INTEGRAL CALCULUS (MATH 106)

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Chapter 4: Indeterminate Forms and Techniques of Integration

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The student is expected to be able to:

1. handles with Indeterminate Forms and uses Hopital's Rule.
2. integrate the functions using integration by parts.
3. handles with Integrals Involving Trigonometric Functions.
4. Applying Trigonometric substitution to integrals.
5. solve integrals of rational functions (Partial fractions).
6. solve integrals of rational functions involving $\sin x$ or $\cos x$.
7. solve integrals involving fraction powers of x.
8. solve integrals involving a square root of a linear factor.
9. deal with improper integrals.

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Theorem (L'Hopital's Rule)

Suppose that f and g are differentiable on the interval (a, b), except possibly at a point $c \in (a, b)$ and that $g'(x) \neq 0$ on (a, b), except possibly at c. Suppose further that $\lim_{x\to c} \frac{f(x)}{g(x)}$ has the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and that $\lim_{x\to c} \frac{f'(x)}{g'(x)} = L(\text{or } \pm\infty)$.

Then, $\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$.

Remark

The conclusion of the theorem also holds if $\lim_{x\to c} \frac{f(x)}{g(x)}$ is replaced with $\lim_{x\to c^-} \frac{f(x)}{g(x)}$, $\lim_{x\to c^+} \frac{f(x)}{g(x)}$, $\lim_{x\to -\infty} \frac{f(x)}{g(x)}$ or $\lim_{x\to +\infty} \frac{f(x)}{g(x)}$. (In each case, we must make appropriate adjustment of the hypothesis.)

Indeterminate Forms and l'Hopital's Rule.

Types of indeterminate forms:

1. $\frac{0}{0}$ or $\frac{\infty}{\infty}$
2. $\infty - \infty$ or $-\infty + \infty$
3. $0.\infty$ or $0(-\infty)$
4. $0^0, 1^{\infty}, 1^{-\infty}$ or ∞^0

Examples

1. $\lim_{x\to 1} \frac{\sqrt{x}}{\ln x} = \frac{0}{0}$
   Apply L'Hopital's rule
   $$\lim_{x\to 1} \frac{\sqrt{x}}{\ln x} = \lim_{x\to 1} \frac{\left(\frac{1}{2\sqrt{x}}\right)}{\left(\frac{1}{x}\right)} = \lim_{x\to 1} \frac{x}{2\sqrt{x}} = \frac{1}{2}$$

2. $\lim_{x\to (\frac{\pi}{2})^-} \frac{2-\sec x}{3\tan x} = \frac{-\infty}{\infty}$
   Apply L'Hopital's rule
   $$\lim_{x\to (\frac{\pi}{2})^-} \frac{2-\sec x}{3\tan x} = \lim_{x\to (\frac{\pi}{2})^-} \frac{-\sec x\tan x}{3\sec^2 x} = \lim_{x\to (\frac{\pi}{2})^-} \frac{-\tan x}{3\sec x} =$$
   $$\lim_{x\to (\frac{\pi}{2})^-} \frac{-\sin x}{3} = -\frac{1}{3}$$

Examples

3. $\lim_{x \to 1^+} (\frac{3}{\ln x} - \frac{2}{x-1}) = (\infty - \infty)$

$\lim_{x \to 1^+} \frac{3(x-1) - 2\ln x}{(x-1)\ln x} = \frac{0}{0}$

Apply L'Hopital's rule

$\lim_{x \to 1^+} \frac{3(x-1) - 2\ln x}{(x-1)\ln x} = \lim_{x \to 1^+} \frac{3 - \frac{2}{x}}{\ln x + (x-1)\frac{1}{x}} =$

$\lim_{x \to 1^+} \frac{3 - \frac{2}{x}}{\ln x + 1 - \frac{1}{x}} = \infty$

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Integration By Parts

It is used to solve integration of a product of two functions using the formula:

$$\int u\ dv = uv - \int v\ du$$

1. $\displaystyle\int xe^x dx$, We put, $u = x \quad dv = e^x\ dx$, Then $du = dx \quad v = e^x$

   $$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + c$$

2. $\displaystyle\int_0^\pi x \sin x\ dx$, We put $u = x \quad dv = \sin x\ dx$, Then,

   $du = dx \quad v = -\cos x$

   $$\int_0^\pi x \sin x\ dx = [-x \cos x]_0^\pi + \int_0^\pi \cos x dx = [-x \cos x]_0^\pi + [\sin x]_0^\pi$$

   $$[(-\pi \cos \pi) - (-(0) \cos 0)] + [\sin \pi - \sin 0] = \pi$$

Integration By Parts: Examples

- $\displaystyle\int xe^x\,dx = (x-1)e^x + c$
- $\displaystyle\int x^2e^x\,dx = (x^2-2x+2)e^x + c$
- $\displaystyle\int x^3e^x\,dx = (x^3-3x^2+6x-6)e^x + c$

