INTEGRAL CALCULUS (MATH 106)

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Chapter 5: Application of the definite integral

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The student is expected to be able to:

1. Calculate the area between curves.
2. Calculate the volume of a solid revolution using the disk method.
3. Calculate the volume of a solid revolution using the washer method.
4. Calculate the volume of a solid revolution using the Cylindrical shells method.
5. Calculate the arc length.
6. Calculate the area of a surface of revolution.

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Area Between Curves.

In this section we are going to look at finding the area between two curves.

QUESTION:

How we can determine the area between $y = f(x)$ and $y = g(x)$ on the interval $[a, b]$

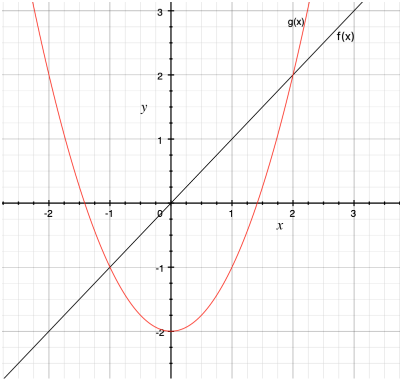
Theorem: Area Between Curves

Let $f(x)$ and $g(x)$ be continuous functions defind on $[a, b]$ where $f(x) \geq g(x)$ for all x in $[a, b]$.
The area of the region bounded by the curves $y = f(x)$, $y = g(x)$ and the lines $x = a$ and $x = b$ is

$$\int_a^b \Big[f(x) - g(x) \Big] \, dx$$

Area Between Curves.

$$A = \int_a^b \text{(upper function) - (lower function)}\, dx, \qquad a \le x \le b$$



$$A = \int_a^b f(x) - g(x)\, dx$$

The steps to calculate the area between curves

1. Find the intersection points between the curves.
2. determinant the upper function and the lower function.
3. Calculate the integral:

$$A = \int_a^b \text{(upper function) - (lower function) } dx$$

Which give us the required area.

Area Between Two Curves (Example)

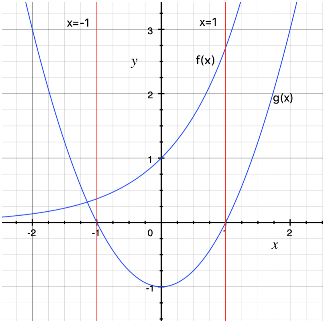
Example

Find the area enclosed between the graphs $y = x$ and $y = x^2 - 2$.

1. Points of intersection between $y = x^2 - 2$ and $y = x$ is:
   $x^2 - 2 = x \Rightarrow x^2 - x - 2 = 0 \Rightarrow (x+1)(x-2) = 0$
   $\Rightarrow x = -1$ and $x = 2$
2. Note that upper function is $y = x$ and lower function is $y = x^2 - 2$
   Note that $y = x^2 - 2$ is a parabola opens upward with vertex $(0, -2)$, and $y = x$ is a straight line passing through the origin.
3. $A = \int_{-1}^{2} x - (x^2 - 2)\, dx = \int_{-1}^{2} x - x^2 + 2\, dx = \left[\frac{x^2}{2} - \frac{x^3}{3} + 2x\right]_{-1}^{2} = \frac{27}{6}$

Example

Find the area enclosed between the graphs $y = e^x, y = x^2 - 1, x = -1,$ and $x = 1$



Area Between Two Curves (Example)

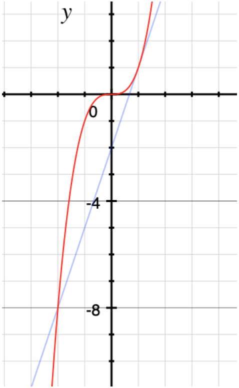
Note that upper function is $y = e^x$ and lower function is $y = x^2 - 1$

$$A = \int_{-1}^{1} e^x - (x^2 - 1)\ dx = \int_{-1}^{1} e^x - x^2 + 1\ dx = \left[e^x - \frac{1}{3}x^3 + x\right]_{-1}^{1}$$

