Engineering Probability & Statistics (AGE 1150)

Chapter 4: Mathematical Expectation
Part 1

Mean of a Random Variable

- **Definition:**

Let X be a random variable with a probability distribution f(x). The mean (or expected value) of X is denoted by μ_X (or E(X)) and is defined by:

$$E(X)=\mu_X=\begin{cases} \sum_{all\ x} x\, f(x)\ ; & if\ X\ is\ discrete \\ \int_{-\infty}^{\infty} x\, f(x)\, dx\ ; & if\ X\ is\ continuous \end{cases}$$

Example: A lot containing 7 components is sampled by a quality inspector; the lot contains 4 good components and 3 defective components. A sample of 3 is taken by the inspector. Find the expected value of the number of good components in this sample.

Sol.:

- Let *X* represent the number of good components in the sample. The probability distribution of *X* is

$$f(x) = \frac{\binom{4}{x}\binom{3}{3-x}}{\binom{7}{3}}, \qquad x = 0, 1, 2, 3.$$

- Simple calculations yield $f(0) = 1/35$, $f(1) = 12/35$, $f(2) = 18/35$, and $f(3) = 4/35$.
- It is a discrete, therefore

$$\mu = E(X) = (0)\left(\frac{1}{35}\right) + (1)\left(\frac{12}{35}\right) + (2)\left(\frac{18}{35}\right) + (3)\left(\frac{4}{35}\right) = \frac{12}{7} = 1.7.$$

Thus, if a sample of size 3 is selected at random over and over again from a lot of 4 good components and 3 defective components, it will contain, on average, 1.7 good components.

Example: (Return to example in Ch 3)

A shipment of 8 similar microcomputers to a retail outlet contains 3 that are defective and 5 are non-defective. If a school makes a random purchase of 2 of these computers, find the expected number of defective computers purchased

Sol. :

Let X = the number of defective computers purchased.

We found previously (in Ch 3) that the probability distribution of X is:

x	0	1	2
f(x)= P(X=x)	$\frac{10}{28}$	$\frac{15}{28}$	$\frac{3}{28}$

$$f(x) = P(X = x) = \begin{cases} \dfrac{\binom{3}{x} \times \binom{5}{2-x}}{\binom{8}{2}}; & x = 0, 1, 2 \\ 0; \ otherwise & \end{cases}$$

$$E(X) = \mu_X = \sum_{x=0}^{2} x\, f(x)$$

$$= (0)\, f(0) + (1)\, f(1) + (2)\, f(2)$$

$$= (0)\, \frac{10}{28} + (1)\, \frac{15}{28} + (2)\, \frac{3}{28}$$

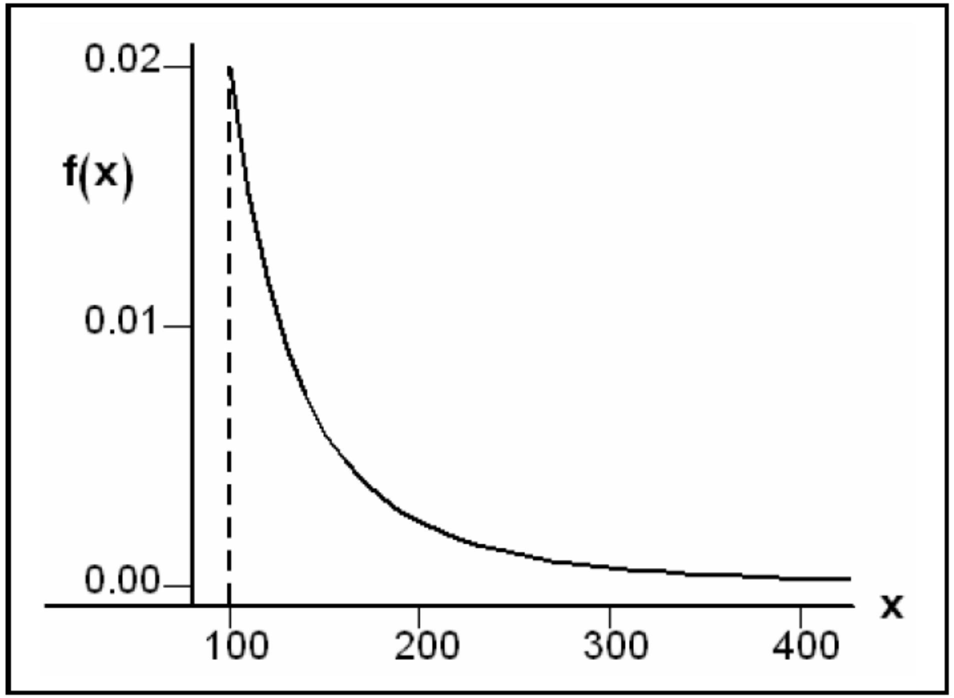
$$= \frac{15}{28} + \frac{6}{28} = \frac{21}{28} = 0.75 \text{ (computers)}$$

Thus, if a sample of size 2 is selected at random over and over again from a lot of 3 defective computers and 5 non-defective computers, it will contain, on average, 0.75 defective computers.