Integration By Parts: Examples

- $\displaystyle\int x\cos x\;dx = x\sin x + \cos x + c$
- $\displaystyle\int x^2\cos x\;dx = (x^2-2)\sin x + 2x\cos x + c$
- $\displaystyle\int x^3\cos x\;dx = (x^3-6x)\sin x + (3x^2-6)\cos x + c$
- $\displaystyle\int x^4\cos x\;dx = (x^4-12x^2+24)\sin x + (4x^3-24x)\cos x + c$

Integration By Parts: Examples

- $\int x \sin x \; dx = -x \cos x + \sin x + c$
- $\int x^2 \sin x \; dx = (-x^2 + 2) \cos x + 2x \sin x + c$
- $\int x^3 \sin x \; dx = (-x^3 + 6x) \cos x + (3x^2 - 6) \sin x + c$
- $\int x^4 \sin x \; dx = (-x^4 + 12x^2 - 24) \cos x + (4x^3 - 24x) \sin x + c$

Integration By Parts: Examples

Evaluate $\displaystyle\int \cos(\ln(x))\ dx$.

Letting: $u = \ln(x)$, we have $du = 1/x\ dx$.

$$du = \frac{1}{x}\ dx \Rightarrow x \cdot du = dx.$$

Since $u = \ln(x)$, we can use inverse functions and conclude that $e^{\ln(x)} = e^u \Rightarrow x = e^u$. therefore we have that $dx = x \cdot du = e^u\ du$.

$$\begin{aligned}\int \cos(\ln(x))\ dx &= \int e^u \cos(u)\ du \\ &= \frac{1}{2}e^u\big(\sin(u) + \cos(u)\big) + C \\ &= \frac{1}{2}e^{\ln(x)}\big(\sin(\ln(x)) + \cos(\ln(x))\big) + C \\ &= \frac{1}{2}x\big(\sin(\ln(x)) + \cos(\ln(x))\big) + C.\end{aligned}$$

$$\int e^x \cos x \, dx$$

$$\begin{array}{ll} u = \cos x & dv = e^x \; dx \\ du = -\sin x \; dx & v = e^x \end{array}$$

$$\int e^x \cos x \, dx = e^x \cos x + \int e^x \sin x \, dx$$

Now to solve $\int e^x \sin x \, dx$

$$\begin{array}{ll} u = \sin x & dv = e^x \; dx \\ du = \cos x \; dx & v = e^x \end{array}$$

Therefore , $\int e^x \cos x \, dx = e^x \cos x + e^x \sin x - \int e^x \cos x \, dx$

$$2 \int e^x \cos x \, dx = e^x \cos x + e^x \sin x$$

$$\int e^x \cos x \, dx = \frac{1}{2} \left[e^x \cos x + e^x \sin x \right] + c \; .$$

Integration By Parts: Examples

$$\int \ln|x|\,dx$$

$$u = \ln|x| \qquad dv = dx$$

$$du = \frac{1}{x}\,dx \qquad v = x$$

$$\int \ln|x|\,dx = x\ln|x| - \int x\,\frac{1}{x}\,dx = x\ln|x| - \int dx = x\ln|x| - x + c$$

$$\int \tan^{-1} x \, dx$$

$$u = \tan^{-1} x \qquad dv = dx$$

$$du = \frac{1}{1+x^2}\, dx \qquad v = x$$

$$\int \tan^{-1} x \, dx = x \tan^{-1} x - \int x \, \frac{1}{1+x^2}\, dx$$

$$\int \tan^{-1} x \, dx = x \tan^{-1} x - \frac{1}{2}\int \frac{2x}{1+x^2}\, dx = x \tan^{-1} x - \frac{1}{2}\ln(1+x^2) + c$$

Integration By Parts: Examples

$$\int \sec^3 x\,dx = \int \sec x \sec^2 x\,dx$$

$$u = \sec x \qquad dv = \sec^2 x\ dx$$

$$du = \sec x \tan x\ dx \qquad v = \tan x$$

$$\int \sec^3 x\,dx = \sec x \tan x - \int \sec x \tan^2 x\,dx$$

$$\int \sec^3 x\,dx = \sec x \tan x - \int \sec x \left(\sec^2 x - 1\right)\ dx$$

$$\int \sec^3 x\,dx = \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx$$

$$2\int \sec^3 x\,dx = \sec x \tan x + \ln|\sec x + \tan x|$$

$$\int \sec^3 x\,dx = \frac{1}{2}\left[\sec x \tan x + \ln|\sec x + \tan x|\right] + c$$

