INTEGRAL CALCULUS (MATH 106)

BEN AMIRA Aymen

abenamira@ksu.edu.sa

Department of Mathematics

College of Sciences

King Saud University

Chapter 3: Inverse Trigonometeric and Hyperbolic Functions

The student is expected to be able to:

1. Find the derivative and integrals The Inverse trigonometric functions.
2. Find the derivative and integrals Hyperbolic functions.
3. Find the derivative and integrals Inverse Hyperbolic functions.

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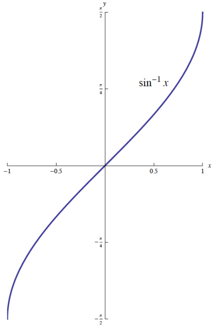
Definition (Inverse if sine)

The inverse sine function is denoted by $\sin^{-1}$ and it is defined as
$y = \sin^{-1} x \Leftrightarrow x = \sin y$, where $x \in [-1, 1]$ and $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$
The domain of the inverse sine function is $[-1, 1]$
The range of the inverse sine function is $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

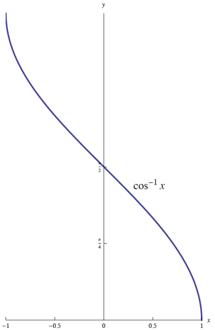


Graph of $\sin^{-1} x$

Inverse Trigonometeric

Definition (Inverse of cosine)

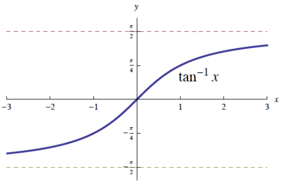
The inverse cosine function is denoted by $\cos^{-1}$ and it is defined as
$y = \cos^{-1} x \Leftrightarrow x = \cos y$, where $x \in [-1, 1]$ and $y \in [0, \pi]$
The domain of the inverse cosine function is $[-1, 1]$
The range of the inverse cosine function is $[0, \pi]$.



Graph of $\cos^{-1} x$

Definition (Inverse of Tangent)

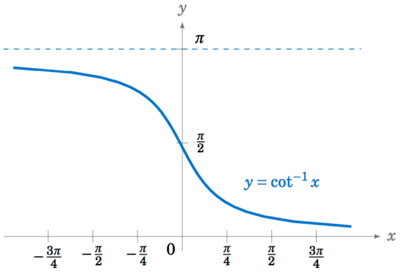
The inverse tangent function is denoted by $\tan^{-1}$ and it is defined as
$y = \tan^{-1} x \Leftrightarrow x = \tan y$, where $x \in \mathbb{R}$ and $y \in (-\frac{\pi}{2}, \frac{\pi}{2})$
The domain of the inverse tangent function is $\mathbb{R}$
The range of the inverse tangent function is $(-\frac{\pi}{2}, \frac{\pi}{2})$.



Graph of $\tan^{-1} x$

Definition (Inverse of cotangent)

The inverse cotangent function is denoted by $\cot^{-1}$ and it is defined as
$\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$, where $x \in \mathbb{R}$
The domain of the inverse cotangent function is $\mathbb{R}$
The range of the inverse cotangent function is $(0, \pi)$.

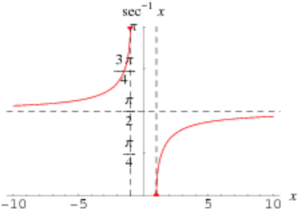


Definition (Inverse secant)

The inverse secant function is denoted by $\sec^{-1}$ and it is defined as $y = \sec^{-1} x \Leftrightarrow x = \sec y$, where $y \in [0, \frac{\pi}{2})$ if $x \geq 1$, and $y \in [\pi, \frac{3\pi}{2})$ if $x \leq -1$

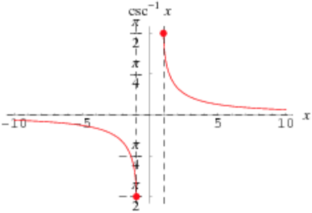
The domain of the inverse secant function is $(-\infty, -1] \cup [1, \infty)$

The range of the inverse secant function is $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$.



Definition (Inverse cosecant)

The inverse cosecant function is denoted by $\csc^{-1}$ and it is defined as $\csc^{-1} x = \frac{\pi}{2} - sec^{-1}x$, where $|x| \geq 1$
The domain of the inverse cosecant function is $(-\infty, -1] \cup [1, \infty)$
The range of the inverse cosecant function is $[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$.



