving A Polar Curve.

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Parametric equations

To this point we've looked almost exclusively at functions in the form $y = f(x)$ or $x = h(y)$
It is easy to write down the equation of a circle centered at the origin with radius r.

$$x^2 + y^2 = r^2$$

However, we will never be able to write the equation of a circle down as a single equation in either of the forms above. Sure we can solve for x or y as the following two formulas show

$$y = \pm\sqrt{r^2 - x^2} \qquad\qquad x = \pm\sqrt{r^2 - y^2}$$

but there are in fact two functions in each of these. Each formula gives a portion of the circle.

$$y = \sqrt{r^2 - x^2} \qquad \text{(top)} \qquad x = \sqrt{r^2 - y^2} \qquad \text{(right side)}$$
$$y = -\sqrt{r^2 - x^2} \qquad \text{(bottom)} \qquad x = -\sqrt{r^2 - y^2} \qquad \text{(left side)}$$

There are also a great many curves out there that we can't even write down as a single equation in terms of only x and y. So, to deal with some of these problems we introduce **parametric equations.**

Instead of defining y in terms of x, $y = f(x)$ or x in terms of y $x = h(y)$ we define both x and y in terms of a third variable called a parameter as follows,

$$x = f(t) \qquad y = g(t)$$

This third variable is usually denoted by t.
Each value of t defines a point $(x, y) = (f(t), g(t))$ that we can plot.
The collection of points that we get by letting t be all possible values is the graph of the parametric equations and is called the **parametric curve.**

Parametric equations

Example

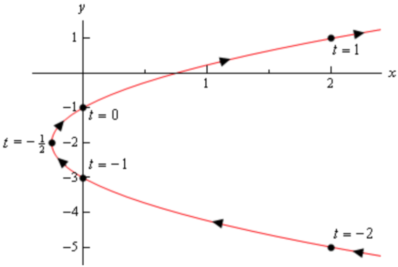
Sketch the parametric curve for the following set of parametric equations.

$$x = t^2 + t \qquad y = 2t - 1 \qquad -2 \leq t \leq 2$$

At this point our only option for sketching a parametric curve is to pick values of t, plug them into the parametric equations and then plot the points. So, let's plug in some t's.

Parametric equations (Example)

t	x	y
-2	2	-5
-1	0	-3
$-\frac{1}{2}$	$-\frac{1}{4}$	-2
0	0	-1
1	2	1

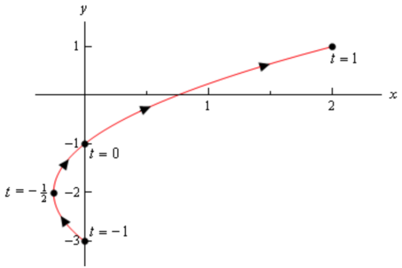


Parametric equations

Example

Sketch the parametric curve for the following set of parametric equations.

$$x = t^2 + t \qquad y = 2t - 1 \qquad -1 \leq t \leq 1$$



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If $C : x = x(t), y = y(t); a \leq t \leq b$ is a differentiable parametric curve then the slope of the tangent line to C at $t_0 \in [a, b]$ is:

$$m = \frac{dy}{dx}\Big|_{t=t_0} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}\Big|_{t=t_0}$$

Remark

1. The tangent line to the parametric curve is horizontal if the slope equals zero, which means that $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$
2. The tangent line to the parametric curve is vertical if $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$

The second derivative is $\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{\left(\frac{dy'}{dt}\right)}{\left(\frac{dx}{dt}\right)}$

Example

Find the slope of the tangent line(s) to the parametric curve given by

$$x = t^5 - 4t^3 \qquad y = t^2 \qquad at \ (0,4)$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{5t^4 - 12t^2} = \frac{2}{5t^3 - 12t}$$

$$0 = t^5 - 4t^3 = t^3\left(t^2 - 4\right) \qquad \Rightarrow \qquad t = 0, \pm 2$$

$$4 = t^2 \qquad \Rightarrow \qquad t = \pm 2$$

The slope of the tangent line to a parametric curve

1. at $t = -2$:

$$m = \left.\frac{dy}{dx}\right|_{t=-2} = -\frac{1}{8}$$

2. at $t = 2$

$$m = \left.\frac{dy}{dx}\right|_{t=2} = \frac{1}{8}$$

The slope of the tangent line to a parametric curve

Example

Find the equation of the tangent line to $C: x = t^3 - 3t,\ y = t^2 - 5t$ at $t = 2$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{2t - 5}{3t^2 - 3}$$

The slope of the tangent line is $\frac{dy}{dx}\big|_{t=2} = -\frac{1}{9}$
At $t = 2 : x = 2$ and $y = -6$
The tangent line to C at $t = 2$ passes through the point $(2, -6)$ and its slope is $-\frac{1}{9}$
therefore its equation is $\dfrac{y+6}{x-2} = -\dfrac{1}{9}$

The slope of the tangent line to a parametric curve

Example

Find the points on $C : x = e^t, \; y = e^{-t}$ at which the slope of the tangent line to C equals $-e^{-2}$

$m = \frac{dy}{dx} = \frac{(\frac{dy}{dt})}{(\frac{dx}{dt})} = \frac{-e^{-t}}{e^t} = -e^{-2t}$

$\Rightarrow m = e^{-2t} \Rightarrow e^{-2t} = -e^{-2} \Rightarrow t = 1.$

At $t = 1 : x = e^1 = e$ and $y = e^{-1} = \frac{1}{e}.$

Hence, the point at which the slope of the tangent line to C equals $-e^{-2}$ is $(e, \frac{1}{e})$

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Arc Length of a Parametric Equations

Definition

If $C : x = x(t),\ y = y(t); a \leq t \leq b$ is a differentiable parametric curve ,then its arc length equals

$$L = \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\ dt$$

Example

Determine the length of the parametric curve given by the following parametric equations.

$$x = 3\sin(3t) \qquad y = 3\cos(3t) \qquad 0 \leq t \leq 2\pi$$

$$\frac{dx}{dt} = 9\cos(3t) \qquad \frac{dy}{dt} = -9\sin(3t)$$

and the length formula gives,

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{81\sin^2(3t) + 81\cos^2(3t)}\ dt \\ &= \int_0^{2\pi} 9\ dt \\ &= 18\pi \end{aligned}$$