Example: Let X be a continuous random variable that represents the life (in hours) of a certain electronic device. The pdf of X is given by:

$$f(x) = \begin{cases} \dfrac{20{,}000}{x^3} \; ; \; x > 100 \\ 0 \; ; \; elsewhere \end{cases}$$

- Find the expected life of this type of devices.



Solution:

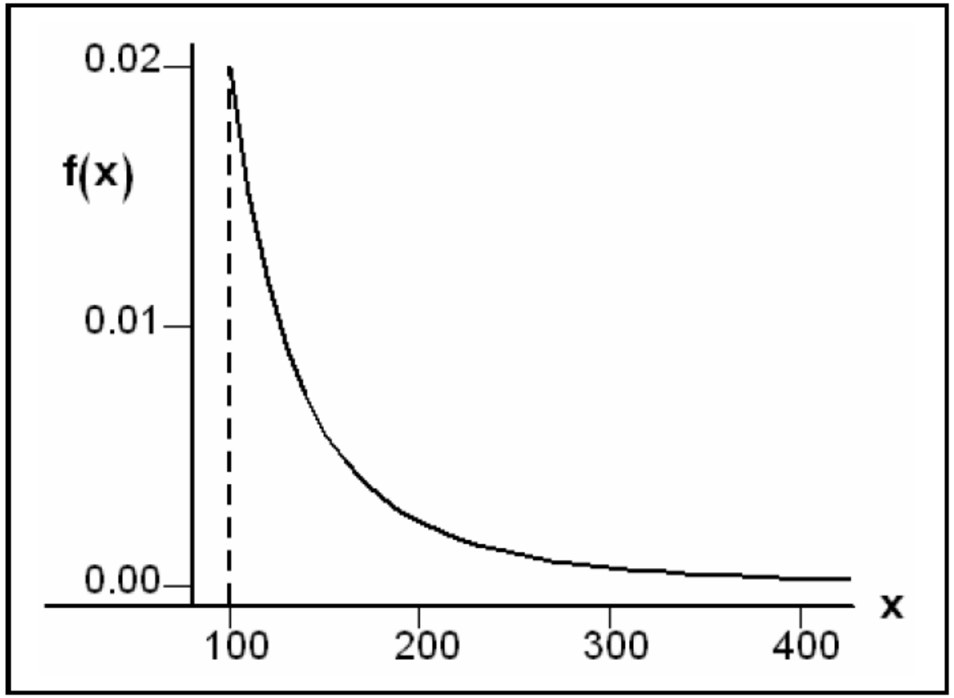
$$E(X)=\mu_X=\int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \int_{100}^{\infty} x \frac{20000}{x^3}\, dx$$

$$= 20000 \int_{100}^{\infty} \frac{1}{x^2}\, dx$$

$$= 20000 \left[-\frac{1}{x} \Big|_{x=100}^{x=\infty} \right]$$

$$= -20000 \left[0 - \frac{1}{100} \right] = 200 \quad \text{(hours)}$$



Therefore, we expect that this type of electronic devices to last, on average, 200 hours.

Theorem

- Let X be a random variable with a probability distribution f(x), and let g(X) be a function of the random variable X. The mean (or expected value) of the random variable g(X) is denoted by $\mu_{g(X)}$ (or E[g(X)]) and is defined by:

$$E[g(X)] = \mu_{g(X)} = \begin{cases} \sum_{all\ x} g(x) f(x) \ ; & if\ X\ is\ discrete \\ \int_{-\infty}^{\infty} g(x) f(x) dx \ ; & if\ X\ is\ continuous \end{cases}$$

Example: Let X be a discrete random variable with the following probability distribution

x	0	1	2
f(x)	$\frac{10}{28}$	$\frac{15}{28}$	$\frac{3}{28}$

Find E[g(X)], where $g(X)=(X-1)^2$.