Inverse Trigonometeric

Derivatives of the inverse trigonometric functions

1. $\dfrac{d}{dx}\sin^{-1}x = \dfrac{1}{\sqrt{1-x^2}}$, where $|x| < 1$
2. $\dfrac{d}{dx}\cos^{-1}x = \dfrac{-1}{\sqrt{1-x^2}}$, where $|x| < 1$
3. $\dfrac{d}{dx}\tan^{-1}x = \dfrac{1}{1+x^2}$
4. $\dfrac{d}{dx}\cot^{-1}x = \dfrac{-1}{1+x^2}$
5. $\dfrac{d}{dx}\sec^{-1}x = \dfrac{1}{x\sqrt{1-x^2}}$, where $|x| > 1$
6. $\dfrac{d}{dx}\csc^{-1}x = \dfrac{-1}{x\sqrt{x^2-1}}$, where $|x| > 1$

Inverse Trigonometeric

Integration of the inverse trigonometric functions

1. $$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}(\frac{x}{a}) + c, \ (|x| < a)$$
   $$\int \frac{f'(x)}{\sqrt{a^2 - [f(x)]^2}} dx = \sin^{-1}(\frac{f(x)}{a}) + c, \ (|f(x)| < a)$$
2. $$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1}(\frac{x}{a}) + c$$
   $$\int \frac{f'(x)}{a^2 + [f(x)]^2} dx = \frac{1}{a} \tan^{-1}(\frac{f(x)}{a}) + c$$
3. $$\int \frac{1}{x\sqrt{x^2 - a^2}} dx = \frac{1}{a} \sec^{-1}(\frac{x}{a}) + c, \ (|x| > a)$$
   $$\int \frac{f'(x)}{f(x)\sqrt{[f(x)]^2 - a^2}} dx = \frac{1}{a} \sec^{-1}(\frac{f(x)}{a}) + c, \ (|f(x)| > a)$$

Examples

1. $$\int \frac{x^2}{5+x^6}dx = \frac{1}{3}\int \frac{3x^2}{(\sqrt{5})^2+(x^3)^2}dx = \frac{1}{3}\frac{1}{\sqrt{5}}\tan^{-1}(\frac{x^3}{\sqrt{5}}) + c.$$
   Here $a=\sqrt{5}$, $f(x)=x^3$ and $f'(x)=3x^2$.
2. $$\int \frac{3x}{\sqrt{9-x^4}}dx = \frac{3}{2}\int \frac{2x}{\sqrt{3^2-(x^2)^2}}dx = \frac{3}{2}\sin^{-1}(\frac{x^2}{3}) + c.$$
3. $$\int \frac{3x}{\sqrt{9-x^2}}dx = \frac{3}{-2}\int (9-x^2)^{-\frac{1}{2}}(-2x)dx = -\frac{3}{2}\frac{(9-x^2)^{\frac{1}{2}}}{\frac{1}{2}} + c$$
4. $$\int \frac{1}{x\sqrt{1-(\ln x)^2}}dx = \int \frac{(\frac{1}{x})}{\sqrt{(1)^2-(\ln x)^2}}dx = \sin^{-1}(\ln x) + c$$
5. $$\int \frac{1}{1+3x^2}dx = \frac{1}{\sqrt{3}}\int \frac{\sqrt{3}}{1^2+(\sqrt{3}x)^2}dx = \frac{1}{\sqrt{3}}\tan^{-1}(\sqrt{3}x) + c.$$

Inverse Trigonometeric

Examples

6. $$\int \frac{e^{2x}}{e^{4x}+16}dx = \frac{1}{2}\int \frac{2e^{2x}}{4^2+(2^{2x})^2}dx = \frac{1}{2*4}\tan^{-1}\left(\frac{e^{2x}}{4}\right)+c.$$
7. $$\int \frac{1}{\sqrt{e^{2x}-36}}dx = \int \frac{e^x}{e^x\sqrt{(e^x)^2-(6)^2}}dx = \frac{1}{6}\sec^{-1}\left(\frac{e^x}{6}\right)+c.$$

Exercises

Solve the following integrals :