Arc Length of a Parametric Equations

Example

Determine the length of the parametric curve given by the following set of parametric equations.

$$x = 8t^{\frac{3}{2}} \qquad y = 3 + (8 - t)^{\frac{3}{2}} \qquad 0 \le t \le 4$$

$$\frac{dx}{dt} = 12t^{\frac{1}{2}} \qquad \frac{dy}{dt} = -\frac{3}{2}(8 - t)^{\frac{1}{2}}$$

$$L = \int_0^4 \sqrt{\left[12t^{\frac{1}{2}}\right]^2 + \left[-\frac{3}{2}(8 - t)^{\frac{1}{2}}\right]^2}\, dt = \int_0^4 \sqrt{144t + \frac{9}{4}(8 - t)}\, dt$$

$$= \int_0^4 \sqrt{\frac{567}{4}t + 18}\, dt = \frac{4}{567}\left(\frac{2}{3}\right)\left(\frac{567}{4}t + 18\right)^{\frac{3}{2}}\Bigg|_0^4$$

$$= \frac{8}{1701}\left(585^{\frac{3}{2}} - 18^{\frac{3}{2}}\right) = 66.1865$$

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Surface Area Generated By Revolving A Parametric Curve

If $C : x = x(t),\ y = y(t); a \leq t \leq b$ is a differentiable parametric curve ,then the surface area generated by revolving C around the $x-$axis is

$$SA = 2\pi \int_a^b |y(t)| \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\ dt$$

The surface area generated by revolving C around the $y-$axis is

$$SA = 2\pi \int_a^b |x(t)| \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\ dt$$

Example

Determine the surface area of the solid obtained by rotating the following parametric curve about the $x-$axis.

$$x = \cos^3\theta \qquad y = \sin^3\theta \qquad 0 \leq \theta \leq \frac{\pi}{2}$$

We'll first need the derivatives of the parametric equations.

$$\frac{dx}{d\theta} = -3\cos^2\theta \sin\theta \qquad \frac{dy}{d\theta} = 3\sin^2\theta \cos\theta$$

$$\begin{aligned}\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} &= \sqrt{9\cos^4\theta \sin^2\theta + 9\sin^4\theta \cos^2\theta}\, d\theta \\ &= 3\left|\cos\theta \sin\theta\right| \sqrt{\cos^2\theta + \sin^2\theta} \\ &= 3\cos\theta \sin\theta\end{aligned}$$

$$
\begin{aligned}
SA &= 2\pi \int_0^{\frac{\pi}{2}} \sin^3\theta \left(3\cos\theta \sin\theta\right) \; d\theta \\
&= 6\pi \int_0^{\frac{\pi}{2}} \sin^4\theta \cos\theta \; d\theta \qquad\qquad u = \sin\theta \\
&= 6\pi \int_0^{1} u^4 \, du \\
&= \frac{6\pi}{5}
\end{aligned}
$$

Example

Determine the surface area of the object obtained by rotating the parametric curve about the $y-$axis.

$$x = 3\cos(\pi t) \qquad y = 5t + 2 \qquad 0 \leq t \leq \frac{1}{2}$$

The first thing we'll need here are the following two derivatives.

$$\frac{dx}{dt} = -3\pi \sin(\pi t) \qquad \frac{dy}{dt} = 5$$

$$\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} = \sqrt{[-3\pi \sin(\pi t)]^2 + [5]^2} = \sqrt{9\pi^2 \sin^2(\pi t) + 25}$$

$$SA = \int_0^{\frac{1}{2}} 2\pi\left(3\cos(\pi t)\right)\sqrt{9\pi^2\sin^2(\pi t) + 25}\,dt$$

$$= 6\pi\int_0^{\frac{1}{2}} \cos(\pi t)\sqrt{9\pi^2\sin^2(\pi t) + 25}\,dt$$

$$u = \sin(\pi t) \quad \rightarrow \quad \sin^2(\pi t) = u^2 \qquad du = \pi\cos(\pi t)$$

$$t = 0: \quad u = \sin(0) = 0 \qquad t = \frac{1}{2}: \quad u = \sin\left(\frac{1}{2}\pi\right) = 1$$

$$SA = 6\int_0^1 \sqrt{9\pi^2 u^2 + 25}\,du$$

$$u = \frac{5}{3\pi}\tan\theta \qquad du = \frac{5}{3\pi}\sec^2\theta\, d\theta$$

$$\sqrt{9\pi^2 u^2 + 25} = \sqrt{25\tan^2\theta + 25} = 5\sqrt{\tan^2\theta + 1} = 5\sqrt{\sec^2\theta} = 5\left|\sec\theta\right|$$

$$u = 0 : 0 = \frac{5}{3\pi}\tan\theta \qquad \rightarrow \tan\theta = 0 \qquad \rightarrow \qquad \theta = 0$$

$$u = 1 : 1 = \frac{5}{3\pi}\tan\theta \qquad \rightarrow \tan\theta = \frac{3\pi}{5} \rightarrow \theta = \tan^{-1}\left(\frac{3\pi}{5}\right) = 1.0830$$

$$
\begin{aligned}
SA &= \int_0^{\frac{1}{2}} 2\pi \left(3\cos\left(\pi t\right)\right) \sqrt{9\pi^2 \sin^2\left(\pi t\right) + 25}\, dt \\
&= 6 \int_0^1 \sqrt{9\pi^2 u^2 + 25}\, du \\
&= 6 \int_0^{1.0830} \left(5\sec\theta\right) \left(\frac{5}{3\pi} \sec^2\theta\right)\, d\theta \\
&= 6 \int_0^{1.0830} \frac{25}{3\pi} \sec^3\theta\, d\theta \\
&= \frac{25}{\pi} \left(\sec\theta tan\theta + ln\left|\sec\theta + \tan\theta\right|\right)\Big|_0^{1.0830} = 43.0705
\end{aligned}
$$

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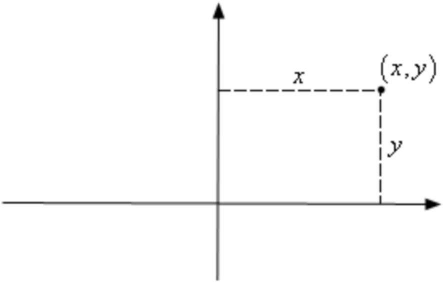
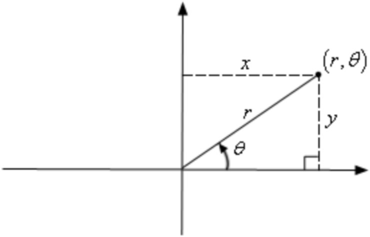
the Cartesian coordinate system at point is given the coordinates (x, y) and we use this to define the point by starting at the origin and then moving x units horizontally followed by y units vertically.



