Engineering Probability & Statistics (AGE 1150)

Chapter 4: Mathematical Expectation
Part 2

Dr. Feras Fraige

Means and Variances of Linear Combinations of Random Variables:

- If $X_1, X_2, \ldots, X_n$ are n random variables and $a_1, a_2, \ldots, a_n$ are constants, then the random variable :

$$Y = \sum_{i=1}^{n} a_i X_i = a_1 X_1 + a_2 X_2 + \cdots + a_n X_n$$

is called a linear combination of the random variables $X_1, X_2, \ldots, X_n$.

- **Theorem:**

If X is a random variable with mean $\mu=E(X)$, and if a and b are constants, then:

$$E(aX \pm b) = a\,E(X) \pm b$$

$$\Leftrightarrow$$

$$\mu_{aX \pm b} = a\,\mu_X \pm b$$

- **Corollary 1:** $E(b) = b$ (a=0 in the Theorem)
- **Corollary 2:** $E(aX) = a\,E(X)$ (b=0 in the Theorem)

Example: Let X be a random variable with the following probability density function:

$$f(x)=\begin{cases}\frac{1}{3}x^2 \ ; -1<x<2\\ 0 \ ; \ elsewhere\end{cases}$$

- Find E(4X+3).
- Sol.

$$\mu = \mathrm{E(X)} = \int_{-\infty}^{\infty} x\, f(x)\,dx = \int_{-1}^{2} x[\frac{1}{3}x^2]\,dx = \frac{1}{3}\int_{-1}^{2} x^3\,dx = \frac{1}{3}\left[\frac{1}{4}x^4 \Big|_{x=-1}^{x=2}\right] = 5/4$$

E(4X+3) = 4 E(X)+3 = 4(5/4) + 3=8

Another solution:

$$\mathrm{E[g(X)]} = \int_{-\infty}^{\infty} g(x)\, f(x)\,dx \qquad ; \mathrm{g(X) = 4X+3}$$

$$\mathrm{E(4X+3)} = \int_{-\infty}^{\infty}(4x+3)\, f(x)\,dx = \int_{-1}^{2}(4x+3)\,[\frac{1}{3}x^2]\,dx = \cdots = 8$$

Theorem:

- If $X_1, X_2, \ldots, X_n$ are n random variables and $a_1, a_2, \ldots, a_n$ are constants, then:

$$E(a_1X_1+a_2X_2+ \ldots +a_nX_n) = a_1E(X_1)+ a_2E(X_2)+ \ldots+a_nE(X_n)$$

$$\Leftrightarrow$$

$$E(\sum_{i=1}^{n} a_i X_i) = \sum_{i=1}^{n} a_i E(X_i)$$

- **Corollary:**

If X, and Y are random variables, then:

$E(X \pm Y) = E(X) \pm E(Y)$

Theorem:

- If X is a random variable with variance $Var(X)=\sigma_X^2$ and if a and b are constants, then:

$$\mathrm{Var}(aX \pm b) = a^2\ \mathrm{Var}(X)$$

$$\Leftrightarrow$$

$$\sigma_{aX+b}^2 = a^2\sigma_X^2$$

- **Theorem :**

If X_1, X_2, …, X_n are n <u>**independent**</u> random variables and a_1, a_2, …, a_n are constants, then:

$$\mathrm{Var}(a_1X_1+a_2X_2+\ldots+a_nX_n) = a_1^2\mathrm{Var}(X_1)+ a_2^2\mathrm{Var}\ (X_2)+\ldots+ a_n^2\mathrm{Var}(X_n)$$

$$\Leftrightarrow$$

$$Var(\sum_{i=1}^{n} a_i X_i) = \sum_{i=1}^{n} a_i^2 Var(X_i)$$

$$\Leftrightarrow$$

$$\sigma_{a_1X_1+a_2X_2+\ldots+a_nX_n}^2 = a_1^2\sigma_{X_1}^2 + a_2^2\sigma_{X_2}^2 + \ldots + a_n^2\sigma_{X_n}^2$$

Corollary: :

- If X, and Y are **independent** random variables, then:
- $Var(aX+bY) = a^2 Var(X) + b^2 Var(Y)$
- $Var(aX-bY) = a^2 Var(X) + b^2 Var(Y)$
- $Var(X \pm Y) = Var(X) + Var(Y)$

Example:

Let X, and Y be two independent random variables such that E(X)=2, Var(X)=4, E(Y)=7, and Var(Y)=1. Find:

1. E(3X+7) and Var(3X+7)
2. E(5X+2Y−2) and Var(5X+2Y−2).

Sol.:

E(X)=2, Var(X)=4, E(Y)=7, and Var(Y)=1. required to find:

1. E(3X+7) and Var(3X+7)
2. E(5X+2Y−2) and Var(5X+2Y−2).

1. $E(3X+7) = 3E(X)+7 = 3(2)+7 = 13$

   $Var(3X+7)= (3)^2\ Var(X)=(3)^2\ (4) =36$

2. $E(5X+2Y-2)= 5E(X) + 2E(Y) -2= (5)(2) + (2)(7) - 2= 22$

   $Var(5X+2Y-2)= Var(5X+2Y)= 5^2\ Var(X) + 2^2\ Var(Y)$

   $= (25)(4)+(4)(1) = 104$