Vector Spaces & Subspaces

A **vector space** is a nonempty set V of objects, called *vectors*, on which are defined two operations, called *addition* and *multiplication by scalars* (real numbers), subject to the ten axioms below. The axioms must hold for all $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in V and for all scalars c and d.

1. $\mathbf{u}+\mathbf{v}$ is in V.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$
4. There is a vector (called the zero vector) $\mathbf{0}$ in V such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$.
5. For each $\mathbf{u}$ in V, there is vector $-\mathbf{u}$ in V satisfying $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
6. $c\mathbf{u}$ is in V.
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$.
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$.
9. $(cd)\mathbf{u}=c(d\mathbf{u})$.
10. $1\mathbf{u}=\mathbf{u}$.

Example 4.2.3 Here is a collection examples of vector spaces:

1. The set $\mathbb{R}$ of real numbers $\mathbb{R}$ is a vector space over $\mathbb{R}$.
2. The set $\mathbb{R}^2$ of all ordered pairs of real numers is a vector space over $\mathbb{R}$.
3. The set $\mathbb{R}^n$ of all ordered $n-$tuples of real numersis a vector space over $\mathbb{R}$.
4. The set $C(\mathbb{R})$ of all continuous functions defined on the real number line, is a vector space over $\mathbb{R}$.
5. The set $C([a, b]))$ of all continuous functions defined on interval $[a, b]$ is a vector space over $\mathbb{R}$.
6. The set $\mathbb{P}$ of all polynomials, with real coefficients is a vector space over $\mathbb{R}$.
7. The set $\mathbb{P}_n$ of all polynomials of degree $\leq n$, with real coefficients is a vector space over $\mathbb{R}$.
8. The set $\mathbb{M}_{m,n}$ of all $m \times n$ matrices, with real entries, is a vector space over $\mathbb{R}$.

Question: Give an example of a non-vector space over the field of real numbers?

Exercise Let V be the set of all fifth-degree polynomials with standared operations. Is it a vector space. Justify your answer.

Solution: In fact, V is not a vector space. Because V is not closed under addition $f = x^5 + x - 1$ and $g = -x^5$ are in V but $f + g = (x^5 + x - 1) - x^5 = x - 1$ is not in V.

Exercise Let $V = \{(x, y) : x \geq 0, y \geq 0\}$ with standared operations. Is it a vector space. Justify your answer.

Solution: In fact, V is not a vector space. Not every element in V has an addditive inverse $-(1, 1) = (-1, -1)$ is not in V.

Theorem 4.2.4 Let V be vector space over the reals $\mathbb{R}$ and $\mathbf{v}$ be an element in V. Also let c be a scalar. Then,

1. $0\mathbf{v} = \mathbf{0}$.
2. $c\mathbf{0} = \mathbf{0}$.
3. If $c\mathbf{v} = \mathbf{0}$, then either $c = 0$ or $\mathbf{v} = \mathbf{0}$.
4. $(-1)\mathbf{v} = -\mathbf{v}$.

Remark. We denote a vector $\mathbf{u}$ in $\mathbb{R}^n$ by a row $\mathbf{u} = (u_1, u_2, \ldots, u_n)$. As I said before, it can be thought of a row matrix

$$\mathbf{u} = \begin{bmatrix} u_1 & u_2 & \ldots & u_n \end{bmatrix}.$$

In some other situation, it may even be convenient to denote it by a column matrix:

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \ldots \\ u_n \end{bmatrix}.$$

Obviosly, we cannot mix the two (in fact, three) different ways.

Exercise 4.1.6 Let $\mathbf{u} = (0, 0, -8, 1)$ and $\mathbf{v} = (1, -8, 0, 7)$. Find $\mathbf{w}$ such that $2\mathbf{u} + \mathbf{v} - 3\mathbf{w} = \mathbf{0}$.

Solution: We have

$$\mathbf{w} = \frac{2}{3}\mathbf{u} + \frac{1}{3}\mathbf{v} = \frac{2}{3}(0, 0, -8, 1) + \frac{1}{3}(1, -8, 0, 7) = (\frac{1}{3}, -\frac{8}{3}, -\frac{16}{3}, 3).$$

Exercise 4.1.7 Let $\mathbf{u_1} = (1, 3, 2, 1)$, $\mathbf{u_2} = (2, -2, -5, 4)$, $\mathbf{u_3} = (2, -1, 3, 6)$. If $\mathbf{v} = (2, 5, -4, 0)$, write $\mathbf{v}$ as a linear combination of $\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}$. If it is not possible say so.