Cartesian coordinate system (or Rectangular, or x-y)

Cartesian coordinate is not the only way to define a point in two dimensional space. Instead of moving vertically and horizontally from the origin to get to the point we could instead go straight out of the origin until we hit the point and then determine the angle this line makes with the positive $x-$axis. We could then use the distance of the point from the origin and the amount we needed to rotate from the positive x-axis as the coordinates of the point.

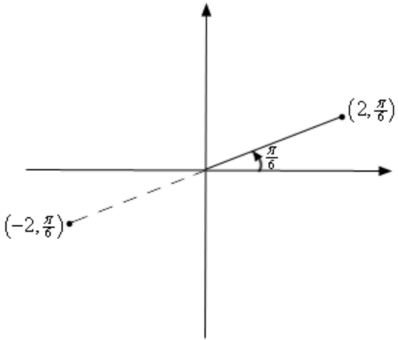


Coordinates in this form are called **polar coordinates.**

Polar Coordinates

Example

The two points $\left(2, \frac{\pi}{6}\right)$ and $\left(-2, \frac{\pi}{6}\right)$



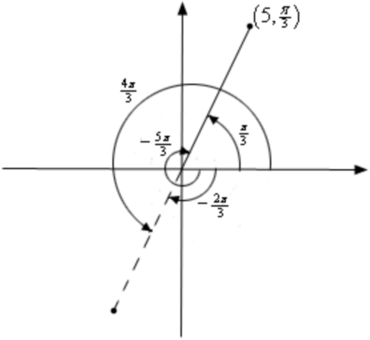
Remark

The polar coordinates of a point is not unique, if $P = (r, \theta)$ then other representations are:

1. $P = (r, \theta + 2n\pi)$, where $n \in \mathbb{Z}$
2. $P = (-r, \theta + \pi)$
3. $P = (-r, \theta + \pi + 2n\pi)$, where $n \in \mathbb{Z}$
4. $P = (-r, \theta - \pi)$
5. $P = (-r, \theta - \pi + 2n\pi)$, where $n \in \mathbb{Z}$

Example

$\left(5, \frac{\pi}{3}\right) = \left(5, -\frac{5\pi}{3}\right) = \left(-5, \frac{4\pi}{3}\right) = \left(-5, -\frac{2\pi}{3}\right)$



Relationship between the polar and the Cartesian coordinates
the following equations that will convert polar coordinates into Cartesian coordinates.

$$x = r\cos\theta \qquad\qquad y = r\sin\theta$$

Converting from Cartesian is almost as easy. Let's first notice the following.

$$\begin{aligned} x^2 + y^2 &= (r\cos\theta)^2 + (r\sin\theta)^2 \\ &= r^2\cos^2\theta + r^2\sin^2\theta \\ &= r^2\left(\cos^2\theta + \sin^2\theta\right) = r^2 \end{aligned}$$

$$r = \sqrt{x^2 + y^2}, \quad \textit{and} \ \frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta$$

Polar Coordinates

Example

1. Convert $\left(-4, \frac{2\pi}{3}\right)$ into Cartesian coordinates.
2. Convert $(-1, -1)$ into polar coordinates.

1. This conversion is easy enough. All we need to do is plug the points into the formulas.

$$x = -4\cos\left(\frac{2\pi}{3}\right) = -4\left(-\frac{1}{2}\right) = 2$$

$$y = -4\sin\left(\frac{2\pi}{3}\right) = -4\left(\frac{\sqrt{3}}{2}\right) = -2\sqrt{3}$$

So, in Cartesian coordinates this point is $\left(2, -2\sqrt{3}\right)$

2. Let's first get r

$r = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$

Now, let's get θ

$\theta = \tan^{-1}\left(\frac{-1}{-1}\right) = \tan^{-1}(1) = \frac{\pi}{4}$

Example

Convert each of the following into an equation in the given coordinate system.

1. Convert $2x - 5x^3 = 1 + xy$ into polar coordinates.
2. Convert $r = -8\cos\theta$ into Cartesian coordinates.

1.

$$2\,(r\cos\theta) - 5(r\cos\theta)^3 = 1 + (r\cos\theta)\,(r\sin\theta)$$
$$2r\cos\theta - 5r^3\cos^3\theta = 1 + r^2\cos\theta\sin\theta$$

2. $r^2 = -8r\cos\theta \Rightarrow x^2 + y^2 = -8x$

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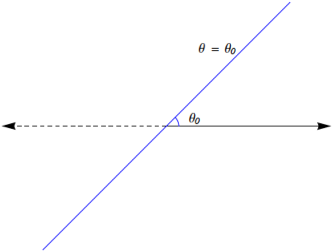
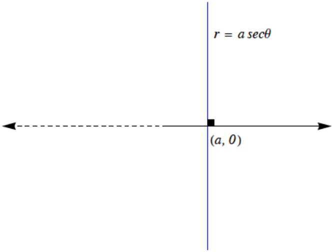
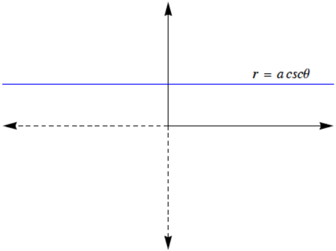
First - Straight Lines:
1-Lines passing through the pole :
Any straight line passing through the pole has the form $\theta = \theta_0$ where θ_0 is the angle between the straight line and the polar axis .
$\theta = \theta_0 \Rightarrow \tan\theta = \tan\theta_0 \Rightarrow \frac{y}{x} = \tan\theta_0 \Rightarrow y = x\tan\theta_0$
The straight line $\theta = \theta_0$ is passing through the pole with a slope equals to $\tan\theta_0$.