$$= e - \tfrac{1}{e} + \tfrac{4}{3}$$

Example

Compute the area oh the region bounded by the curves
$y = x^3$ and $y = 3x - 2$



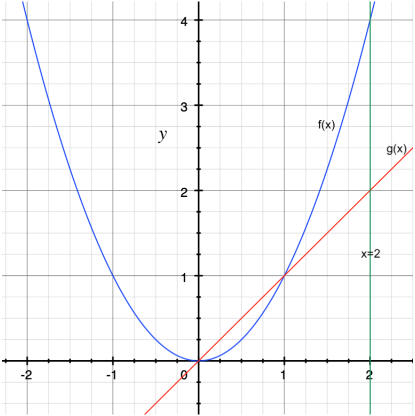
Area Between Two Curves (Example)

1. Points of intersection between $y = x^3$ and $y = 3x - 2$
$x^3 - 3x + 2 = 0 \Rightarrow (x-1)(x^2 + x - 2) = 0$
$\Rightarrow x = -2$ and $x = 1$
2. Note that upper function is $y = x^3$ and lower function is $y = 3x - 2$
3. $$A = \int_{-2}^{1} x^3 - (3x-2)\ dx = \int_{-2}^{1} x^3 - 3x + 2\ dx$$
$$= \left[\frac{x^4}{4} - \frac{3}{2}x^2 + 2x\right]_{-2}^{1}$$
$$= \frac{3}{4} + 6 = \frac{27}{4}$$

Area Between Two Curves (Example)

Example

Find the area enclosed between the graphs
$f(x) = x^2$ and $g(x) = x$ between $x = 0$, and $x = 2$.



Area Between Two Curves (Example)

1. we see that the two graphs intersect at $(0, 0)$ and $(1, 1)$.
2. In the interval $[0, 1]$, we have $g(x) = x \geq f(x) = x^2$, and in the interval $[1, 2]$, we have $f(x) = x^2 \geq g(x) = x$
3. Therefore the desired area is:

$$A = \int_0^1 (x - x^2)\ dx + \int_1^2 (x^2 - x)\ dx = \left[\frac{x^2}{2} - \frac{x^3}{0}\right]_0^1 + \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_1^2$$

$$= \frac{1}{6} + \frac{5}{6} = 1$$

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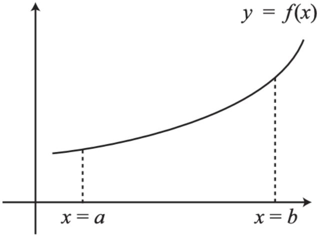
Volume Of A Solid Revolution (The Disk Method)

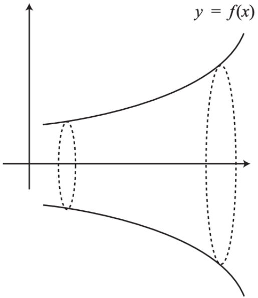
Suppose we have a curve $y = f(x)$



Imagine that the part of the curve between the ordinates $x = a$ and $x = b$ is rotated about the x-axis through 360 degree.

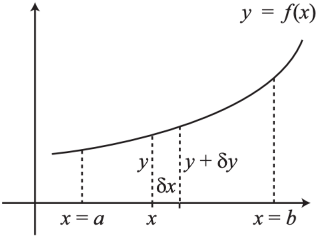
Volume Of A Solid Revolution (The Disk Method)

Now if we take a cross-section of the solid, parallel to the y-axis, this cross-section will be a circle.



But rather than take a cross-section, let us take a thin disc of thickness δx, with the face of the disc nearest the y-axis at a distance x from the origin.

Volume Of A Solid Revolution (The Disk Method)



The radius of this circular face will then be y. The radius of the other circular face will be $y + \delta y$, where δy is the change in y caused by the small positive increase in $x, \delta x$.