Solution: Let $\mathbf{v} = a\mathbf{u_1} + b\mathbf{u_2} + c\mathbf{u_3}$. We need to solve for a, b, c. Writing the equation explicitly, we have

$$(2, 5, -4, 0) = a(1, 3, 2, 1) + b(2, -2, -5, 4) + c(2, -1, 3, 6).$$

Therefore

$$(2, 5, -4, 0) = (a + 2b + 2c, 3a - 2b - c, 2a - 5b + 3c, a + 4b + 6c)$$

Equating entry-wise, we have system of linear equation

$$\begin{array}{rrrl} a & +2b & +2c & = 2 \\ 3a & -2b & -c & = 5 \\ 2a & -5b & +3c & = -4 \\ a & +4b & +6c & = 0 \end{array}$$

We write the augmented matrix:

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 3 & -2 & -1 & 5 \\ 2 & -5 & 3 & -4 \\ 1 & 4 & 6 & 0 \end{bmatrix}$$

We use TI, to reduce this matrix to Gauss-Jordan form:

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So, the system is consistent and $a = 2, b = 1, c = -1$. Therefore

$$\mathbf{v} = 2\mathbf{u_1} + \mathbf{u_2} - \mathbf{u_3},$$

Subspaces of Vector spaces

Definition 4.3.1 A nonempty subset W of a vector space V is called a subspace of V if W is a vector space under the operations addition and scalar multiplication defined in V.

Theorem 4.3.3 Suppose V is a vector space over $\mathbb{R}$ and $W \subseteq V$ is a **nonempty** subset of V. Then W is a subspace of V if and only if the following two closure conditions hold:

1. If $\mathbf{u}, \mathbf{v}$ are in W, then $\mathbf{u} + \mathbf{v}$ is in W.
2. If $\mathbf{u}$ is in W and c is a scalar, then $c\mathbf{u}$ is in W.

Example Here are some obvious examples:

1. Let $W = \{(x, 0) : x \text{ is real number}\}$. Then $W \subseteq \mathbb{R}^2$. (*The notation* $\subseteq$ *reads as 'subset of'*.) It is easy to check that W is a subspace of $\mathbb{R}^2$.
2. Let W be the set of all points on any given line $y = mx$ through the origin in the plane $\mathbb{R}^2$. Then, W is a subspace of $\mathbb{R}^2$.
3. Let P_2, P_3, P_n be vector space of polynomials, respectively, of degree less or equal to $2, 3, n$. (See example 4.2.3.) Then P_2 is a subspace of P_3 and P_n is a subspace of P_{n+1}.

Spanning sets and linear indipendence

The main point here is to write a vector as linear combination of a give set of vectors.

Definition 4.4.1 A vector $\mathbf{v}$ in a vector space V is called a **linear combination** of vectors $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_k}$ in V if $\mathbf{v}$ can be written in the form

$$\mathbf{v} = c_1\mathbf{u_1} + c_2\mathbf{u_2} + \cdots + c_k\mathbf{u_k},$$

where $c_1, c_2, \ldots, c_k$ are scalars.

Definition 4.4.2 Let V be a vector space over $\mathbb{R}$ and $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$ be a subset of V. We say that S is a **spanning set** of V if every vector $\mathbf{v}$ of V can be written as a liner combination of vectors in S. In such cases, we say that S **spans** V.

$$span(S) = \{c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_k\mathbf{v_k} : c_1, c_2, \ldots, c_k \ are\ scalars\}.$$

1. The span of S is denoted by $span(S)$ as above or $span\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$.
2. If $V = span(S)$, then say V is spanned by S or S spans V.

Theorem 4.4.4 Let V be a vector space over $\mathbb{R}$ and $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$ be a subset of V. Then $span(S)$ is a subspace of V.

Further, $span(S)$ is the smallest subspace of V that contains S. This means, if W is a subspace of V and W contains S, then $span(S)$ is contained in W.

Proof.

we only need to show that $span(S)$ is closed under addition and scalar multiplication. So, let $\mathbf{u}, \mathbf{v}$ be two elements in $span(S)$. We can write

$$\mathbf{u} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_k\mathbf{v_k} \quad \textit{and} \quad \mathbf{v} = d_1\mathbf{v_1} + d_2\mathbf{v_2} + \cdots + d_k\mathbf{v_k}$$

where $c_1, c_2, \ldots, c_k, d_1, d_2, \ldots, d_k$ are scalars. It follows

$$\mathbf{u} + \mathbf{v} = (c_1 + d_1)\mathbf{v_1} + (c_2 + d_2)\mathbf{v_2} + \cdots + (c_k + d_k)\mathbf{v_k}$$

and for a scalar c, we have

$$c\mathbf{u} = (cc_1)\mathbf{v_1} + (cc_2)\mathbf{v_2} + \cdots + (cc_k)\mathbf{v_k}.$$

So, both $\mathbf{u} + \mathbf{v}$ and $c\mathbf{u}$ are in $span(S)$, because the are linear combination of elements in S. So, $span(S)$ is closed under addition and scalar multiplication, hence a subspace of V.