Integration By Parts: Examples

$$\int \ln(1+x^2)\,dx$$

$$u = \ln(1+x^2) \qquad dv = dx$$

$$du = \frac{2x}{1+x^2}\,dx \qquad v = x$$

$$\int \ln(1+x^2)\,dx = x\ln(1+x^2) - \int \frac{2x^2}{1+x^2}\,dx$$

$$\int \ln(1+x^2)\,dx = x\ln(1+x^2) - \int \frac{(2x^2+2)-2}{1+x^2}\,dx$$

$$\int \ln(1+x^2)\,dx = x\ln(1+x^2) - \int \frac{2(x^2+1)}{1+x^2}\,dx + 2\int \frac{1}{1+x^2}\,dx$$

$$\int \ln(1+x^2)\,dx = x\ln(1+x^2) - 2x + 2\tan^{-1}x + c$$

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Integrals Involving Trigonometric Functions

First form

Integrals of the forms

$$\int \sin ax \cos bx \ dx, \quad \int \sin ax \sin bx \ dx, \quad \int \cos ax \cos bx \ dx$$

Where $a, b \in \mathbb{Z}$

1. The integral $\int \sin ax \cos bx \ dx$ can be solved using the formula $\sin ax \cos bx = \frac{1}{2}[\sin(ax + bx) + \sin(ax - bx)]$
2. The integral $\int \sin ax \sin bx \ dx$ can be solved using the formula $\sin ax \sin bx = \frac{1}{2}[\cos(ax - bx) - \cos(ax + bx)]$
3. The integral $\int \cos ax \cos bx \ dx$ can be solved using the formula $\cos ax \cos bx = \frac{1}{2}[\cos(ax + bx) + \cos(ax - bx)]$

1. $\displaystyle\int \sin 3x \cos 2x \; dx = \frac{1}{2}\int [\sin 5x + \sin x] dx =$
$\displaystyle\frac{1}{2}\int \sin 5x \; dx + \frac{1}{2}\int \sin x \; dx = -\frac{1}{10}\cos 5x - \frac{1}{2}\cos x + c$
2. $\displaystyle\int \sin x \sin 3x \; dx = \frac{1}{2}\int [\cos 2x - \cos 4x] dx =$
$\displaystyle\frac{1}{2}\int \cos 2x \; dx - \frac{1}{2}\int \cos 4x \; dx = \frac{1}{4}\sin 2x - \frac{1}{8}\sin 4x + c$
3. $\displaystyle\int \cos 5x \cos 2x \; dx = \frac{1}{2}\int [\cos 7x + \cos 3x] dx =$
$\displaystyle\frac{1}{2}\int \cos 7x \; dx + \int \cos 3x \; dx = \frac{1}{4}\sin 7x + \frac{1}{6}\sin 3x + c$

Second form

Integrals of the forms

$$\int \sin^n x \cos^m x \ dx, \quad \int \sinh^n x \cosh^m x \ dx, \qquad \textit{Where } n, m \in \mathbb{N}$$

The above two integrals can be solved by substitution if n or m is odd.

1. If n is odd: The substitution $u = \cos x$ can be used to solve $\int \sin^n x \cos^m x \ dx$
   The substitution $u = \cosh x$ can be used to solve $\int \sinh^n x \cosh^m x \ dx$
2. If m is odd: The substitution $u = \sin x$ can be used to solve $\int \sin^n x \cos^m x \ dx$
   The substitution $u = \sinh x$ can be used to solve $\int \sinh^n x \cosh^m x \ dx$

Integrals Involving Trigonometric Functions (Examples)

1. Evaluate $I = \int \sin^5 x \cos^4 x dx$

$$\int \sin^5 x \cos^4 x dx = \int (\sin^2 x)^2 \cos^4 x \sin x dx$$
$$= \int (1 - \cos^2 x) \cos^4 x \sin x \; dx$$

to solve this integral put

$$u = \cos x \Rightarrow -du = \sin x \; dx$$

$$I = -\int (1 - u^2)^2 u^4 du = -\int u^4 - 2u^6 + u^8 du$$
$$= -\left[\frac{u^5}{5} - \frac{2u^7}{7} + \frac{u^9}{9}\right] + c = -\left[\frac{\cos^5 x}{5} - \frac{2\cos^7 x}{7} + \frac{\cos^9 x}{9}\right] + c$$

Integrals Involving Trigonometric Functions (Examples)