Volume Of A Solid Revolution (The Disk Method)

The volume δV of the disc is then given by the volume of a cylinder, $\pi r^2 h$, so that

$$\delta V = \pi r^2 \delta x$$

So the volume V of the solid of revolution is given by

$$V = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} \delta V = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} \pi y^2 \delta x = \pi \int_a^b [f(x)]^2 \, dx$$

Examples

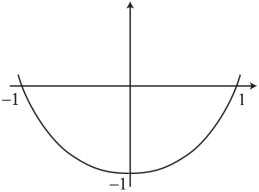
Example

The curve $y = x^2 - 1$ is rotated about the x-axis through 360 degree. Find the volume of the solid generated when the area contained between the curve and the x-axis is rotated about the x-axis by 360 degree.

$$V = \pi \int_a^b [f(x)]^2 \, dx = \pi \int_{-1}^{1} [x^2 - 1]^2 \, dx$$

$$= \pi \int_{-1}^{1} (x^4 - 2x^2 + 1) \, dx$$

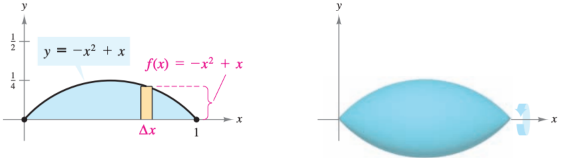
$$= \left[\frac{x^5}{5} - \frac{2x^3}{3} + x \right]_{-1}^{1} = \frac{16\pi}{15}$$



The graph of $y = x^2 - 1$

Example

Find the volume of the solid formed by revolving the region bounded by the graph of $f(x) = -x^2 + x$ and the x-axis about the x-axis.



Using the Disk Method, you can find the volume of the solid of revolution.

$$V = \pi \int_0^1 [f(x)]^2 \, dx = \pi \int_0^1 [(-x^2 + x)^2 \, dx = \pi \int_0^1 (x^4 - 2x^3 + x^2) \, dx$$

$$= \pi \left[\frac{x^5}{5} - \frac{2x^4}{4} + \frac{x^3}{3} \right]_0^1 = \frac{\pi}{30}$$

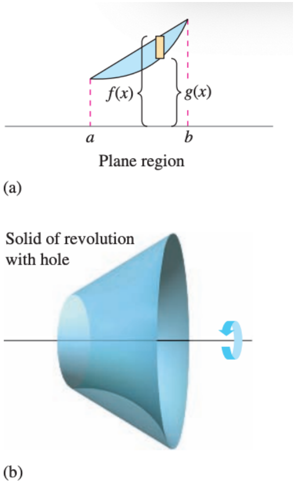
Volume Of A Solid Revolution (The Washer Method)

The Washer Method

Let f and g be continuous and nonnegative on the closed interval $[a, b]$, if $f(x) \geq g(x)$ for all x in the interval, then the volume of the solid formed by revolving the region bounded by the graphs of $f(x)$ and $g(x)$ $(a \leq x \leq b)$, about the x-axis is:

$$V = \pi \int_a^b \left\{ [f(x)]^2 - [g(x)]^2 \right\} dx$$

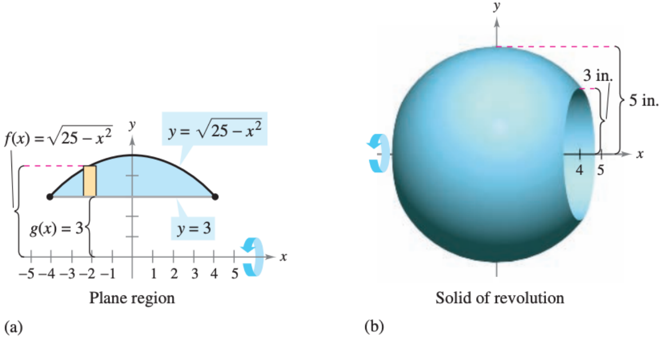
$f(x)$ is the **outer radius**
and $g(x)$ is the **inner radius**.



