Engineering Probability & Statistics (AGE 1150)

Chapter 5: Some Discrete Probability Distributions

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Discrete Uniform Distribution

- If the discrete random variable X assumes the values x_1, x_2, …, x_k with equal probabilities, then X has the discrete uniform distribution given by:

$$f(x) = P(X = x) = f(x;k) = \begin{cases} \frac{1}{k} & ; x = x_1, x_2, \cdots, x_k \\ 0 ; & elsewhere \end{cases}$$

- Note:
- f(x)=f(x;*k*)=P(X=x)
- *k* is called the parameter of the distribution.

Example 5.2:

- Experiment: tossing a balanced die.
- Sample space: S={1,2,3,4,5,6}
- Each sample point of S occurs with the same probability 1/6.
- Let X= the number observed when tossing a balanced die.
- The probability distribution of X is:

$$f(x) = P(X = x) = f(x;6) = \begin{cases} \frac{1}{6} \; ; x = 1, 2, \cdots, 6 \\ 0 \; ; \; elsewhere \end{cases}$$

Theorem 5.1:

- If the discrete random variable X has a discrete uniform distribution with parameter k, then the mean and the variance of X are:

$$\mathrm{E(X)} = \mu = \frac{\sum_{i=1}^{k} x_i}{k}$$

$$\mathrm{Var(X)} = \sigma^2 = \frac{\sum_{i=1}^{k} (x_i - \mu)^2}{k}$$

Example 5.3:

Find E(X) and Var(X) in Example 5.2.

$$E(X) = \mu = \frac{\sum_{i=1}^{k} x_i}{k} = \frac{1+2+3+4+5+6}{6} = 3.5$$

$$Var(X) = \sigma^2 = \frac{\sum_{i=1}^{k} (x_i - \mu)^2}{k} = \frac{\sum_{i=1}^{k} (x_i - 3.5)^2}{6}$$

$$= \frac{(1-3.5)^2 + (2-3.5)^2 + \cdots + (6-3.5)^2}{6} = \frac{35}{12}$$

Binomial Distribution

- **Bernoulli Trial:**
- Bernoulli trial is an experiment with only two possible outcomes.
- The two possible outcomes are labeled:

success (s) and failure (f)

- The probability of success is $P(s)=p$ and the probability of failure is $P(f)= q = 1-p$.
- Examples:
  - 1. Tossing a coin (success=H, failure=T, and p=P(H))
  - 2. Inspecting an item (success=defective, failure=non-defective, and p=P(defective))

Bernoulli Process

- Bernoulli process is an experiment that must satisfy the following properties:

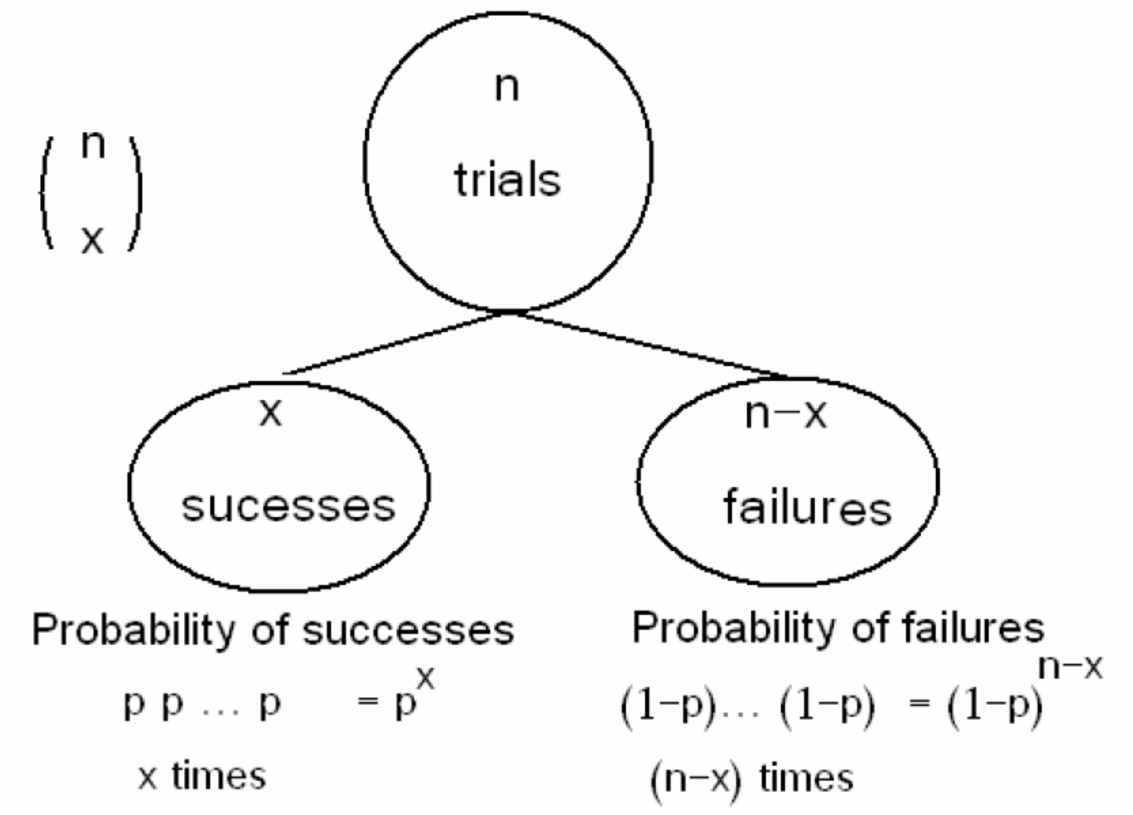
1. The experiment consists of *n* repeated Bernoulli trials.
2. The probability of success, P(*s*)=*p*, remains constant from trial to trial.
3. The repeated trials are independent; that is the outcome of one trial has no effect on the outcome of any other trial