Solution:

$g(X)=(X-1)^2$

$$E[g(X)]=\mu_{g(X)}=\sum_{x=0}^{2} g(x)f(x)=\sum_{x=0}^{2}(x-1)^2 f(x)$$

$$=(0-1)^2 f(0)+(1-1)^2 f(1)+(2-1)^2 f(2)$$

$$=(-1)^2 \frac{10}{28}+(0)^2\frac{15}{28}+(1)^2\frac{3}{28}$$

$$=\frac{10}{28}+0+\frac{3}{28}=\frac{13}{28}$$

Example (See expected life of electronic device example): Find $E\left(\frac{1}{X}\right)$. {note: $g(X)=\frac{1}{X}$}

Solution:

$$f(x)=\begin{cases}\frac{20,000}{x^3} \; ; \; x>100 \\ 0 \; ; \; elsewhere\end{cases}$$

$$g(X)=\frac{1}{X}$$

$$E\left(\frac{1}{X}\right) = E[g(X)]=\mu_{g(X)} = \int_{-\infty}^{\infty} g(x)f(x)dx = \int_{-\infty}^{\infty} \frac{1}{x} f(x)dx$$

$$= \int_{100}^{\infty} \frac{1}{x}\frac{20000}{x^3}dx = 20000\int_{100}^{\infty} \frac{1}{x^4}dx = \frac{20000}{-3}\left[\frac{1}{x^3}\Big|_{x=100}^{x=\infty}\right]$$

$$= \frac{-20000}{3}\left[0-\frac{1}{1000000}\right] = 0.0067$$

Variance (of a Random Variable):

- The most important measure of variability of a random variable X is called the variance of X and is denoted by Var(X) or .
- **Definition:**

Let X be a random variable with a probability distribu σ_X^2 f(x) and mean μ. The variance of X is defined by:

$$\mathrm{Var}(X)=\sigma_X^2=E[(X-\mu)^2]=\begin{cases}\sum\limits_{all\ x}(x-\mu)^2 f(x)\ ; & if\ X\ is\ discrete\\ \int\limits_{-\infty}^{\infty}(x-\mu)^2 f(x)\,dx\ ; & if\ X\ is\ continuous\end{cases}$$

- **Definition:**
- The positive square root of the variance of X, $\sigma_X=\sqrt{\sigma_X^2}$, is called the standard deviation of X.

Note:

$\mathrm{Var}(X)=E[g(X)]$, where $g(X)=(X-\mu)^2$

- **Theorem:**

The variance of the random variable X is given by:

$$\mathrm{Var}(X) = \sigma_X^2 = E(X^2) - \mu^2$$

$$\text{where } E(X^2) = \begin{cases} \sum_{all\ x} x^2 f(x)\,; & if\ X\ is\ discrete \\ \int_{-\infty}^{\infty} x^2 f(x)\,dx\,; & if\ X\ is\ continuous \end{cases}$$

- **Example:**
- Let X be a discrete random variable with the following probability distribution. Find Var(X)= σ_X^2

x	0	1	2	3
f(x)	0.51	0.38	0.10	0.01

Solution:

$$\mu = \sum_{x=0}^{3} x\, f(x) = (0)\ f(0) + (1)\ f(1) + (2)\ f(2) + (3)\ f(3)$$
$$= (0)\ (0.51) + (1)\ (0.38) + (2)\ (0.10) + (3)\ (0.01)$$
$$= 0.61$$

1. First method:

$$\text{Var}(X) = \sigma_X^2 = \sum_{x=0}^{3} (x-\mu)^2 f(x)$$
$$= \sum_{x=0}^{3} (x-0.61)^2 f(x)$$
$$= (0-0.61)^2\ f(0) + (1-0.61)^2\ f(1) + (2-0.61)^2\ f(2) + (3-0.61)^2\ f(3)$$
$$= (-0.61)^2\ (0.51) + (0.39)^2\ (0.38) + (1.39)^2\ (0.10) + (2.39)^2\ (0.01)$$
$$= 0.4979$$

2. Second method:

$$\text{Var}(X) = \sigma_X^2 = E(X^2) - \mu^2$$
$$E(X^2) = \sum_{x=0}^{3} x^2 f(x) = (0^2)\ f(0) + (1^2)\ f(1) + (2^2)\ f(2) + (3^2)\ f(3)$$
$$= (0)\ (0.51) + (1)\ (0.38) + (4)\ (0.10) + (9)\ (0.01)$$
$$= 0.87$$
$$\text{Var}(X) = \sigma_X^2 = E(X^2) - \mu^2 = 0.87 - (0.61)^2 = 0.4979$$

- **Example:** Let X be a continuous random variable with the following pdf, Find the mean and the variance of X.

$$f(x) = \begin{cases} 2(x-1) \ ; \ 1 < x < 2 \\ 0 \ ; \ elsewhere \end{cases}$$

Solution:

$$\mu = E(X) = \int_{-\infty}^{\infty} x\, f(x) dx \ = \ \int_{1}^{2} x[2(x-1)]\, dx \ = \ 2\int_{1}^{2} x(x-1)\, dx = 5/3$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2\, f(x) dx \ = \ \int_{1}^{2} x^2 [2(x-1)]\, dx \ = \ 2\int_{1}^{2} x^2 (x-1)\, dx = 17/6$$

$$Var(X) = \sigma_X^2 = E(X^2) - \mu^2 = 17/6 - (5/3)^2 = 1/18$$