2. Evaluate $I = \int \sin^7 \cos^3 x \; dx$

$\int \sin^7 \cos^3 x \; dx = \int \sin^7 x(1 - \sin^2 x) \cos x \; dx$
to solve this integral put

$$u = \sin x \Rightarrow du = \cos x \; dx$$

$$I = \int u^7(1 - u^2) \; du = \int u^7 - u^9 \; du = \frac{u^8}{8} - \frac{u^{10}}{10} + c$$

$$= \frac{\sin^8 x}{8} - \frac{sin^{10}}{10} + c$$

3. $\displaystyle\int \sinh^3 x \cosh^2 x \; dx$ to solve this integral put
$u = \cosh x \Rightarrow du = \sinh x$
$\displaystyle\int \sinh^3 x \cosh^2 x \; dx = \int (\cosh^2 x - 1) \cosh^2 x \sinh x \; dx =$
$\displaystyle\int (u^2 - 1)u^2 du = \int (u^4 - u^2) du = \frac{u^5}{5} - \frac{u^3}{3} + c$

$$= \frac{\cosh^5 x}{5} - \frac{\cosh^3 x}{3} + c$$

$$\int \sqrt{\sin x}\ \cos^3 x\, dx = \int \sqrt{\sin x}\ \cos^2 x\ \cos x\, dx$$

$$= \int (\sin x)^{\frac{1}{2}}\ \left(1 - \sin^2 x\right)\ \cos x\, dx$$

Put $u = \sin x \Rightarrow du = \cos x\ dx$

$$\int \sqrt{\sin x}\ \cos^3 x\, dx = \int u^{\frac{1}{2}}(1 - u^2)\ du = \int \left(u^{\frac{1}{2}} - u^{\frac{5}{2}}\right)\ du$$

$$= \frac{2u^{\frac{3}{2}}}{3} - \frac{2u^{\frac{7}{2}}}{7} + c = \frac{2\,(\sin x)^{\frac{3}{2}}}{3} - \frac{2\,(\sin x)^{\frac{7}{2}}}{7} + c$$

Integrals Involving Trigonometric Functions (Examples)

$$\int \frac{\sin^3 x}{\cos^2 x}\,dx = \int \sin^2 x\ \cos^{-2} x\ \sin x\,dx = \int \left(1-\cos^2 x\right)\ \cos^{-2} x\ \sin x\,dx$$

Put $u = \cos x \Rightarrow -du = \sin x\ dx$

$$\int \frac{\sin^3 x}{\cos^2 x}\,dx = -\int \left(1-u^2\right)\ u^{-2}\ du = -\int \left(u^{-2}-1\right)\,du$$

$$= -\frac{u^{-1}}{-1} + u + c = \frac{1}{u} + u + c = \sec x + \cos x + c$$

$$\int \sin^7 x \ \cos^3 x \, dx = \int \sin^7 x \ \cos^2 x \ \cos x \, dx$$

$$= \int \sin^7 x \ (1 - \sin^2 x) \ \cos x \, dx$$

Put $u = \sin x \Rightarrow du = \cos x \ dx$

$$\int \sin^7 x \ \cos^3 x \, dx = \int u^7 \ (1 - u^2) \ du = \int (u^7 - u^9) \ du$$

$$= \frac{u^8}{8} - \frac{u^{10}}{10} + c = \frac{\sin^8 x}{8} - \frac{\sin^{10} x}{10} + c$$

Special cases :

1. $\int \sin^2 x \; dx = \frac{1}{2}\int [1 - \cos 2x] \; dx = \frac{1}{2}[x - \frac{\sin 2x}{2}] + c$
2. $\int \cos^2 x \; dx = \frac{1}{2}\int [1 + \cos 2x] \; dx = \frac{1}{2}[x + \frac{\sin 2x}{2}] + c$

Integrals Involving Trigonometric Functions

Third form

Integrals of the forms

$$\int \sec^n x \tan^m x \ dx, \quad \int csc^n x \cot^m x \ dx,$$

$$\int sech^n x \tanh^m x \ dx, \quad \int csch^n x \coth^m x \ dx$$

The above four integrals can be solved by substitution if n is even or m is odd.

Integrals Involving Trigonometric Functions

1. If n is even:
   The substitution $u = \tan x$ can be used to solve $\int \sec^n x \tan^m x \; dx$.
   The substitution $u = \cot x$, $u = \tanh x$ and $u = \coth x$ can be used to solve the other three integrals respectively.
2. If m is odd:
   The substitution $u = \sec x$ can be used to solve $\int \sec^n x \tan^m x \; dx$.
   The substitutions $u = csc \;\; x, u = sech \;\; x$ and $u = csch \;\; x$ can be used to solve the other theree integrals respictively.