2-Lines perpendicular to the polar axis : Any straight line perpendicular to the polar axis has the form $r = a\sec\theta$, where $a \in \mathbb{R}^*$ and $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$r = a\sec\theta \Rightarrow r = \frac{a}{\cos\theta} \Rightarrow r\cos\theta = a \Rightarrow x = a$



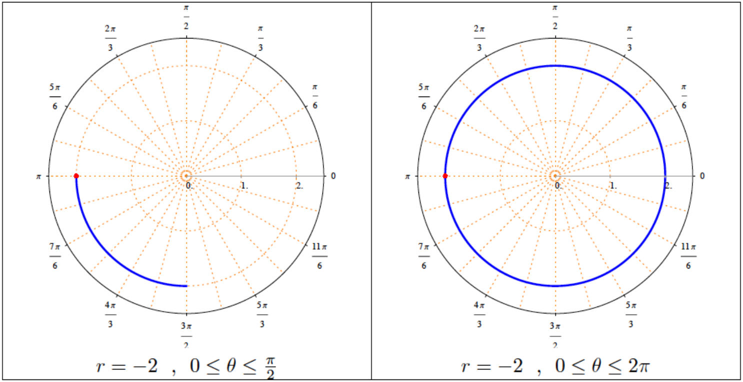
3-Lines parallel to the polar axis : Any straight line parallel to the polar axis has the form $r = a \csc \theta$, where $a \in \mathbb{R}^*$ and $\theta \in (0, \pi)$
$r = a \csc \theta \Rightarrow r = \frac{a}{\sin \theta} \Rightarrow r \sin \theta = a \Rightarrow y = a$



Second- Circles
1-Circles of the form $r = a$, where $a \in \mathbb{R}^*$
$r = a \Rightarrow r^2 = a^2 \Rightarrow x^2 + y^2 = a^2$
Therefore, $r = a$ represents a circle with center $= (0, 0)$ and radius equals $|a|$

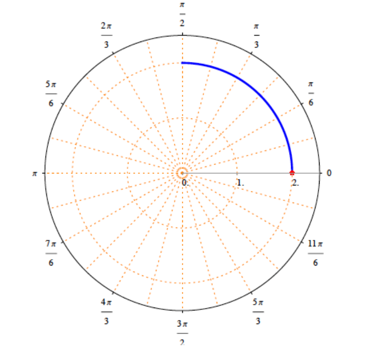
Example

$r = -2$ represents a circle with center $= (0,0)$ and radius to 2.

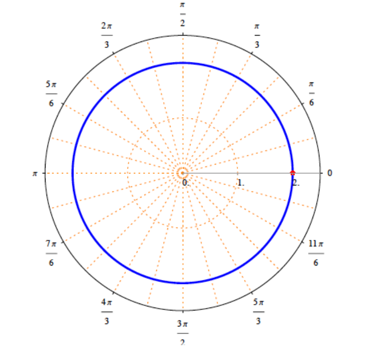


Example

$r = 2$ represents a circle with center $= (0, 0)$ and radius to 2.



$r = 2 \ , \ 0 \leq \theta \leq \frac{\pi}{2}$



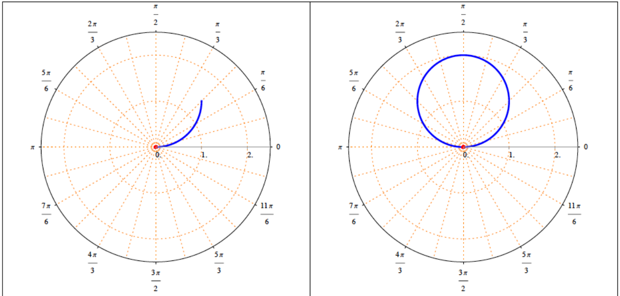
$r = 2 \ , \ 0 \leq \theta \leq 2\pi$

2-Circles of the form $r = a\sin\theta$, where $a \in \mathbb{R}^*$ and $0 \leq \theta \leq \pi$
$r = a\sin\theta \Rightarrow r^2 = a\ r\sin\theta \Rightarrow x^2 + y^2 = ay \Rightarrow x^2 + y^2 - ay = 0 \Rightarrow$
$x^2 + (y^2 - ay + \frac{a^2}{4}) = \frac{a^2}{4} \Rightarrow x^2 + (y - \frac{a}{2})^2 = \frac{a^2}{4}$
Therefore, $r = a\sin\theta$ represents a circle with center $= (0, \frac{a}{2})$ and radius equals to $\frac{|a|}{2}$

Polar Curves

Example

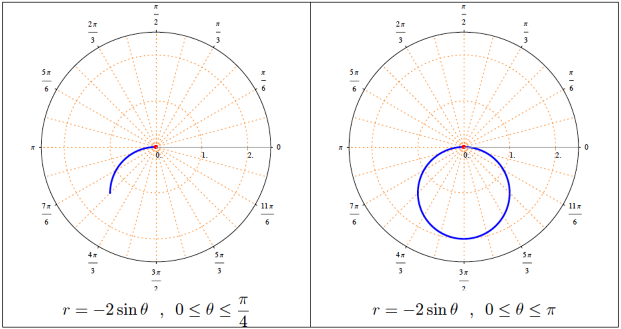
$r = 2\sin\theta$ represents a circle with center $= (0, 1)$ and radius equals to 1



$r = 2\sin\theta \;,\; 0 \leq \theta \leq \frac{\pi}{4}$ $\qquad$ $r = 2\sin\theta \;,\; 0 \leq \theta \leq \pi$

Example

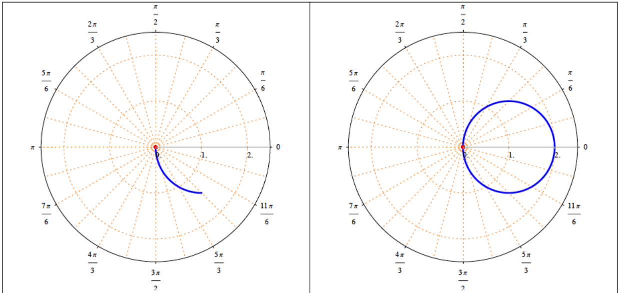
$r = -2\sin\theta$ represents a circle with center $= (0, -1)$ and radius equals to 1.