Example

Find the volume of the solid formed by revolving the region bounded by the graphs of $f(x) = \sqrt{25 - x^2}$ and $g(x) = 3$

We sketch the bounding region and the solid of revolution:



Examples

First find the points of intersection of f and g, by setting $f(x)$ equal to $g(x)$ and solving for x.
$\sqrt{25 - x^2} = 3 \Rightarrow 25 - x^2 = 9 \Rightarrow x^2 = 16 \Rightarrow x = \pm 4$
Using $f(x)$ as the outer radius and $g(x)$ as the inner radius, you can find the volume of the solid as shown.

$$V = \pi \int_a^b \left\{ [f(x)]^2 - [g(x)]^2 \right\} dx = \pi \int_{-4}^{4} (\sqrt{25 - x^2})^2 - (3)^2 \ dx$$

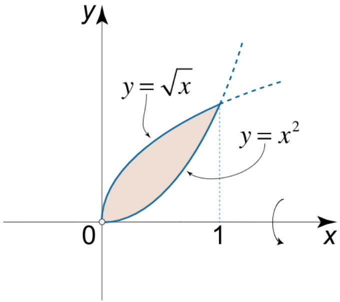
$$= \pi \int_{-4}^{4} (16 - x^2) \ dx = \pi \left[16x - \frac{x^3}{3} \right]_{-4}^{4} = \frac{256\pi}{3}$$

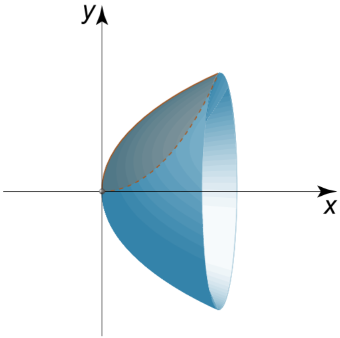
Examples

Example

Calculate the volume of the solid obtained by rotating the region bounded by the parabola $y = x^2$ and the square root function $y = \sqrt{x}$ around the $x-$axis

We sketch the bounding region and the solid of revolution:





Examples

Both curves intersect at the points $x = 0$ and $x = 1$. Using the washer method, we have

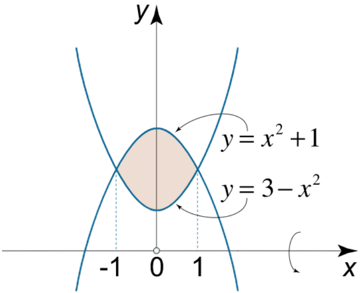
$$V = \pi \int_a^b \left\{ [f(x)]^2 - [g(x)]^2 \right\} dx = \pi \int_0^1 (\sqrt{x})^2 - (x^2)^2 \ dx$$

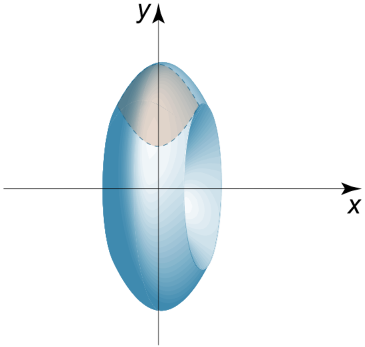
$$= \pi \int_0^1 (x - x^4) \ dx = \pi \left[\frac{x^2}{2} - \frac{x^5}{5} \right]_0^1 = \pi \left[\frac{1}{2} - \frac{1}{5} \right] = \frac{3\pi}{10}$$

Example

Find the volume of the solid obtained by rotating the region bounded by two parabolas $y = x^2 + 1$ and $y = 3 - x^2$ about the $x-$axis.

We sketch the bounding region and the solid of revolution:





Examples

First we determine the boundaries a and b:
$x^2+1=3-x^2 \Rightarrow 2x^2=2 \Rightarrow x^2=1 \Rightarrow x=\pm 1$
Hence the limits of integration are $a=1$ and $b=-1$.
Using the washer method, we find the volume of the solid:

$$V=\pi\int_a^b \left\{ [f(x)]^2-[g(x)]^2 \right\} dx$$

$$=\pi\int_{-1}^{1} \left[(3-x^2)^2-(x^2+1)^2\right] \ dx=\pi\int_{-1}^{1} \left(8-8x^2\right) \ dx$$

$$=8\pi\int_{-1}^{1} \left(1-x^2\right) \ dx=8\pi\left[x-\frac{x^3}{3}\right]_{-1}^{1}=\frac{32\pi}{3}$$

