Theory of statistics 2

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March 17, 2020

Testing Hypothesis

Let $f(x;\theta)$ be the pdf of X. The parameter θ belongs to a parameter space Ω. We may doubt that θ belongs to a subset $\Omega_0 \subseteq \Omega$ or to its complement $\Omega_1 = \Omega \setminus \Omega_0$. In this case, there are two hypotheses H_0 and H_1:

$$\gamma = \begin{cases} H_0 : \theta \in \Omega_0 \\ \\ H_1 : \theta \in \Omega_1 \end{cases}$$

Definitions

1. The error type 1 of the test γ is to reject H_0 when it is true, and the error type 2 of the test γ is to accept H_0 when it is false.
2. The power function of the test γ is the probability of rejecting H_0 conditional θ:

$$\pi_\gamma(\theta) = \mathbf{P}(\text{reject } H_0 | \theta).$$

3. The size of the test γ is given by:

$$\alpha = \sup_{\theta \in \Omega_0} \pi_\gamma(\theta).$$

Simple hypothesis

In the simple case, the null hypothesis is given by $H_0 : \theta = \theta_0$. The size α becomes $\alpha = \pi_\gamma(\theta_0)$ and it represents the probability of the error type 1. Moreover, the probability of the error type 2 of the alternative simple case $H_1 : \theta = \theta_1$ is usually given by the symbol $\beta = 1 - \pi_\gamma(\theta_1)$.

Definition: The most powerful test

A test γ_{MP} with size α_{MP} is called Most Powerful Test (MPT) for the hypothesis $H_0 : \theta = \theta_0$ vs $H_1 : \theta = \theta_1$ if, for all other test γ with size α, we have

$$\alpha \leq \alpha_{MP}.$$

Consequently, $\pi_{\gamma_{MP}}(\theta_1) \geq \pi_{\gamma}(\theta_1)$ $(\beta_{MP} \leq \beta)$.

Neyman-Pearson lemma

The test γ_{MP} of size α_{MP} is found by the following steps:

1. Take the Likelihood Ratio (LR) $\lambda = \dfrac{\ell(\underline{X};\theta_0)}{\ell(\underline{X};\theta_1)}$.
2. Reject $H_0 : \theta = \theta_0$ if $\lambda < k$.
3. Find k by solving the implicit equation $\alpha_{MP} = \mathbf{P}(\lambda < k|\theta_0)$.

Remark

In general, any test γ uses LR to reject $H_0 : \theta = \theta_0$ if $\lambda < k$ is called Likelihood Ratio Test (LRT). Thus, γ_{MP} is LRT. However, there are another two LRT: Minimax test γ_{MM} and Bayes test γ_B. In these tests, we reject if $\lambda < k$ but k is found in a different way.

Definition

A distribution $f(x;\theta)$ belongs to the class of exponential families if, it is written in the form:

$$f(x;\theta) = e^{a(\theta)+b(x)+d(x)c(\theta)}.$$

Example 1: The normal distribution

Let $f(x;\theta)$ be the normal density with mean θ and known variance σ^2. Thus,

$$f(x;\theta) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\theta}{\sigma}\right)^2} = e^{-\frac{1}{2}\left(\frac{\theta}{\sigma}\right)^2 - \frac{1}{2}\left(\log(2\pi\sigma^2) + \left(\frac{x}{\sigma}\right)^2\right) + x\frac{\theta}{\sigma^2}}.$$

Hence $a(\theta) = -\frac{1}{2}\left(\frac{\theta}{\sigma}\right)^2$, $b(x) = \frac{1}{2}\left(\log(2\pi\sigma^2) + \left(\frac{x}{\sigma}\right)^2\right)$, $c(\theta) = \frac{\theta}{\sigma^2}$ and $d(x) = x$.

Example 2

Let $f(x;\theta) = \theta x^{\theta-1}, \quad 0 < x < 1$. Thus,

$$f(x;\theta) = \theta x^{\theta-1} = e^{\log(\theta)+\theta\log(x)-\log(x)}.$$

Hence $a(\theta) = \log(\theta)$, $b(x) = -\log(x)$, $c(\theta) = \theta$ and $d(x) = \log(x)$.

Example 3

Let $f(x;\theta) = \dfrac{\theta}{(1+x)^{\theta+1}}, \quad x > 0$. Thus,

$$f(x;\theta) = \frac{\theta}{(1+x)^{\theta+1}} = e^{\log(\theta) - \theta\log(1+x) - \log(1+x)}.$$

Hence $a(\theta) = \log(\theta)$, $b(x) = -\log(1+x)$, $c(\theta) = \theta$ and $d(x) = \log(1+x)$.