1. Evaluate $I = \int \tan^3 x \sec^3 x \; dx$

   to solve this integral put: $u = \sec x \Rightarrow du = \sec x \tan x \; dx$

   $I = \int \tan^3 x \sec^3 x \; dx = \int (\sec^2 x - 1) \sec^2 x \sec x \tan x \; dx$

   $$= (u^2 - 1)u^2 \; du = \int u^4 - u^2 \; du$$

   $$= \frac{u^5}{5} - \frac{u^3}{3} + c$$

   $$= \frac{\sec^5}{x} 5 - \frac{\sec^3}{x} 3 + c$$

2. Evaluate $I = \int tanh^3 x \ sech \ x \ dx$

to solve this integral put: $u = sech \ x \Rightarrow -du = sech \ x \tanh x \ dx$

$$I = \int tanh^3 x \ sech \ x \ dx = \int (1 - sech^2 x) sech \ x \ tanh \ x \ dx$$

$$= -\int (1 - u^2) du = -u + \frac{u^3}{3} + c$$

$$= -sech \ x + \frac{sech^3 x}{3} + c$$

$$\int \csc^4 x \ \cot^4 x \, dx$$

$$= \int \csc^2 x \ \cot^4 x \ \csc^2 x \, dx = \int (1+\cot^2 x) \cot^4 x \ \csc^2 x \, dx$$

Put $u = \cot x \Rightarrow -du = \csc^2 x \ dx$

$$\int \csc^4 x \ \cot^4 x \, dx = -\int (1+u^2)u^4 \ du = -\int (u^4+u^6) \ du$$

$$= -\frac{u^5}{5} - \frac{u^7}{7} + c = -\frac{\cot^5 x}{5} - \frac{\cot^7 x}{7} + c$$

Integrals Involving Trigonometric Functions (Examples)

$$\int \frac{\sec^4 x}{\sqrt{\tan x}}\,dx$$

$$\int \sec^2 x\ (\tan x)^{-\frac{1}{2}}\ \sec^2 x\,dx = \int (1+\tan^2 x)\ (\tan x)^{-\frac{1}{2}}\ \sec^2 x\,dx$$

Put $u = \tan x \Rightarrow du = \sec^2 x\ dx$

$$\int \frac{\sec^4 x}{\sqrt{\tan x}}\,dx = \int (1+u^2)u^{-\frac{1}{2}}\ du = \int \left(u^{-\frac{1}{2}} + u^{\frac{3}{2}}\right)\ du$$

$$= 2u^{\frac{1}{2}} + \frac{2u^{\frac{5}{2}}}{5} + c = 2(\tan x)^{\frac{1}{2}} + \frac{2(\tan x)^{\frac{5}{2}}}{5} + c$$

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Trigonometric Substitutions

If the integrand contains a term of the form
$\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$ or $\sqrt{x^2 - a^2}$ where $a > 0$,
then trigonometric substitutions can be used to solve the integral.

1. An integral involving $\sqrt{a^2 - x^2}$ use the substitution $x = a \sin\theta$ where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ to solve the integral.
2. An integral involving $\sqrt{a^2 + x^2}$ use the substitution $x = a \tan\theta$ where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ to solve the integral.
3. An integral involving $\sqrt{x^2 - a^2}$ use the substitution $x = a \sec\theta$ where $0 \leq \theta < \frac{\pi}{2}$ to solve the integral.

Solve the following integral: $\displaystyle\int \frac{1}{x^2\sqrt{16-x^2}}dx$

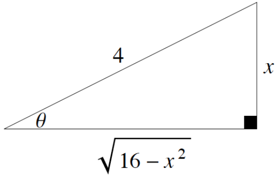
$$\int \frac{1}{x^2\sqrt{16-x^2}}dx = \int \frac{1}{x^2\sqrt{(4)^2-x^2}}dx,$$

Put $x = 4\sin\theta \Rightarrow dx = 4\cos\theta d\theta$

$$I = \int \frac{4\cos\theta}{16\sin^2\theta\sqrt{16-16\sin\theta}}d\theta$$

$$= \int \frac{4\cos\theta}{16\sin^2\theta\ 4\cos\theta}d\theta$$

$$= \frac{1}{16}\int \frac{1}{\sin^2\theta}d\theta = \frac{1}{16}\int \csc^2\theta d\theta = \frac{1}{16}\cot\theta + c$$



$$\int \frac{1}{x^2\sqrt{16-x^2}}dx = -\frac{1}{16}\frac{\sqrt{16-x^2}}{x} + c$$

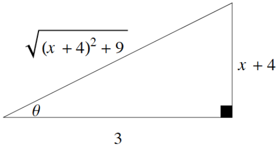
Trigonometric Substitutions (examples)