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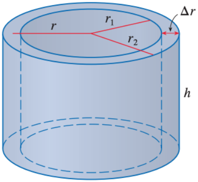
The method of cylindrical shells

the cylindrical shell with inner radius r_1, outer radius r_2 , and height h. Its volume V is calculated by subtracting the volume V_1 of the inner cylinder from the volume V_2 of the outer cylinder:

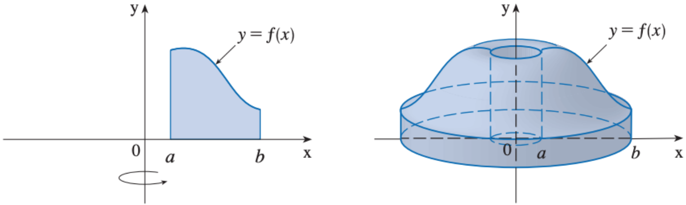
$$V = V_2 - V_1 = \pi r_2^2 h - \pi r_1^2 h$$

$$= \pi(r_2^2 - r_1^2)h = \pi(r_2 - r_1)(r_2 + r_1)h$$

$$= 2\pi \frac{r_2 + r_1}{2} h(r_2 - r_1) \Rightarrow V = 2\pi r h \Delta r$$

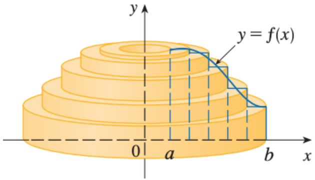


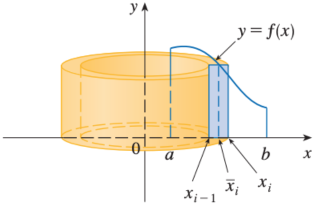
let be the solid obtained by rotating about the -axis the region bounded by $y = f(x)$,
where $f(x) \geq 0$, $y = 0$, $x = a$ and $x = b$, where $b > a \geq 0$.



We divide the interval into n subintervals $[x_{i-1}, x_{i+1}]$ of equal width and let $\overline{x_i}$ be the midpoint of the i th subinterval. If the rectangle with base $[x_{i-1}, x_i]$ and height $f(\overline{x}_i)$ is rotated about the $y-$ axis then the result is a cylindrical shell with average radius $\overline{x}_i$ height $f(\overline{x}_i)$ and thickness Δx so its volume is:

$$V_i = (2\pi)\overline{x}_i[f(\overline{x}_i)]\Delta x$$





Volume Of A Solid Revolution (Cylindrical shells method)

An approximation to the volume of is given by the sum of the volumes of these shells:

$$V \approx \sum_{i=1}^{n} V_i = \sum_{i=1}^{n} 2\pi \overline{x}_i [f(\overline{x}_i)] \Delta x$$

This approximation appears to become better as $n \to \infty$ But, from the definition of an integral, we know that

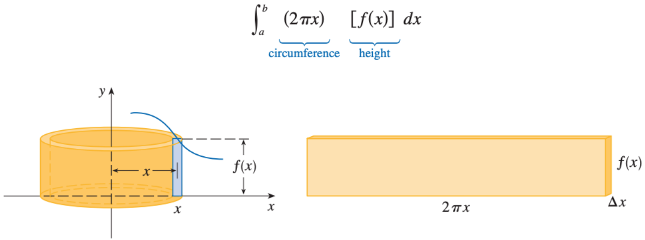
$$\lim_{n\to\infty} \sum_{i=1}^{n} 2\pi \overline{x}_i [f(\overline{x}_i)] \Delta x = \int_a^b 2\pi x f(x)\, dx$$

Volume Of A Solid Revolution (Cylindrical shells method)

The volume of the solid, obtained by rotating about the $y-$axis the region under the curve $y = f(x)$ from a to b, is

$$V = \int_a^b 2\pi x f(x) dx \qquad \textit{where} \;\; 0 \leq a < b$$

The best way to remember the last Formula is to think of a typical shell, cut and flattened as in Figure with radius x, circumference $2\pi x$, height $f(x)$ and thickness Δx or dx :