Theorem

If $f(x;\theta)$ belongs to the class of exponential families, then the test γ_{MP} for $H_0 : \theta = \theta_0$ vs $H_1 : \theta = \theta_1$ rejects H_0 is reduced as follows:

	$\theta_0 < \theta_1$	$\theta_0 > \theta_1$
$c(\theta) \nearrow$	$\sum d(x_i) > k$	$\sum d(x_i) < k$
$c(\theta) \searrow$	$\sum d(x_i) < k$	$\sum d(x_i) > k$
	Alike	Inverse

k solves the equation

$$\alpha_{MP} = \mathbf{P}\left(Reject \quad H_0|\theta_0\right).$$

Example 1: The normal distribution $N(\theta, \sigma^2)$ when σ^2 is known

Let X be a normal random variable with distribution $N(\theta, 16)$. Let $X_1, \ldots, X_{25}$ be 25 copies of X. Our aim is to test the hypotheses $H_0 : \theta = 3$ vs $H_1 : \theta = 5$ by the γ_{MP} with size $\alpha_{MP} = 0.05$. Note that $f(x; \theta)$ belongs to the class of exponential families with $c(\theta) = \dfrac{\theta}{\sigma^2}$. Since $c(\theta)$ is an increasing function, then γ_{MP} rejects H_0 if $\sum d(x_i) = \sum x_i > k$. Let us determine k.

$$
\begin{aligned}
\alpha_{MP} &= 0.05 = \mathbf{P}\left(\sum x_i > k \middle| \theta = 3\right) = \mathbf{P}\left(\overline{x} > \frac{k}{25} \middle| \theta = 3\right) \\
&= \mathbf{P}\left(\frac{\overline{x} - \theta}{16/\sqrt{25}} > \frac{\frac{k}{25} - \theta}{16/\sqrt{25}} \middle| \theta = 3\right) = \mathbf{P}\left(Z > \frac{\frac{k}{25} - 3}{16/\sqrt{25}}\right).
\end{aligned}
$$

Example 1: The normal distribution $N(\theta, \sigma^2)$ when σ^2 is known

This implies that $\dfrac{\frac{k}{25} - \theta}{16/\sqrt{25}} = 1.645$. Thus $k = 107.9$. The error type 2 β_{MP} is equal to

$$\beta_{MP} = \mathbf{P}\left(\sum x_i < 107.9 \middle| \theta = 5\right) = \mathbf{P}\left(Z < -0.855\right) = 0.1949.$$

Example 2: The normal distribution $N(\theta, \sigma^2)$ when θ is known

Let X be a normal random variable with distribution $N(\theta, \sigma^2)$. Let $X_1, \ldots, X_{10}$ be 10 copies of X. Our aim is to test the hypotheses $H_0 : \sigma = 2$ vs $H_1 : \sigma = 4$ by the γ_{MP} with size $\alpha_{MP} = 0.05$. Note that $f(x; \sigma)$ belongs to the class of exponential families with $c(\sigma^2) = -\frac{1}{2\sigma^2}$. Since $c(\sigma)$ is an increasing function, then γ_{MP} rejects H_0 if $\sum d(x_i) = \sum (x_i - \theta)^2 > k$. Let us determine k.

$$\begin{aligned}\alpha_{MP} &= \mathbf{P}\left(\sum (x_i - \theta)^2 > k \middle| \sigma = 2\right) = \mathbf{P}\left(\frac{1}{4}\sum (x_i - \theta)^2 > \frac{k}{4}\right) \\ &= \mathbf{P}\left(U > \frac{k}{4}\right) = \int_{\frac{k}{4}}^{+\inf} \mathcal{X}^2_{10}(dt)\end{aligned}$$

Example 2: The normal distribution $N(\theta, \sigma^2)$ when θ is known

This implies that $\dfrac{k}{4} = 18.31$. Thus 73.24. The error type 2 β_{MP} is equal to

$$\begin{aligned} \beta_{MP} &= \mathbf{P}\left(\sum (x_i - \theta)^2 < 73.24 \middle| \sigma = 4\right) \\ &= \mathbf{P}\left(\frac{1}{16}\sum (x_i - \theta)^2 < \frac{73.24}{16}\right) \\ &= \int_0^{\frac{73.24}{16}} \mathcal{X}^2_{10}(dt) = 0.082441. \end{aligned}$$

Example 3: The exponential distribution

Let X be an exponential random variable with distribution $\exp(\theta)$. Let $X_1, \ldots, X_{10}$ be 10 copies of X. Our aim is to test the hypotheses $H_0 : \theta = 2$ vs $H_1 : \theta = 3$ by the γ_{MP} with size $\alpha_{MP} = 0.05$. Note that $f(x;\theta)$ belongs to the class of exponential families with $c(\theta) = -\theta$. Since $c(\theta)$ is a decreasing function, then γ_{MP} rejects H_0 if $S = \sum d(x_i) = \sum x_i < k$. Using the fact that X_i are exponentially distributed, then $U = 2\theta S$ is Chi-squared distributed $\mathcal{X}^2_{2n}$ Let us determine k.

$$\alpha_{MP} = \mathbf{P}\left(S < k \middle| \theta = 2\right) = \mathbf{P}\left(U < 2\theta k \middle| \theta = 2\right) = \mathbf{P}\left(U < 4k\right).$$

This implies that $4k = 10.851$. Thus $k = 2.713$. The error type 2 β_{MP} is equal to

$$\beta_{MP} = \mathbf{P}\left(S > 2.713 \middle| \theta = 3\right) = \mathbf{P}\left(U > 16.277\right) = 0.7.$$

Thank you