



324 Stat
Lecture Notes

**(3) Mathematical
Expectation**

(Book: Chapter 4 ,pg 111-137)

Mean of a Random Variable:

Definition:

Let \mathbf{X} be a random variable with probability distribution $\mathbf{f}(\mathbf{x})$.

The mean or expected value of \mathbf{X} is:


$$\mu = E(X) = \sum_{\forall X} X f(X) \quad \text{if } X \text{ is discrete}$$

$$\mu = E(X) = \int_{-\infty}^{\infty} X f(X) dX \quad \text{if } X \text{ is continuous} \quad (1)$$



Properties of the Expectation:

1. $E(a) = a$, where a is a constant
2. $E(aX) = aE(X)$
3. $E(aX + b) = aE(X) + b$



Ex (1):

Find the expected number of chemists on a committee of 3 selected at random from 4 chemists and 3 biologists.

Find: $E(X)$, $E(X^2)$, $E(X-1)$

Solution:

Let \mathbf{X} represent the number of chemists on the committee.

The probability distribution of \mathbf{X} is given by:

$$f(x) = \frac{\binom{4}{x} \binom{3}{3-x}}{\binom{7}{3}}, \quad X = 0,1,2,3$$

$$f(0) = \frac{\binom{4}{0}\binom{3}{3}}{\binom{7}{3}} = \frac{1}{35} ,$$

$$f(1) = \frac{\binom{4}{1}\binom{3}{2}}{\binom{7}{3}} = \frac{12}{35} ,$$

$$f(2) = \frac{\binom{4}{2}\binom{3}{1}}{\binom{7}{3}} = \frac{18}{35} ,$$

$$f(3) = \frac{\binom{4}{3}\binom{3}{0}}{\binom{7}{3}} = \frac{4}{35}$$

X	0	1	2	3	Σ
f(x)	1/35	12/35	18/35	4/35	1
x f(x)	0	12/35	36/35	12/35	60/35=1.71

$$E(X) = \mu_x = \sum xf(x) = 60/35 = 1.71$$

$$E(5) = 5$$

See Ex 4.1
pg 113

$$E(3x) = 3E(x) = 3(60/35) = 5.143$$

$$E(2x - 1) = 2E(x) - 1 = 2(60/35) - 1 = 2.429$$

Ex 4.3 pg114:

Let X be a random variable that denotes the life in hours of a certain electronic device. The probability density function is given by:

$$f(x) = \left\{ \begin{array}{ll} \frac{20000}{X^3} & , \quad X > 100 \\ 0 & \textit{otherwise} \end{array} \right\}$$

Find the expected life of this type of device

Solution:

$$\begin{aligned}\mu = E(X) &= \int_{-\infty}^{\infty} xf(x)dx = \int_{100}^{\infty} X \left(\frac{20000}{X^3}\right) dX \\ &= \int_{100}^{\infty} \frac{20000}{X^2} dX = 20000 \int_{100}^{\infty} X^{-2} dX \\ &= 20000 \left(\frac{X^{-1}}{-1}\right)_{100}^{\infty} = 20000[(100)^{-1} - (\infty)^{-1}] \\ &= \frac{20000}{100} - \frac{20000}{\infty} = 200 - 0 = 200\end{aligned}$$

EX 4.4 pg 115:

Suppose that the number of cars X that pass through a car wash between 4 P.M. and 5 P.M. on any sunny Friday has the following probability distribution:

X	4	5	6	7	8	9
$f(X)$	1/12	1/12	1/4	1/4	1/6	1/6

Let $g(x) = 2x - 1$ represent the amount of money in dollars, paid to the attendant by the manager. Find the attendant's expected earning for this particular time period.

Solution:

X	4	5	6	7	8	9	Σ
f(X)	1/12	1/12	1/4	1/4	1/6	1/6	1
X f(x)	4/12	5/12	6/4	7/4	8/6	9/6	164/24

$$\begin{aligned} E(g(x)) &= E(2x - 1) = 2E(X) - 1 = \\ &= 2\left(\frac{164}{24}\right) - 1 = 12.67 \end{aligned}$$

Ex (4.5 pg 115):

Let X be a random variable with density function:

$$f(x) = \left\{ \begin{array}{ll} \frac{x^2}{3} & , \quad -1 < x < 2 \\ 0 & \quad \textit{otherwise} \end{array} \right\}$$

Find the expected value of $g(x) = 4x+3$

Solution:

$$\begin{aligned} E(X) &= \int_{-1}^2 x \left(\frac{x^2}{3} \right) dx = \int_{-1}^2 \left(\frac{x^3}{3} \right) dx = \left(\frac{x^4}{12} \right)_{-1}^2 \\ &= \frac{1}{12} [2^4 - (-1)^4] = \frac{1}{12} (16 - 1) = \frac{15}{12} \end{aligned}$$

$$E(g(x)) = E(4x + 3) = 4E(X) + 3 = 4\left(\frac{15}{12}\right) + 3 = 8$$

Variance:

Definition:

Let \mathbf{X} be a random variable with probability distribution $\mathbf{f}(\mathbf{x})$ and mean μ . The variance of \mathbf{X} is denoted by $V(\mathbf{x})$ or σ_x^2 :

$$V(x) = \sigma^2 = E(x - \mu)^2 = \sum_{\forall x} (x - \mu)^2 f(x) = E(X^2) - (E(X))^2 \quad \text{if } x \text{ is discrete (2)}$$

$$V(x) = \sigma^2 = E(x - \mu)^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx = E(X^2) - (E(X))^2 \quad \text{if } x \text{ is continuous (3)}$$

where:

$$E(x^2) = \left\{ \begin{array}{l} \sum x^2 f(x) \quad \text{if } x \text{ is discrete} \\ \int_{-\infty}^{\infty} x^2 f(x) dx \quad \text{if } x \text{ is continuous} \end{array} \right\}$$

Properties of the variance:

1. $V(a) = 0$ where a is a constant
2. $V(aX) = a^2V(X)$
3. $V(aX + b) = a^2V(X) + 0$

The Standard Deviation:

The positive square root of the variance, σ is called the standard deviation of \mathbf{X} which is given by:

$$\sigma_X = \sqrt{V(x)} = \sqrt{E(x - \mu_X)^2}$$

Ex (4.8 pg 120):

The probability distribution for company **A** is given by:

X	1	2	3
f(x)	0.3	0.4	0.3

and for company **B** is given by:

Y	0	1	2	3	4
f(y)	0.2	0.1	0.3	0.3	0.1

Show that the variance of the probability distribution for company **B** is greater than that of company **A**.

Solution:

X	1	2	3	Σ
f(x)	0.3	0.4	0.3	1
x f(x)	0.3	0.8	0.9	2
f(x)x ²	0.3	1.6	2.7	4.6

$$\sigma^2 = E(x^2) - (E(x))^2 = 4.6 - 4 = 0.6, \sigma = .77$$

Y	0	1	2	3	4	Σ
f(y)	0.2	0.1	0.3	0.3	0.1	1
Y f(y)	0	0.1	0.6	0.9	0.4	2
$y^2 f(y)$	0	0.1	1.2	2.7	1.6	5.6

$$\sigma^2 = E(y^2) - (E(y))^2 = 5.6 - 4 = 1.6, \sigma = 1.26$$

Note that σ_y^2 is greater than σ_x^2 .

Ex (4.10 pg 121):

The weekly demand for a drinking-water product, in thousands of liters from a local chain of efficiency stores having the probability density:

$$f(x) = \begin{cases} 2(x-1) & , \quad 1 < X < 2 \\ 0 & \textit{otherwise} \end{cases}$$

Find the mean and variance of x .

Solution:

$$\begin{aligned}\mu &= \int_1^2 2x(x-1) dx = 2 \int_1^2 (x^2 - x) dx = 2 \left(\frac{x^3}{3} - \frac{x^2}{2} \right) \Big|_1^2 = 2 \left[\left(\frac{8}{3} - 2 \right) - \left(\frac{1}{3} - \frac{1}{2} \right) \right] \\ &= 2 \left(\frac{8-6}{3} - \frac{2-3}{6} \right) = 2 \left(\frac{2}{3} + \frac{1}{6} \right) = \frac{5}{3}\end{aligned}$$

$$\begin{aligned}E(X^2) &= \int_1^2 2x^2(x-1) dx = 2 \int_1^2 (x^3 - x^2) dx = 2 \left(\frac{x^4}{4} - \frac{x^3}{3} \right) \Big|_1^2 \\ &= 2 \left[\left(4 - \frac{2}{8} \right) - \left(\frac{1}{4} - \frac{1}{3} \right) \right] = 17/6\end{aligned}$$

$$\sigma^2 = E(x^2) - (E(x))^2 = \frac{17}{6} - \left(\frac{5}{3} \right)^2 = 1/18$$

Ex 4.18 pg 129:

Let \mathbf{X} be a random variable having the density function:

$$f(x) = \begin{cases} \frac{x^2}{3} & , \quad -1 < x < 2 \\ 0 & \textit{otherwise} \end{cases}$$

Find the variance of the random variable **$g(\mathbf{x}) = 4\mathbf{x}+3$** .