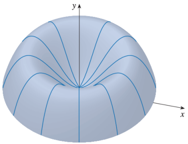
Example

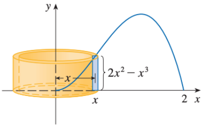
Find the volume of the solid obtained by rotating about the $y-$axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$

by the shell method, the volume is

$$V = \int_0^2 (2\pi x)(2x^2 - x^3)\ dx = 2\pi \int_0^2 (2x^3 - x^4)\ dx = 2\pi \left[\frac{x^4}{2} - \frac{x^5}{5}\right]_0^2$$

$$= 2\pi(8 - \tfrac{32}{5}) = \tfrac{16}{5}\pi$$





Examples

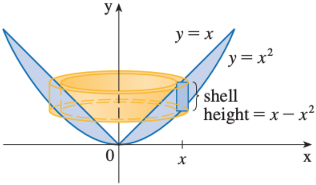
Example

Find the volume of the solid obtained by rotating about the $y-$axis the region between $y = x$ and $y = x^2$.

$$V = \int_0^1 (2\pi x)(x - x^2)\ dx$$

$$= 2\pi \int_0^1 (x^2 - x^3)\ dx$$

$$= 2\pi \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{\pi}{6}$$



Example

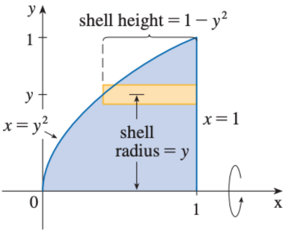
Use cylindrical shells to find the volume of the solid obtained by rotating about the $x-$axis the region under the curve $y = \sqrt{x}$ from 0 to 1.

For rotation about the $x-$axis we see that a typical shell has radius y, circumference $2\pi y$, and height $1 - y^2$. So the volume is

$$V = \int_0^1 (2\pi y)(1 - y^2)\ dy$$

$$= 2\pi \int_0^1 (y - y^3)\ dy$$

$$= 2\pi \left[\frac{y^2}{2} - \frac{y^4}{4}\right]_0^1 = \frac{\pi}{2}$$

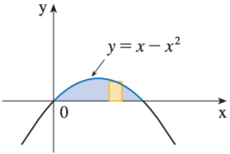


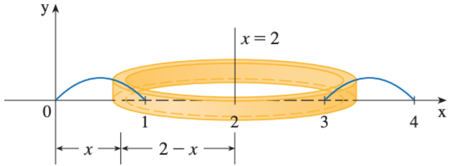
Example

Find the volume of the solid obtained by rotating the region bounded by $y = x - x^2$ and $y = 0$ about the line $x = 2$.

the region and a cylindrical shell formed by rotation about the line $x = 2$. It has radius $2 - x$, circumference $2\pi(2 - x)$, and height $x - x^2$.

$$V = \int_0^1 2\pi(2-x)(x-x^2)\, dx = 2\pi \int_0^1 (x^3 - 3x^2 + 2x)\, dx$$
$$= 2\pi\left[\frac{x^4}{4} - x^3 + x^2\right]_0^1 = \frac{\pi}{2}$$





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Arc Length

Definition

1. If $f(x)$ is continuous function on the interval $[a, b]$, then the arc length of $f(x)$ from $x = a$ to $x = b$ is:

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

2. If $g(y)$ is continuous function on the interval $[c, d]$, then the arc length of $g(y)$ from $y = c$ to $y = d$ is:

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy$$

Examples

example

Determine the length of $y = \ln(\sec x)$ between $0 \le x \le \frac{\pi}{4}$

$$f'(x) = \frac{\sec x \tan x}{\sec x} = \tan x \Rightarrow [f'(x)]^2 = \tan^2 x$$

$$\sqrt{1 + [f'(x)]^2} = \sqrt{1 + \tan^2 x} = \sqrt{\sec^2 x} = |\sec x| = \sec x$$