3-Circles of the form $r = a\cos\theta$, where $a \in \mathbb{R}^*$ and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
$r = a\cos\theta \Rightarrow r^2 = a\ r\cos\theta \Rightarrow x^2 + y^2 = ax \Rightarrow x^2 - ax + y^2 = 0 \Rightarrow$
$(x^2 - ax + \frac{a^2}{4}) + y^2 = \frac{a^2}{4} \Rightarrow (x - \frac{a}{2})^2 + y^2 = \frac{a^2}{4}$
Therefore, $r = a\cos\theta$ represents a circle with center $= (\frac{a}{2}, 0)$ and radius equals to $\frac{|a|}{2}$

Example

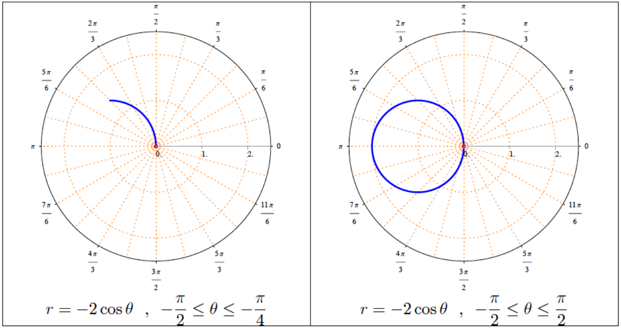
$r = 2\cos\theta$ represents a circle with center $= (1, 0)$ and radius equals to 1.



$r = 2\cos\theta \ , \ -\frac{\pi}{2} \le \theta \le -\frac{\pi}{4}$ $\qquad$ $r = 2\cos\theta \ , \ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

Example

$r = -2\cos\theta$ represents a circle with center $= (-1, 0)$ and radius equals to 1.



Third - Limacon curves:

The general form of a Limacon curve is

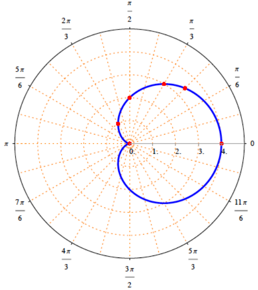
$r(\theta) = a + b\sin\theta$ or $r(\theta) = a + b\cos\theta$, where $a, b \in \mathbb{R}^*$ and $0 \leq \theta \leq 2\pi$

1-Cardioid (Heart-shaped): It has the form $r(\theta) = a + a\sin\theta$ or $r(\theta) = a + a\cos\theta$, where $a \in \mathbb{R}^*$ and $0 \leq \theta \leq 2\pi$

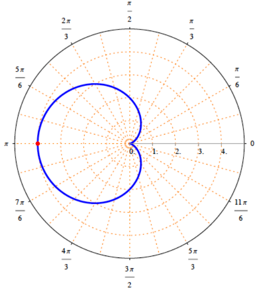
Example

$r(\theta) = 2 + 2\cos\theta$

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	4	$2+\sqrt{2}$	3	2	1	0



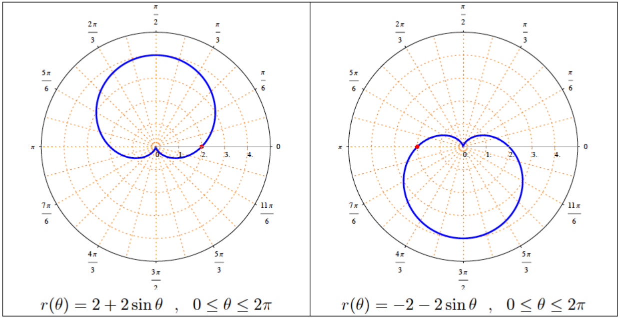
$r(\theta) = 2 + 2\cos\theta \quad , \quad 0 \le \theta \le 2\pi$



$r(\theta) = -2 - 2\cos\theta \quad , \quad 0 \le \theta \le 2\pi$

Example

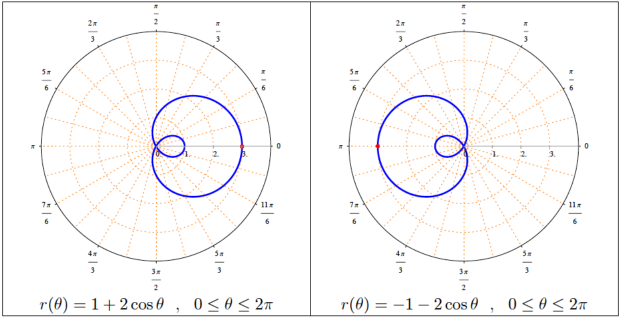
$r(\theta) = 2 + 2\sin\theta$ and $r(\theta) = -2 - 2\sin\theta$



2-Limacon with inner loop: It has the form $r(\theta) = a + b\sin\theta$ or $r(\theta) = a + b\cos\theta$, where $a, b \in \mathbb{R}^*$, $|a| < |b|$ and $0 \leq \theta \leq 2\pi$