Solution:

$$V(g(x)) = V(4x+3) = 16 V(x) = 16[E(x^2) - (E(x))^2]$$

$$E(x) = \int_{-1}^2 X \frac{X^2}{3} dx = \frac{X^4}{12} \Big|_{-1}^2 = \frac{16}{12} - \frac{1}{12} = \frac{15}{12} = \frac{5}{4}$$

$$E(x^2) = \int_{-1}^2 x^2 \left(\frac{x^2}{3}\right) dx = \frac{x^5}{15} \Big|_{-1}^2 = \frac{32}{15} - \frac{(-1)}{15} = \frac{11}{5}$$

$$V(x) = E(x^2) - (E(x))^2 = \frac{11}{5} - \left(\frac{5}{4}\right)^2 = \frac{11}{5} - \frac{25}{16} = \frac{176 - 125}{80} = 0.6375$$

$$V(g(x)) = V(4x + 3) = 16V(x) + 0 = 16(0.6375) = 10.2$$

4.3 Means and Variance of Linear Combinations of Random Variables (pg 128):

The expected value of the sum or difference of two or more functions of a random variable \mathbf{X} is the sum or difference of the expected values of the functions. That is

$$E(g(x) \pm h(x)) = E(g(x)) \pm E(h(x)) \quad (6)$$

Ex4.19 pg 129 :

Let **X** be a random variable with probability distribution as follows:

X	0	1	2	3
f(x)	1/3	1/2	0	1/6

Find the expected value of $y = (x-1)^2$.

Solution:

$$E(y) = E(x-1)^2 = E(x^2 - 2x + 1) = E(x^2) - 2E(x) + 1$$

X	0	1	2	3	Σ
f(x)	1/3	1/2	0	1/6	1
X f(x)	0	1/2	0	3/6	1
X ² f(x)	0	1/2	0	9/6	2

$$E(y) = 2 - 2(1) + 1 = 1$$

See ex 4.17 pg
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Ex 4.20 pg 130:

Find the expected value for $g(x) = x^2 + x - 2$,
where \mathbf{X} has the density function:

$$f(x) = \left\{ \begin{array}{ll} 2(x-1) & , \quad 1 < x < 2 \\ 0 & \textit{otherwise} \end{array} \right\}$$

Solution:

$$\begin{aligned} E(x) &= \int_1^2 x 2(x-1) dx = 2 \int_1^2 (x^2 - x) dx = 2 \left(\frac{x^3}{3} - \frac{x^2}{2} \right) \Big|_1^2 = \\ &= 2 \left(\frac{2}{3} + \frac{1}{6} \right) = \frac{5}{3} \end{aligned}$$

$$\begin{aligned} E(x^2) &= \int_1^2 2x^2(x-1) dx = 2 \int_1^2 (x^3 - x^2) dx = 2 \left(\frac{x^4}{4} - \frac{x^3}{3} \right) \Big|_1^2 \\ &= 2 \left[\left(4 - \frac{8}{3} \right) - \left(\frac{1}{4} - \frac{1}{3} \right) \right] = 2 \left(\frac{12-8}{3} - \frac{3-4}{12} \right) = 2 \left(\frac{4}{3} + \frac{1}{12} \right) = \frac{17}{6} \end{aligned}$$

$$E(x^2 + x - 2) = E(x^2) + E(x) - 2 = \frac{17}{6} + \frac{5}{3} - 2 = \frac{5}{2}$$

4.4 Chebyshev's Theorem (pg 135):

The probability that any random variable \mathbf{X} will assume a value within \mathbf{K} standard deviations of the mean μ_x is at least $(1 - \frac{1}{K^2})$.

That is:

$$P(\mu - K\sigma < X < \mu + K\sigma) \geq 1 - \frac{1}{K^2} \quad (7)$$

Ex (4.27 pg 137):

A random variable \mathbf{X} has a mean $\mu = 8$, a variance $\sigma^2 = 9$ and an unknown probability distribution. Find:

(a) $P(-4 < X < 20)$

(b) $P(|X - 8| \geq 6)$

Solution:

$$(a) P(-4 < X < 20) = P[8 - (k)(3) < X < 8 + (k)(3)] \rightarrow$$

$$8 - 3k = -4 \rightarrow 8 + 4 = 3k \rightarrow 12 = 3k \rightarrow k = 4$$

$$P(-4 < X < 20) \geq 1 - \frac{1}{16} \rightarrow P(-4 < X < 20) \geq \frac{15}{16}$$

$$(b) P(|X - 8| \geq 6) = 1 - P(|x - 8| < 6) = 1 - P(-6 < (X - 8) < 6)$$

$$= 1 - P(-6 + 8 < X < 6 + 8) = 1 - P(2 < X < 14)$$

$$1 - P(2 < X < 14) \geq 1 - \frac{1}{k^2} \rightarrow 1 - 1 + \frac{1}{k^2} \geq P(2 < X < 14)$$

$$\rightarrow \frac{1}{k^2} \geq P(2 < X < 14) \rightarrow P(2 < X < 14) \leq \frac{1}{k^2}$$

$$2 = 8 - 3K \rightarrow 3K = 8 - 2 = 6 \rightarrow K = 2 \quad \text{or}$$

$$14 = 8 + 3k \rightarrow 14 - 8 = 3K \rightarrow 6 = 3K \rightarrow K = 2$$

$$P(2 < x < 14) \leq \frac{1}{4}$$