1. $$\int \frac{x+\sin^{-1}x}{\sqrt{1-x^2}}dx$$
2. $$\int \frac{x+1}{x^2+1}dx$$

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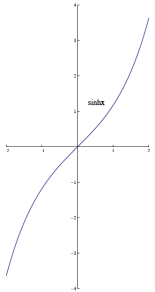
Hyperbolic Functions

Definition (The hyperbolic sine function)

It is denoted by $\sinh x$ and it is defined as $\sinh x = \frac{e^x - e^{-x}}{2}$

Notes

1. The domain of $\sinh x$ is $\mathbb{R}$ and the range of $\sinh x$ is $\mathbb{R}$.
2. It is an odd function and $\sinh(0) = 0$



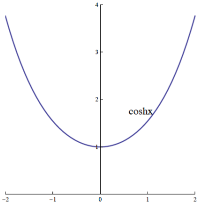
Hyperbolic Functions

Definition (The hyperbolic cosine function)

It is denoted by $\cosh x$ and it is defined as $\cosh x = \frac{e^x + e^{-x}}{2}$

Notes

1. The domain of $\cosh x$ is $\mathbb{R}$ and the range of $\cosh x$ is $[1, \infty]$.
2. It is an even function and $\cosh(0) = 1$



Hyperbolic Functions

Definitions

1. The hyperbolic tangent function is denoted by $\tanh x$ and it is defined as $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ for every $x \in \mathbb{R}$
2. The hyperbolic cotangent function is denoted by $\coth x$ and it is defined as $\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^x + e^x}{e^x - e^{-x}}$ for every $x \in \mathbb{R} - \{0\}$
3. The hyperbolic secant function is denoted by *sech* x and it is defined as $sech\ x = \dfrac{1}{\cosh x} = \dfrac{2}{e^x - e^{-x}}$ for every $x \in \mathbb{R}$
4. The hyperbolic cosecant function is denoted by *csch* x and it is defined as $csch\ x = \dfrac{1}{\sinh x} = \dfrac{2}{e^x - e^{-x}}$ for every $x \in \mathbb{R} - \{0\}$

Hyperbolic Functions

Notes

1. $cosh^2\ x - sinh^2\ x = 1$ for every $x \in \mathbb{R}$
2. $1 - tanh^2\ x = sech^2\ x$ for every $x \in \mathbb{R}$
3. $coth^2\ x - 1 = csch^2\ x$ for every $x \in \mathbb{R} - \{0\}$

Hyperbolic Functions

Derivatives of the hyperbolic functions

1. $\dfrac{d}{dx}\sinh x = \cosh x,$
2. $\dfrac{d}{dx}\cosh x = \sinh x,$
3. $\dfrac{d}{dx}tanh\ x = sech^2\ x$
4. $\dfrac{d}{dx}coth\ x = -csch^2\ x$
5. $\dfrac{d}{dx}sech\ x = -sech\ x\ tanh\ x$
6. $\dfrac{d}{dx}csch\ x = -csch\ x\ coth\ x$

Derivatives of the hyperbolic functions

1. $$\frac{d}{dx}\sinh(f(x)) = \cosh(f(x))f'(x)$$
2. $$\frac{d}{dx}\cosh(f(x)) = \sinh(f(x))f'(x)$$
3. $$\frac{d}{dx}tanh\ (f(x)) = sech^2\ (f(x))f'(x)$$
4. $$\frac{d}{dx}coth\ (f(x)) = -csch^2\ (f(x))f'(x)$$
5. $$\frac{d}{dx}sech\ (f(x)) = -sech\ (f(x))\ tanh\ (f(x))f'(x)$$
6. $$\frac{d}{dx}csch\ (f(x)) = -csch\ (f(x))\ coth\ (f(x))f'(x)$$

Examples

1. Find the value of $f(0)$ if $f(x) = \ln[cosh(3x)]$.
   $f(0) = \ln[cosh(0)] = \ln(1) = 0$.
2. Find the value of $f'(0)$ if $f(x) = \ln|1 + sinh(x)|$.
   $$f'(x) = \frac{cosh(x)}{1 + sinh(x)} \rightarrow f'(0) = \frac{cosh(0)}{1 + sinh(0)} = \frac{1}{1+0} = 1$$
3. Find $f'(x)$ if $f(x) = e^{sinh(x)}$.
   $f'(x) = e^{sinh(x)} cosh(x)$.
4. Find $f'(x)$ if $f(x) = \tan^{-1}(sinh(x))$.
   $$f'(x) = \frac{cosh(x)}{1 + (sinh(x))^2} = \frac{cosh(x)}{(cosh(x))^2} = \frac{1}{cosh(x)} = sech(x).$$