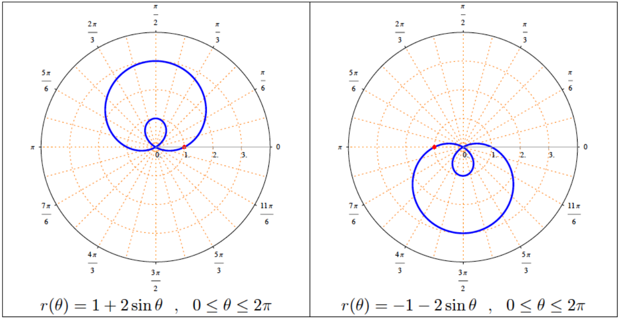
Example

$r(\theta) = 1 + 2\cos\theta$ and $r(\theta) = -1 - 2\cos\theta$



Example

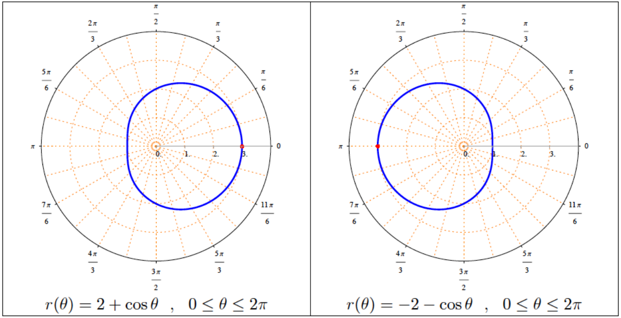
$r(\theta) = 1 + 2\sin\theta$ and $r(\theta) = -1 - 2\sin\theta$



3-Dimpled Limacon : It has the form $r(\theta) = a + b\sin\theta$ or $r(\theta)a + b\cos\theta$, where $a, b \in \mathbb{R}^*$, $|a| > |b|$ and $0 \leq \theta \leq 2\pi$

Example

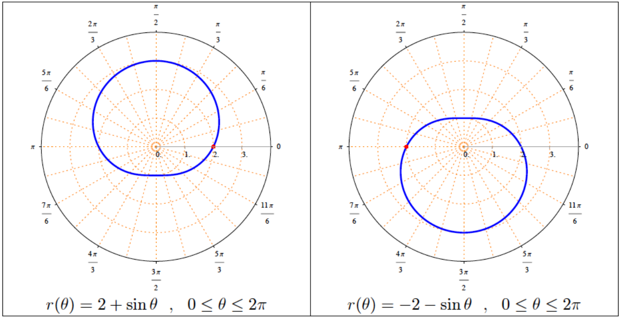
$r(\theta) = 2 + \cos\theta$ and $r(\theta) = -2 - \cos\theta$



Polar Curves

Example

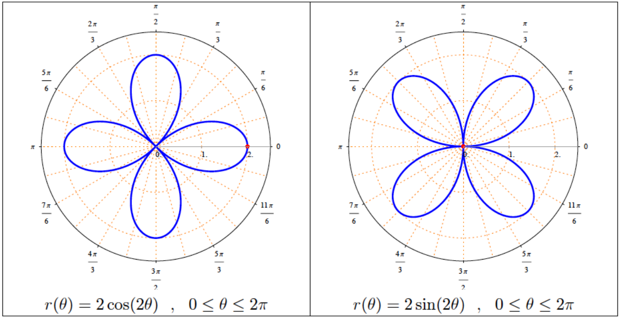
$r(\theta) = 2 + \sin\theta$ and $r(\theta) = -2 - \sin\theta$



Fourth - Rose curves:
It has the form $r(\theta) = a\cos(n\theta)$ or $r(\theta) = a\sin(n\theta)$, where $a \in \mathbb{R}^*$, $n \in \mathbb{N}$ and $n \geq 2$
1-n is even: In this case the number of loops (or leaves) is 2n.

Example

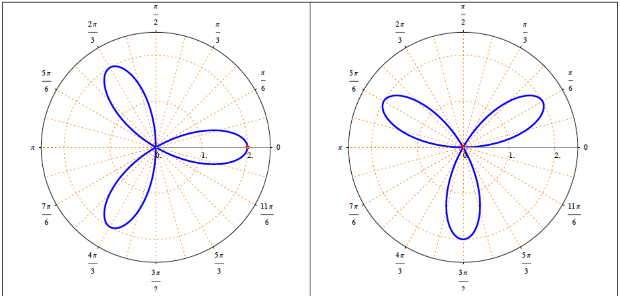
$r(\theta) = 2\cos(2\theta)$ or $r(\theta) = 2\sin(2\theta), 0 \leq \theta \leq 2\pi$



n is odd: In this case the number of loops (or leaves) is n .

Example

$r(\theta) = 2\cos(3\theta)$ or $r(\theta) = 2\sin(3\theta), 0 \leq \theta \leq \pi$



$r(\theta) = 2\cos(3\theta)$, $0 \leq \theta \leq \pi$ $\quad$ $r(\theta) = 2\sin(3\theta)$, $0 \leq \theta \leq \pi$

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Slope Of The Tangents Line With Polar Coordinates

If $r = r(\theta)$ is a smooth polar curve, then the slope of the tangent line to $r = r(\theta)$ is $m = \frac{dy}{dx}$ where $(x = r\cos\theta, \quad y = r\sin\theta)$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\dfrac{dr}{d\theta}\sin\theta + r\cos\theta}{\dfrac{dr}{d\theta}\cos\theta - r\sin\theta}$$

Example

Example

Determine the equation of the tangent line to
$r = 3 + 8\sin\theta$ at $\theta = \dfrac{\pi}{6}$

We'll first need the following derivative. $\frac{dr}{d\theta} = 8\cos\theta$
The formula for the derivative $\frac{dy}{dx}$ becomes,

$$\frac{dy}{dx} = \frac{8\cos\theta\sin\theta + (3 + 8\sin\theta)\cos\theta}{8\cos^2\theta - (3 + 8\sin\theta)\sin\theta} = \frac{16\cos\theta\sin\theta + 3\cos\theta}{8\cos^2\theta - 3\sin\theta - 8\sin^2\theta}$$

The slope of the tangent line is,

$$m = \left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{6}} = \frac{4\sqrt{3} + \frac{3\sqrt{3}}{2}}{4 - \frac{3}{2}} = \frac{11\sqrt{3}}{5}$$

Example

Now, at $\theta = \frac{\pi}{6}$ we have $r = 7$ We'll need to get the corresponding $x - y$ coordinates so we can get the tangent line.