Solve the following integral:

$$\int \frac{1}{[x^2+8x+25]^{\frac{3}{2}}}\ dx$$

$$I = \int \frac{1}{[(x+4)^2+3^2]^{\frac{3}{2}}}\ dx.$$

Put $x+4 = 3\tan\theta \Rightarrow dx = 3\sec^2\theta\ d\theta$



$$\int \frac{1}{[x^2+8x+25]^{\frac{3}{2}}}\ dx = \int \frac{3\sec^2\theta}{(9\tan^2\theta+9)^{\frac{3}{2}}}\ d\theta$$

$$= \int \frac{3\sec^{\theta}}{(9\sec^2\theta)^{\frac{3}{2}}}\ d\theta = \int \frac{3\sec^2\theta}{27\sec^3\theta} d\theta$$

$$= \frac{1}{9}\int \frac{1}{\sec\theta} d\theta = \frac{1}{9}\sin\theta + c$$

$$\int \frac{1}{[x^2+8x+25]^{\frac{3}{2}}}\ dx = \frac{1}{9}\frac{x+4}{\sqrt{x^2+8x+25}} + c$$

Trigonometric Substitutions (examples)

Solve the following integral: $\displaystyle\int \frac{\sqrt{x^2-4}}{x^2}\, dx$

Put $x = 2\sec\theta \Rightarrow dx = 2\sec\theta\tan\theta\ d\theta$

$$\int \frac{\sqrt{x^2-4}}{x^2}\, dx = \int \frac{\sqrt{4\sec^2\theta - 4}\ 2\sec\theta\tan\theta}{4\sec^2\theta} d\theta$$

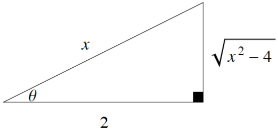
$$= \int \frac{(2\tan\theta)(2\sec\theta\tan\theta)}{4\sec^2\theta} d\theta$$

$$= \int \frac{\tan^2\theta}{\sec\theta} d\theta = \int \frac{\sec^2\theta - 1}{\sec\theta} d\theta$$

$$= \int \frac{\sec^2\theta}{\sec\theta} d\theta - \int \frac{1}{\sec\theta} d\theta$$

$$= \int \sec\theta d\theta - \int \cos\theta d\theta$$

$$= \ln|\sec\theta + \tan\theta| - \sin\theta + c$$



$$\int \frac{\sqrt{x^2-4}}{x^2}\, dx = \ln\left|\frac{x}{2} + \frac{\sqrt{x^2-4}}{2}\right| - \frac{\sqrt{x^2-4}}{x} + c$$

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Method of Partial fractions

Definition: linear factor

A linear factor is a polynomial of degree 1. It has the form $ax + b$ where $a, b \in \mathbb{R}$ and $a \neq 0$.
Such $x, 3x$, and $2x - 7$

Definition: irreducible quadratic

An irreducible quadratic is a polynomial of degree 2. It has the form $ax^2 + bx + c$, where $a, b, c \in \mathbb{R}$ and $b^2 - 4ac < 0$.
Such $x^2 + 9$ and $x^2 + x + 1$.

Method of Partial fractions

What is the Partial Fraction?

It is re-expressing a rational function (a ratio of polynomial function) as a sum of simpler fraction.

Let $h(x) = \frac{P(x)}{Q(x)} = \dfrac{a_0x^n + a_1x^{n-1} + \cdots + a - n - 1x + a_n}{b_ox^m + b_1x^{m-1} + \cdots + b_{m-1}x + b_m}$ be a rational function, where $P(x), Q(x)$ are polynomials, we have two cases:

1. degree $P(x) <$ degree $Q(x)$ use method of partial fractions.
2. degree $P(x) \geq$ degree $Q(x)$ use long division of polynomials, then use method of partial fractions.

Method of Partial fractions

How do we create partial functions?

1. If we can write $Q(x)$ as a linear factors
$b_o x^m + b_1 x^{m-1} + \cdots + b_{m-1}x + b_m = (x-a)^m, \;\; a \in \mathbb{R}, \;\; m \in \mathbb{N}$
Then: $h(x) = \dfrac{A_0}{(x-a)^m} + \dfrac{A_1}{(x-a)^{m-1}} + \cdots + \dfrac{A_{m-1}}{x-a}, m \in \mathbb{N}$
2. If we can write $Q(x)$ as a irreducible quadratic factors
$b_o x^m + b_1 x^{m-1} + \cdots + b_{m-1}x + b_m = (ax^2+bx+c)^n, \;\; a, b, c \in \mathbb{N}$
and $b^2 - 4ac < 0$
Then:
$$h(x) = \frac{B_0 x + C_0}{(ax^2+bx+c)^n} + \frac{B_1 x + C_1}{(ax^2+bx+c)^{n-1}} + \cdots + \frac{B_{n-1}x + C_{n-1}}{ax^2+bx+c}.$$

Method of Partial fractions

Some time we can write $Q(x)$ as a product of linear factors and irreducible quadratics.