Linear dependence and independence

Definition 4.4.5 Let V be a vector space. A set of elements (vectors) $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots \mathbf{v_k}\}$ is said to be **linearly independent** if the equation

$$c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_k\mathbf{v_k} = \mathbf{0}$$

has only trivial solution

$$c_1 = 0, c_2 = 0, \ldots, c_k = 0.$$

We say S is **linearly dependent**, if S in not linearly independent.

Exercise Let $S = \{(6,2,1), (-1,3,2)\}$. Determine, if S is linearly independent or dependent?

Solution: Let

$$c(6,2,1) + d(-1,3,2) = (0,0,0).$$

If this equation has only trivial solutions, then it is linealry independent. This equaton gives the following system of linear equations:

$$\begin{aligned} 6c \quad -d &= 0 \\ 2c \quad +3d &= 0 \\ c \quad +2d &= 0 \end{aligned}$$

The augmented matrix for this system is

$$\begin{bmatrix} 6 & -1 & 0 \\ 2 & 3 & 0 \\ 1 & 2 & 0 \end{bmatrix}. \quad its\ gauss - Jordan\ form: \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So, $c = 0, d = 0$. The system has only trivial (i.e. zero) solution. We conclude that S is linearly independent.

Exercise Let

$$S = \left\{ \left(\frac{3}{4}, \frac{5}{2}, \frac{3}{2}\right), \left(3, 4, \frac{7}{2}\right), \left(-\frac{3}{2}, 6, 2\right) \right\}.$$

Determine, if S is linearly independent or dependent?

Solution: Let

$$a\left(\frac{3}{4}, \frac{5}{2}, \frac{3}{2}\right) + b\left(3, 4, \frac{7}{2}\right) + c\left(-\frac{3}{2}, 6, 2\right) = (0,0,0).$$

If this equation has only trivial solutions, then it is linealry independent. This equaton gives the following system of linear equations:

$$\begin{aligned} \tfrac{3}{4}a \quad +3b \quad -\tfrac{3}{2}c &= 0 \\ \tfrac{5}{2}a \quad +4b \quad +6c &= 0 \\ \tfrac{3}{2}a \quad +\tfrac{7}{2}b \quad +2c &= 0 \end{aligned}$$

This system is homogenous with square matrix A. Notice that |A| is not equal zero. Therefore, the system has a unique solution which is the zero solution. Hence, S is Linearly independent.

Exercise Let

$$S = \{(1,0,0),(0,4,0),(0,0,-6),(1,5,-3)\}.$$

Determine, if S is linearly independent or dependent?

Solution: Let

$$c_1(1,0,0) + c_2(0,4,0) + c_3(0,0,-6) + c_4(1,5,-3) = (0,0,0).$$

If this equation has only trivial solutions, then it is linealry independent. This equaton gives the following system of linear equations:

$$\begin{array}{cccc} c_1 & & & +c_4 & = 0 \\ & 4c_2 & & 5c_4 & = 0 \\ & & -6c_3 & -3c_4 & = 0 \end{array}$$

The augmented matrix for this system is

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 4 & 0 & 5 & 0 \\ 0 & 0 & -6 & -3 & 0 \end{bmatrix}. \quad its\,Gaus{-}Jordan\,form \quad \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1.25 & 0 \\ 0 & 0 & 1 & .5 & 0 \end{bmatrix}.$$

Correspondingly:

$$c_1 + c_4 = 0, \quad c_2 + 1.25c_4 = 0, \quad c_3 + .5c_4 = 0.$$

So, after elimination, the number of equations is not equal the number of variables.

Therefore, there are infinite many solutions (i.e. there is a non-zero solution) which means that S is not independent.

Theorem 4.4.9 Let V be a vector space and $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots \mathbf{v_k}\}, k \geq 2$ a set of elements (vectors) in V. Then S is linearly dependent if and only if one of the vectors v_j can be written as a linear combination of the other vectors in S.

Basis and Dimension

Definition 4.5.1 Let V be a vector space and $S = \{\mathbf{v_1}, \mathbf{v_2}, \dots \mathbf{v_k}\}$ be a set of elements (vectors)in V. We say that S is a **basis** of V if

1. S spans V and
2. S is linearly independent.

Remark.

We say that a vector space V is **finite dimensional,** if V has a basis consisting of finitely many elements. Otherwise, we say that V is **infinite dimensional.**

The vector space P of all polynomials (with real coefficients) has infinite dimension.

Example Most standard example of basis is the **standard basis** of $\mathbb{R}^n$.

1. Consider the vector space $\mathbb{R}^2$. Write

$$\mathbf{e_1} = (1, 0), \mathbf{e_2} = (0, 1).$$

   Then, $\mathbf{e_1}, \mathbf{e_2}$ form a basis of $\mathbb{R}^2$.