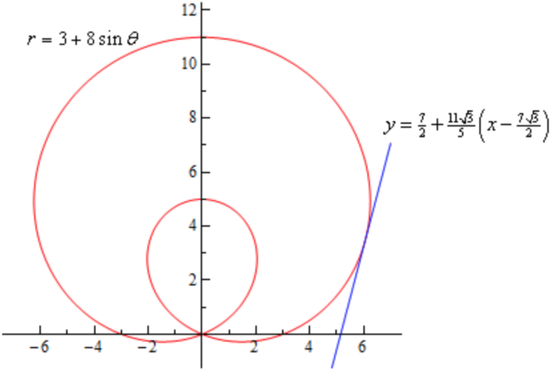
$$x = 7\cos\left(\frac{\pi}{6}\right) = \frac{7\sqrt{3}}{2} \qquad y = 7\sin\left(\frac{\pi}{6}\right) = \frac{7}{2}$$

The tangent line is then,

$$y = \frac{7}{2} + \frac{11\sqrt{3}}{5}\left(x - \frac{7\sqrt{3}}{2}\right)$$

Example

For the sake of completeness here is a graph of the curve and the tangent line.



Example

Example

Find the points on the polar curve $r(\theta) = 1 + \cos\theta,\ 0 \leq \theta \leq 2\pi$ at which the tangent line to r is horizontal.

The tangent line to $r = r(\theta)$ is horizontal if $\frac{dy}{d\theta} = 0$ and $\frac{dx}{d\theta} \neq 0$

$x = r(\theta)\cos\theta \Rightarrow x = \cos\theta(1 + \cos\theta) = \cos\theta + \cos^2\theta$

$y = r(\theta)\sin\theta \Rightarrow y = \sin\theta(1 + \cos\theta) = \sin\theta + \sin\theta\cos\theta = \sin\theta + \frac{1}{2}\sin 2\theta$

$\frac{dx}{d\theta} = -\sin\theta - 2\cos\theta\sin\theta = -\sin\theta - \sin 2\theta$

$\frac{dy}{d\theta} = \cos\theta + \cos 2\theta$

$\frac{dy}{d\theta} = 0 \Rightarrow \cos\theta + \cos 2\theta = 0 \Rightarrow 2\cos^2\theta - 1 + \cos\theta = 0 \Rightarrow$

$(2\cos\theta - 1)(\cos\theta + 1) = 0 \Rightarrow \cos\theta = -1$ or $\cos\theta = \frac{1}{2}$

$\Rightarrow \theta = \pi$ or $\theta = \frac{\pi}{3}, \theta = \frac{5\pi}{3}$

Example

For $\theta = \pi$, $\frac{dx}{d\theta} = 0$.
For $\theta = \frac{\pi}{3}, \theta = \frac{5\pi}{3} \in [0, 2\pi]$ and $\frac{dx}{d\theta} \neq 0$.
At $\theta = \frac{\pi}{3}$: $r(\frac{\pi}{3}) = 1 + \frac{1}{2} = \frac{3}{2}$
At $\theta = \frac{5\pi}{3}$: $r(\frac{5\pi}{3}) = 1 + \frac{1}{2} = \frac{3}{2}$
The points on $r(\theta) = 1 + \cos\theta, 0 \leq \theta \leq 2\pi$ at which the tangent line to r is horizontal are $(\frac{3}{2}, \frac{\pi}{3}), (\frac{3}{2}, \frac{5\pi}{3})$

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Area Inside-Between Polar Curves

The area of the region bounded by the graphs of the polar curves $r = r(\theta), \theta = \theta_1$ and $\theta = \theta_2$ is

$$A = \frac{1}{2}\int_{\theta_1}^{\theta_2} [r(\theta)]^2 \, d\theta$$

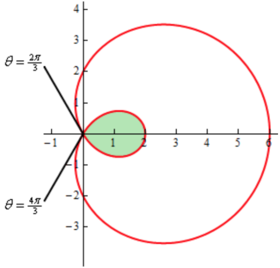
Area Inside-Between Polar Curves

Example

Determine the area of the inner loop of $r = 2 + 4\cos\theta$

$$0 = 2 + 4\cos\theta$$

$$\cos\theta = -\frac{1}{2} \qquad \Rightarrow \qquad \theta = \frac{2\pi}{3}, \frac{4\pi}{3}$$



Area Inside-Between Polar Curves

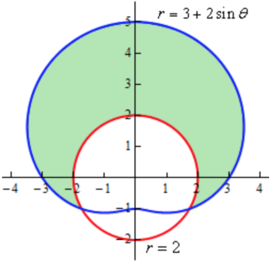
$$
\begin{aligned}
A &= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{1}{2}(2+4\cos\theta)^2 \, d\theta \\
&= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{1}{2}\left(4+16\cos\theta+16\cos^2\theta\right) \, d\theta \\
&= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} 2+8\cos\theta+4\left(1+\cos(2\theta)\right) \, d\theta \\
&= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} 6+8\cos\theta+4\cos(2\theta) \, d\theta \\
&= \left(6\theta+8\sin\theta+2\sin(2\theta)\right)\Big|_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \\
&= 4\pi-6\sqrt{3}=2.174
\end{aligned}
$$

Example

Determine the area that lies inside $r = 3 + 2\sin\theta$ and outside $r = 2$

$$3 + 2\sin\theta = 2$$

$$\sin\theta = -\frac{1}{2} \qquad \Rightarrow \qquad \theta = \frac{7\pi}{6}, \frac{11\pi}{6}$$



Area Inside-Between Polar Curves

$$A = \int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \frac{1}{2}\left((3+2\sin\theta)^2 - (2)^2\right)\, d\theta$$

$$= \int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \frac{1}{2}\left(5 + 12\sin\theta + 4\sin^2\theta\right)\, d\theta$$

$$= \int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \frac{1}{2}\left(7 + 12\sin\theta - 2\cos(2\theta)\right)\, d\theta$$