Then

$$h(x) = \frac{A_0}{(x-a)^m} + \frac{A_1}{(x-a)^{m-1}} + \cdots + \frac{A_{m-1}}{x-a}$$

$$+ \frac{B_0 x + C_0}{(ax^2+bx+c)^n} + \frac{B_1 x + C_1}{(ax^2+bx+c)^{n-1}} + \cdots + \frac{B_{n-1}x + C_{n-1}}{ax^2+bx+c}.$$

Method of Partial fractions (Examples)

Determine the partial fraction for: $\dfrac{x-3}{x^2-4}$

$\dfrac{x-3}{(x-2)(x+2)} = \dfrac{A}{x-2} + \dfrac{B}{x+2} \Rightarrow x-3 = A(x+2) + B(x-2)$

$x = -2 \Rightarrow -5 = -4B \Rightarrow B = \frac{5}{4}$

$x = 2 \Rightarrow -1 = 4A \Rightarrow A = \frac{-1}{4}$

So: $\dfrac{x-3}{(x-2)(x+2)} = \dfrac{-1}{4(x-2)} + \dfrac{5}{4(x+2)}$

Now Integrate:

Determine $\displaystyle\int \frac{x-3}{x^2-4}dx$

$$\int \frac{x-3}{x^2-4}dx = \int \left[\frac{-1}{4(x-2)} + \frac{5}{4(x+2)}\right]dx$$

$$= -\frac{-1}{4}\ln\left|x-2\right| + \frac{5}{4}\ln\left|x+2\right| + c$$

Determine $\displaystyle\int \frac{x-3}{x^2+4x}dx$

Note that degree $P(x) <$ degree $Q(x)$

$\dfrac{x-3}{x^2+4x} = \dfrac{x-3}{x(x+4)} = \dfrac{A}{x} + \dfrac{B}{x+4} \Rightarrow x-3 = A(x+4) + Bx$

$x = -4 \Rightarrow -7 = -4B \Rightarrow B = \frac{7}{4}$

$x = 0 \Rightarrow -3 = 4A \Rightarrow A = \frac{-3}{4}$

Now we can write:

$$\frac{x-3}{x^2-4x} = \frac{-3}{4x} + \frac{7}{4(x+4)}.$$

$$\int \frac{x-3}{x^2-4x}dx = \int \frac{-3}{4x}dx + \int \frac{7}{4(x+4)}dx$$

$$= -\frac{3}{4}\ln|x| + \frac{7}{4}\ln|x+4| + C$$

$$= \frac{\ln|x+4|^{\frac{7}{4}}}{\ln|x|^{\frac{3}{4}}} + C$$

Method of Partial fractions (Examples)

Determine $\int \frac{x^2-2}{x^3+3x^2+2x}dx$

Note that degree $P(x) <$ degree $Q(x)$

$$\frac{x^2-2}{x^3+3x^2+2x} = \frac{x^2-2}{x(x+2)(x+1)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x+1}$$

$\Rightarrow x^2-2 = A(x+2)(x+1) + Bx(x+1) + Cx(x+2)$

$x = 0 \Rightarrow -2 = 2A \Rightarrow A = -1$

$x = -2 \Rightarrow 2 = 2B \Rightarrow b = 1$

$x = -1 \Rightarrow -1 = -C \Rightarrow c = 1$

Now we can write:

$$\frac{x^2-2}{x^3+3x^2+2x} = \frac{-1}{x} + \frac{1}{x+2} + \frac{1}{x+1}.$$

$$\int \frac{x^2-2}{x^3+3x^2+2x}dx = \int \frac{-1}{x}dx + \int \frac{1}{x+2}dx + \int \frac{1}{x+1}$$

$$= -\ln|x| + \ln|x+2| + \ln|x+1| + c$$

Method of Partial fractions (Examples)

Determine $\displaystyle\int \frac{x^3 - 2x^2 + x + 1}{x^4 + 5x^2 + 4} dx$

Note that degree $P(x) <$ degree $Q(x)$.

We can write: $x^4 + 5x^2 + 4 = (x^2 + 4)(x^2 + 1)$

$$\frac{x^3 - 2x^2 + x + 1}{x^4 + 5x^2 + 4} = \frac{Ax + b}{x^2 + 4} + \frac{Cx + D}{x^2 + 1}$$

$$\begin{aligned}\Rightarrow x^3 - 2x^2 + x + 1 &= (Ax + B)(x^2 + 1) + (Cx + D)(x^2 + 4) \\ &= (A + C)x^3 + (B + D)x^2 + (A + 4C)x + (B + 4D)\end{aligned}$$