- **Binomial Random Variable:**
- Consider the random variable :
- *X* = The number of successes in the *n* trials in a Bernoulli process
- The random variable *X* has a binomial distribution with parameters *n* (number of trials) and *p* (probability of success), and we write:

$$X \sim \text{Binomial}(n,p) \quad \text{or} \quad X \sim \text{b}(x;n,p)$$

- The probability distribution of *X* is given by:

$$f(x) = P(X = x) = b(x; n, p) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & ; \ x = 0, 1, 2, \ldots, n \\ 0 \ ; & otherwise \end{cases}$$



- We can write the probability distribution of *X* as a table as follows

x	f(x)=P(X=x)=b(x;n,p)
0	$\binom{n}{0}p^0(1-p)^{n-0}=(1-p)^n$
1	$\binom{n}{1}p^1(1-p)^{n-1}$
2	$\binom{n}{2}p^2(1-p)^{n-2}$
$\vdots$	$\vdots$
$n-1$	$\binom{n}{n-1}p^{n-1}(1-p)^1$
n	$\binom{n}{n}p^n(1-p)^0=p^n$
Total	1.00

Example:

Suppose that 25% of the products of a manufacturing process are defective. Three items are selected at random, inspected, and classified as defective (D) or non-defective (N). Find the probability distribution of the number of defective items.

Solution:

- Experiment: selecting 3 items at random, inspected, and classified as (D) or (N).
- The sample space is

S={DDD,DDN,DND,DNN,NDD,NDN,NND,NNN}

- Let *X* = the number of defective items in the sample
- We need to find the probability distribution of *X*.

(1) First Solution:

Outcome	Probability	x
NNN	$\frac{3}{4}\times\frac{3}{4}\times\frac{3}{4}=\frac{27}{64}$	0
NND	$\frac{3}{4}\times\frac{3}{4}\times\frac{1}{4}=\frac{9}{64}$	1
NDN	$\frac{3}{4}\times\frac{1}{4}\times\frac{3}{4}=\frac{9}{64}$	1
NDD	$\frac{3}{4}\times\frac{1}{4}\times\frac{1}{4}=\frac{3}{64}$	2
DNN	$\frac{1}{4}\times\frac{3}{4}\times\frac{3}{4}=\frac{9}{64}$	1
DND	$\frac{1}{4}\times\frac{3}{4}\times\frac{1}{4}=\frac{3}{64}$	2
DDN	$\frac{1}{4}\times\frac{1}{4}\times\frac{3}{4}=\frac{3}{64}$	2
DDD	$\frac{1}{4}\times\frac{1}{4}\times\frac{1}{4}=\frac{1}{64}$	3

The probability distribution .of *X* is

.x	.f(x)=P(X=x)
0	$\frac{27}{64}$
1	$\frac{9}{64}+\frac{9}{64}+\frac{9}{64}=\frac{27}{64}$
2	$\frac{3}{64}+\frac{3}{64}+\frac{3}{64}=\frac{9}{64}$
3	$\frac{1}{64}$

(2) Second Solution:

- Bernoulli trial is the process of inspecting the item. The results are success=D or failure=N, with probability of success P(s)=25/100=1/4=0.25.
- The experiments is a Bernoulli process with:
- number of trials: n=3
- Probability of success: p=1/4=0.25
- X ~ Binomial(n,p)=Binomial(3,1/4)
- The probability distribution of X is given by:

$$f(x)=P(X=x)=b(x;3,\frac{1}{4})=\begin{cases}\binom{3}{x}(\frac{1}{4})^{x}(\frac{3}{4})^{3-x}; & x=0,1,2,3\\ 0\ ; & otherwise\end{cases}$$

$$f(0)=P(X=0)=b(0;3,\frac{1}{4})=\binom{3}{0}(\frac{1}{4})^{0}(\frac{3}{4})^{3}=\frac{27}{64}$$