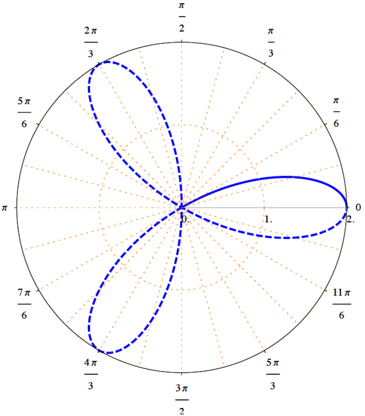
$$= \frac{1}{2}\left(7\theta - 12\cos\theta - \sin(2\theta)\right)\Big|_{-\frac{\pi}{6}}^{\frac{7\pi}{6}}$$

$$= \frac{11\sqrt{3}}{2} + \frac{14\pi}{3} = 24.187$$

Area Inside-Between Polar Curves

Example

Find the area inside one leaf of the rose curve $r = 2\cos 3\theta$



The rose curve $r = 2\cos 3\theta, 0 \leq \theta \leq \pi$ starts at $(r, \theta) = (2, 0)$ and reaches the pole when $r = 0$

$r = 0 \Rightarrow 2\cos 3\theta = 0 \Rightarrow 3\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{6}$ Since the desired area is symmetric with respect to the polar axis , then

$$\begin{aligned} A &= 2\left(\frac{1}{2}\int_0^{\frac{\pi}{6}} (2\cos 3\theta)^2 d\theta\right) \\ &= 4\int_0^{\frac{\pi}{6}} \cos^2 3\theta \, d\theta \\ &= 4\int_0^{\frac{\pi}{6}} \frac{1}{2}(1+\cos 6\theta)\, d\theta \\ &= 2\int_0^{\frac{\pi}{6}} (1+\cos 6\theta)\, d\theta \\ &= 2\left[\theta + \frac{\sin 6\theta}{6}\right]_0^{\frac{\pi}{6}} = \frac{\pi}{3} \end{aligned}$$

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Arc Length Of A Polar Curve

The arc length of the polar curve $r = r(\theta)$ from θ_1 to θ_2 is

$$L = \int_{\theta_1}^{\theta_2} \sqrt{(r(\theta))^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Example

Determine the length of the following polar curve. $r = -4\sin\theta,\ 0 \leq \theta \leq \pi$

$\frac{dr}{d\theta} = -4\cos\theta$

$$L = = \int_0^\pi \sqrt{[-4\sin\theta]^2 + [-4\cos\theta]^2}\, d\theta$$

$$= \int_0^\pi \sqrt{16\sin^2\theta + 16\cos^2\theta}\, d\theta = 4\sqrt{\sin^2\theta + \cos^2\theta}\, d\theta = \int_0^\pi 4 d\theta$$

$$L = \int_0^\pi 4\, d\theta = [4\theta]_0^\pi = 4\pi$$

Example

Find the arc length of the following polar curve: $r = e^{-\theta}$

$\frac{dr}{d\theta} = -e^{-\theta}$

$$L == \int_0^{\pi} \sqrt{(e^{-\theta})^2 + (-e^{-\theta})^2}\, d\theta$$

$$= \int_0^{\pi} \sqrt{e^{-2\theta} + e^{-2\theta}}\, d\theta = \int_0^{\pi} \sqrt{2e^{-2\theta}}\, d\theta = \sqrt{2} \int_0^{\pi} e^{-\theta}\, d\theta$$

$$L = \sqrt{2}\left[-e^{-\theta}\right]_0^{\pi} = \sqrt{2}[-e^{-\pi} + e^{0}] = \sqrt{2}(1 - e^{-\pi})$$

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Surface Area Generated By Revolving A Polar Curve

The surface area generated by revolving the polar curve $r = r(\theta),\ \theta_1 \leq \theta \leq \theta_2$ around the polar axis is

$$SA = 2\pi \int_{\theta_1}^{\theta_2} |r(\theta)\sin\theta| \sqrt{(r(\theta))^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

The surface area generated by revolving the polar curve $r = r(\theta),\ \theta_1 \leq \theta \leq \theta_2$ around the line $\theta = \frac{\pi}{2}$ is

$$SA = 2\pi \int_{\theta_1}^{\theta_2} |r(\theta)\cos\theta| \sqrt{(r(\theta))^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Example

Find the surface area generated by revolving the following polar curve: $r = 2 + 2\cos\theta, 0 \le \theta \le \frac{\pi}{2}$ around the polar axis.

$\frac{dr}{d\theta} = -2\sin\theta$

$$SA = 2\pi \int_0^{\frac{\pi}{2}} |(2 + 2\cos\theta)\sin\theta| \sqrt{(2 + 2\cos\theta)^2 + (-2\sin\theta)^2}\, d\theta$$

$$= 2\pi \int_0^{\frac{\pi}{2}} (2 + 2\cos\theta)\sin\theta \sqrt{4\,(2 + 2\cos\theta)}\, d\theta$$

$$= 4\pi \int_0^{\frac{\pi}{2}} (2 + 2\cos\theta)^{\frac{3}{2}} \sin\theta\, d\theta$$

$\frac{\pi}{2}$

$$SA = -2\pi \int_0^{\frac{\pi}{2}} (2 + 2\cos\theta)^{\frac{3}{2}} \left(-2\sin\theta\right)\, d\theta$$

$$= -2\pi \left[\frac{2}{5}(2 + 2\cos\theta)^{\frac{5}{2}}\right]_0^{\frac{\pi}{2}}$$

$$= -2\pi\frac{2}{5}\left[4\sqrt{2} - 32\right] = \frac{16\pi}{5}\left(8 - \sqrt{2}\right)$$

Example

Find the surface area generated by revolving the following polar curve: $r = 2\sin\theta, 0 \leq \theta \leq \frac{\pi}{2}$ around the line $\theta = \frac{\pi}{2}$

$\frac{dr}{d\theta} = 2\cos\theta$

$$SA = 2\pi \int_0^{\frac{\pi}{2}} |2\sin\theta\cos\theta| \sqrt{(2\sin\theta)^2 + (2\cos\theta)^2}\, d\theta$$

$$= 2\pi \int_0^{\frac{\pi}{2}} \sin 2\theta \sqrt{4}\, d\theta$$

$$SA = 4\pi \left[-\frac{\cos 2\theta}{2} \right]_0^{\frac{\pi}{2}} = 4\pi$$