The arc length is then,

$$\int_0^{\frac{\pi}{4}} \sec x \, dx = \left[\ln |\sec x + \tan x| \right]_0^{\frac{\pi}{4}} = \ln(\sqrt{2} + 1)$$

Examples

example

Determine the length of $x = \frac{2}{3}(y-1)^{\frac{3}{2}}$ between $1 \leq y \leq 4$

$\frac{dx}{dy} = (y-1)^{\frac{1}{2}} \Rightarrow \sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + y - 1} = \sqrt{y}$

The arc length is then,

$$L = \int_1^4 \sqrt{y}\, dy = \frac{2}{3} y^{\frac{3}{2}} \Big|_1^4 = \frac{14}{3}$$

Examples

example

Determine the length of $x = \frac{1}{2}y^2$ between $0 \leq x \leq \frac{1}{2}$. Assume that y is positive.

$\frac{dx}{dy} = y \quad \Rightarrow \quad \sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + y^2}$

Before writing down the length notice that we were given x limits and we will need y limits. $0 \leq y \leq 1$

The integral for the arc length is then,

$$L = \int_0^1 \sqrt{1 + y^2}\, dy$$

Examples

$$L = \int_0^1 \sqrt{1 + y^2}\, dy$$

This integral will require the following trig substitution.
$y = \tan\theta \qquad dy = \sec^2\theta\, d\theta$

$$y = 0 \quad \Rightarrow \quad 0 = \tan\theta \quad \Rightarrow \quad \theta = 0$$
$$y = 1 \quad \Rightarrow \quad 1 = \tan\theta \quad \Rightarrow \quad \theta = \frac{\pi}{4}$$

$$\sqrt{1 + y^2} = \sqrt{1 + \tan^2\theta} = \sqrt{\sec^2\theta} = |\sec\theta| = \sec\theta$$

Examples

The length is then,

$$\begin{aligned} L &= \int_0^{\frac{\pi}{4}} \sec^3\theta \, d\theta \\ &= \frac{1}{2} \left(\sec\theta \tan\theta + \ln\left|\sec\theta + \tan\theta\right|\right)\Big|_0^{\frac{\pi}{4}} \\ &= \frac{1}{2}\left(\sqrt{2} + \ln\left(1 + \sqrt{2}\right)\right) \end{aligned}$$

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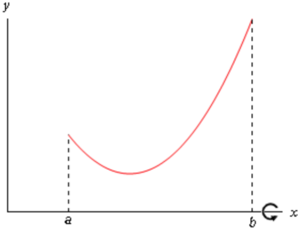
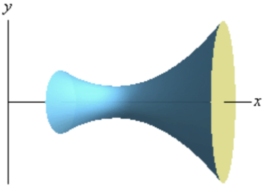
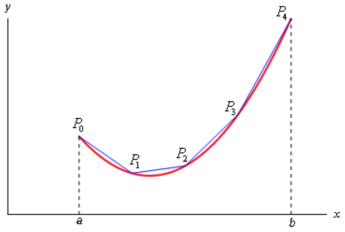
Let $f(x)$ be a nonnegative smooth function over the interval $[a, b]$. We wish to find the surface area of the surface of revolution created by revolving the graph of $y = f(x)$ around the $x-$axis as shown in the following figure.



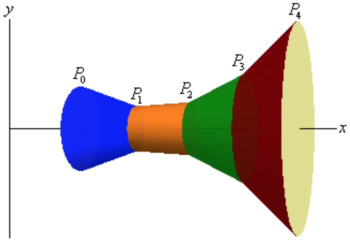


1. We'll start by dividing the interval into n equal subintervals of width Δx
2. On each subinterval we will approximate the function with a straight line that agrees with the function at the endpoints of each interval.
3. Here is a sketch of that for our representative function using $n = 4$



Area of a Surface of Revolution

Now, rotate the approximations about the $x-$axis and we get the following solid.



The approximation on each interval gives a distinct portion of the solid and to make this clear each portion is colored differently.

Area of a Surface of Revolution

The area of each of these is:

$$A = 2\pi r l$$

where,

$$r = \frac{1}{2}(r_1 + r_2) \qquad \begin{aligned} r_1 &= \text{radius of right end} \\ r_2 &= \text{radius of left end} \end{aligned}$$

and l is the length of the slant of each interval.