$A = 1, \quad B = -3, \quad C = 0, \quad D = 1$

Now we can write:

$$\frac{x^3 - 2x^2 + x + 1}{x^4 + 5x^2 + 4} = \frac{x-3}{x^2+4} + \frac{1}{x^2+1}$$

$$\int \frac{x^3 - 2x^2 + x + 1}{x^4 + 5x^2 + 4} dx = \int [\frac{x-3}{x^2+4} + \frac{1}{x^2+1}] dx$$

$$= \frac{1}{2} \ln |x^2 + 4| - \frac{3}{2} \tan^{-1} \frac{x}{2} + \tan^{-1} x + c$$

Method of Partial fractions (Examples)

Determine $\int \frac{x^2+3}{x^2-x-2}dx$

Note that degree $P(x) \geq$ degree $Q(x)$

Here Divide First

$$\frac{x^2+3}{x^2-x-2} = 1 + \frac{x+5}{x^2-x-2}$$

$$\frac{x+5}{x^2-x-2} = \frac{x+5}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1} \Rightarrow$$

$x+5 = A(x+1) + B(x-2) x = 2 \Rightarrow 7 = 3A \Rightarrow A = \frac{7}{3}$

$x = -1 \Rightarrow 4 = -3B \Rightarrow B = \frac{-4}{3}$

Now we can write:

$$\frac{x^2+3}{x^2-x-2} = 1 + \frac{7}{3(x-2)} - \frac{4}{3(x+1)}$$

$$\int \frac{x^2+3}{x^2-x-2}dx = \int \left[1 + \frac{7}{3(x-2)} - \frac{4}{3(x+1)}\right]dx$$
$$= x + \frac{7}{3}\ln\left|x-2\right| - \frac{4}{3}\ln\left|x+1\right| + c$$

Method of Partial fractions (Examples)

Determine $I = \int \frac{x^4+1}{(x+1)(x^2+x+1)} dx$

Note that degree $P(x) \geq$ degree $Q(x)$

$$\frac{x^4+1}{(x+1)(x^2+x+1)} = (x-2) + \frac{2x^2+3x+3}{(x+1)(x^2+x+1)}$$

$$I = \underbrace{\int (x-2)\ dx}_{I_1} + \underbrace{\int \frac{2x^2+3x+3}{(x+1)(x^2+x+1)} dx}_{I_2}$$

Method of Partial fractions (Examples)

Now for I_2 we have de to applied method of partial fraction

$$\frac{2x^2+3x+3}{(x+1)(x^2+x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+x+1},$$

$$\begin{aligned}\Rightarrow 2x^2+3x+3 &= A(x^2+x+1)+(Bx+C)(x+1)\\ &= Ax^2+Ax+A+Bx^2+Bx+Cx+C\\ &= (A+B)x^2+(A+B+C)x+(A+C)\end{aligned}$$

$A+B=2$

$A+B+C=3$

$A+C=3$

So: $A=2,\ B=0,$ and $C=1.$

Method of Partial fractions (Examples)

Now we can write:

$$\frac{2x^2+3x+3}{(x+1)(x^2+x+1)} = \frac{2}{x+1} + \frac{1}{x^2+x+1}$$

$$\begin{aligned} I = I_1 + I_2 &= \int (x-2)\ dx + \int \frac{2x^2+3x+3}{(x+1)(x^2+x+1)} dx \\ &= \underbrace{\int (x-2)\ dx}_{I_1} + \underbrace{\underbrace{\int \frac{2}{x+1} dx}_{J_1} + \underbrace{\int \frac{1}{x^2+x+1} dx}_{J_2}}_{I_2} \end{aligned}$$

$$J_2 = \int \frac{1}{x^2+x+1}dx = \int \frac{1}{x^2+x+\frac{1}{4}+\frac{3}{4}}dx$$
$$= \int \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}dx = \frac{2}{\sqrt{3}}\tan^{-1}(\frac{2x+1}{\sqrt{3}}) + c$$

So,

$$\int \frac{x^4+1}{(x+1)(x^2+x+1)}dx = I_1 + \underbrace{J_1 + J_2}_{I_2}$$
$$= \underbrace{x^2-2x}_{I_1} + \underbrace{2\ln|x+1|}_{J_1} + \underbrace{\frac{2}{\sqrt{3}}\tan^{-1}(\frac{2x+1}{\sqrt{3}})}_{J_2} + c$$

Method of Partial fractions (Examples)

Exercises

1. $$\int \frac{6x+7}{(x+2)^2}dx$$
2. $$\int \frac{-2x+4}{(x^2+1)(x-1)^2}dx$$
3. $$\int \frac{x}{x^2+2x-3}dx$$
4. $$\int \frac{x^2}{(x-1)^2(x+1)}dx$$
5. $$\int \frac{x^3-5x+7}{x^2+x-6}dx$$
6. $$\int \frac{x^3-11x-26}{x^2-2x-8}dx$$
7. $$\int \frac{1}{x(x^2+1)^2}$$