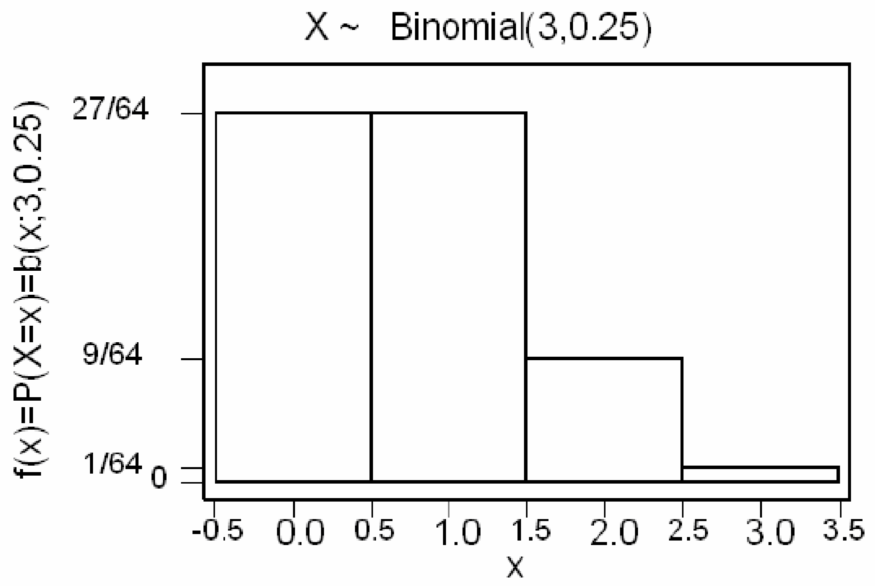
$$f(1)=P(X=1)=b(1;3,\frac{1}{4})=\binom{3}{1}(\frac{1}{4})^{1}(\frac{3}{4})^{2}=\frac{27}{64}$$

$$f(2)=P(X=2)=b(2;3,\frac{1}{4})=\binom{3}{2}(\frac{1}{4})^{2}(\frac{3}{4})^{1}=\frac{9}{64}$$

$$f(3)=P(X=3)=b(3;3,\frac{1}{4})=\binom{3}{3}(\frac{1}{4})^{3}(\frac{3}{4})^{0}=\frac{1}{64}$$

The probability distribution of *X* is

x	f(x)=P(X=x) =b(x;3,1/4)
0	27/64
1	27/64
2	9/64
3	1/64



- **Theorem 5.2:**

The mean and the variance of the binomial distribution b(x;*n*,*p*) are:

$$\mu = n\,p$$
$$\sigma^2 = n\,p\,(1 - p)$$

- **Example:**

In the previous example, find the expected value (mean) and the variance of the number of defective items.

- X = number of defective items
- We need to find $E(X)=\mu$ and $Var(X)=\sigma^2$
- We found that $X \sim$ Binomial(n,p)=Binomial(3,1/4)
- .n=3 and p=1/4

The expected number of defective items is

- $E(X)=\mu = n\ p = (3)\ (1/4) = 3/4 = 0.75$

The variance of the number of defective items is

- $Var(X)=\sigma^2 = n\ p\ (1 - p) = (3)\ (1/4)\ (3/4) = 9/16 = 0.5625$

Example:

In the previous example, find the following probabilities:

(1) The probability of getting at least two defective items.

(2) The probability of getting at most two defective items.

- X ~ Binomial(3,1/4)

$$f(x) = P(X=x) = b(x;3,\frac{1}{4}) = \begin{cases} \binom{3}{x} (\frac{1}{4})^x (\frac{3}{4})^{3-x} & for\ \ x = 0, 1, 2, 3 \\ 0 & otherwise \end{cases}$$

.x	.f(x)=P(X=x)=b(x;3,1/4)
0	27/64
1	27/64
2	9/64
3	1/64

(1) The probability of getting at least two defective items:

$$P(X \geq 2) = P(X=2) + P(X=3) = f(2) + f(3) = \frac{9}{64} + \frac{1}{64} = \frac{10}{64}$$

(2) The probability of getting at most two defective item:

$$P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$$

$$= f(0) + f(1) + f(2) = \frac{27}{64} + \frac{27}{64} + \frac{9}{64} = \frac{63}{64}$$

or

$$P(X \leq 2) = 1 - P(X > 2) = 1 - P(X=3) = 1 - f(3) = 1 - \frac{1}{64} = \frac{63}{64}$$