We know from the previous section that,

$$|P_{i-1}\ P_i| = \sqrt{1 + \left[f'\left(x_i^*\right)\right]^2}\ \Delta x \quad \text{where } x_i^* \text{ is some point in } [x_{i-1}, x_i]$$

Before writing down the formula for the surface area we are going to assume that Δx is "small" and since $f(x)$ is continuous we can then assume that,

$$f(x_i) \approx f(x_i^*) \qquad \text{and} \qquad f(x_{i-1}) \approx f(x_i^*)$$

Area of a Surface of Revolution

So, the surface area of each interval $[x_{i-1}, x_i]$ is approximately,

$$A_i = 2\pi \left(\frac{f(x_i) + f(x_{i-1})}{2} \right) |P_{i-1}\ P_i|$$

$$\approx 2\pi f(x_i^*) \sqrt{1 + \left[f'(x_i^*)\right]^2}\ \Delta x$$

The surface area of the whole solid is then approximately,

$$S \approx \sum_{i=1}^{n} 2\pi f(x_i^*) \sqrt{1 + \left[f'(x_i^*)\right]^2}\ \Delta x$$

and we can get the exact surface area by taking the limit as n goes to infinity.

$$S = 2\pi \lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i^*\right) \sqrt{1 + \left[f'\left(x_i^*\right)\right]^2}\ \Delta x$$
$$= 2\pi \int_a^b f\left(x\right) \sqrt{1 + \left[f'\left(x\right)\right]^2}\, dx$$

If we wanted to we could also derive a similar formula for rotating $x = h\left(y\right)$ on $[c, d]$ about the $y-$axis. This would give the following formula.

$$S = 2\pi \int_c^d h\left(y\right) \sqrt{1 + \left[h'\left(y\right)\right]^2}\, dy$$

Area of a Surface of Revolution (Example)

Example

Determine the surface area of the solid obtained by rotating $y = \sqrt{9 - x^2}, -2 \leq x \leq 2$ about the $x-$axis.

$$S = 2\pi \int_a^b f(x) \sqrt{1 + [f'(x)]^2}\, dy$$

$$\frac{dy}{dx} = \frac{1}{2}\left(9 - x^2\right)^{-\frac{1}{2}}(-2x) = -\frac{x}{(9 - x^2)^{\frac{1}{2}}}$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{x^2}{9 - x^2}} = \sqrt{\frac{9}{9 - x^2}} = \frac{3}{\sqrt{9 - x^2}}$$

Area of a Surface of Revolution (Example)

Here's the integral for the surface area,

$$S = 2\pi \int_{-2}^{2} f(x) \frac{3}{\sqrt{9 - x^2}}\, dx$$

$$S = 2\pi \int_{-2}^{2} \sqrt{9 - x^2} \frac{3}{\sqrt{9 - x^2}}\, dx$$
$$= 6\pi \int_{-2}^{2} dx = 24\pi$$

Area of a Surface of Revolution (Example)

Example

Determine the surface area of the solid obtained by rotating $y = \sqrt[3]{x}, 1 \leq y \leq 2$ about the $y-$axis.

Solution

$$S = 2\pi \int_c^d h(y)\sqrt{1 + [h'(y)]^2}\, dy$$

$$x = h(y) = y^3 \qquad \frac{dx}{dy} = 3y^2$$

$$\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + 9y^4}$$

Area of a Surface of Revolution (Example)

The surface area is then,

$$S = 2\pi \int_1^2 h(y)\sqrt{1+9y^4}\,dy$$

$$\begin{aligned} S &= 2\pi \int_1^2 y^3\sqrt{1+9y^4}\,dy \qquad\qquad u = 1+9y^4 \\ &= \frac{\pi}{18}\int_{10}^{145}\sqrt{u}\,du \\ &= \frac{\pi}{27}\left(145^{\frac{3}{2}} - 10^{\frac{3}{2}}\right) = 199.48 \end{aligned}$$