Hyperbolic Functions

Integration of the hyperbolic functions

1. $\displaystyle\int \sinh x \; dx = \cosh x + c,$
2. $\displaystyle\int \sinh(f(x))f'(x)dx = \cosh(f(x)) + c$
3. $\displaystyle\int \cosh x \; dx = \sinh x + c,$
4. $\displaystyle\int \cosh(f(x))f'(x)dx = \sinh(f(x)) + c$
5. $\displaystyle\int sech^2 x \; dx = \tanh x + c$
6. $\displaystyle\int sech^2(f(x))f'(x)dx = \tanh(f(x)) + c$

Integration of the hyperbolic functions

7. $$\int csch^2 x \; dx = -\coth x + c$$
8. $$\int csch^2(f(x))f'(x)dx = -\coth(f(x)) + c$$

Examples

1. $$\int x^2 \cosh x^3 \; dx = \frac{1}{3}\int \cosh x^3 \; (3x^2)dx = \frac{1}{3}\sinh x^3 + c$$
2. $$\int (e^x - e^{-x})sech^2(e^x + e^{-x})dx = \tanh(e^x + e^{-x}) + c$$
3. $$\int \frac{\sinh x}{1+\sinh^2 x}dx = \int \frac{\sinh x}{\cosh^2 x}dx = \int \frac{1}{\cosh x}\frac{\sinh x}{\cosh x}dx$$
$$= \int sech \; x \tanh x \; dx = -sech \; x + c$$
4. $$\int \frac{1}{sech \; x\sqrt{4-\sinh^2 x}}dx = \int \frac{\cosh x}{\sqrt{(2)^2 - (\sinh x)^2}}dx$$
$$= \sin^{-1}(\frac{\sinh x}{2}) + c$$

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The Inverse Hyperbolic Functions

Definition

The inverse hyperbolic sine function is denoted by $\sinh^{-1}$ and it is defined as $y = \sinh^{-1} x \Leftrightarrow x = \sinh y$, where $x \in \mathbb{R}$ and $y \in \mathbb{R}$

Definition

The inverse hyperbolic cosine function is denoted by $\cosh^{-1}$ and it is defined as $y = \cosh^{-1} x \Leftrightarrow x = \cosh y$, where $x \in [1, \infty)$ and $y \in [0, \infty)$

Definition

The inverse hyperbolic tangent function is denoted by $\tanh^{-1}$ and it is defined as $y = \tanh^{-1} x \Leftrightarrow x = \tanh y$, where $x \in [-1, 1]$ and $y \in \mathbb{R}$

The Inverse Hyperbolic Functions

Definition

The inverse hyperbolic cotangent function is denoted by $\coth^{-1}$ and it is defined as $y = \coth^{-1} x \Leftrightarrow x = \coth y$, where $|x| > 1$ and $y \in \mathbb{R}$.

Definition

The inverse hyperbolic secant function is denoted by $sech^{-1}$ and it is defined as $y = sech^{-1}x \Leftrightarrow x = sech\ y$, where $x \in [0, 1]$ and $y \in [0, \infty)$

Definition

The inverse hyperbolic cosecant function is denoted by $csch^{-1}$ and it is defined as $y = csch^{-1}x \Leftrightarrow x = csch\ y$, where $x \in \mathbb{R}$ and $y \in \mathbb{R} - \{0\}$

The Inverse Hyperbolic Functions

Derivatives of the inverse hyperbolic functions

- $\dfrac{d}{dx}\sinh^{-1} x = \dfrac{1}{\sqrt{1+x^2}}$,
- $\dfrac{d}{dx}\sinh^{-1} f(x) = \dfrac{f'(x)}{\sqrt{1+f((x))^2}}$.
- $\dfrac{d}{dx}\cosh^{-1} x = \dfrac{1}{\sqrt{x^2-1}}$, where $x > 1$
- $\dfrac{d}{dx}\cosh^{-1} f(x) = \dfrac{f'(x)}{\sqrt{(f(x))^2-1}}$, where $|f(x)| > 1$
- $\dfrac{d}{dx}\tanh^{-1} x = \dfrac{1}{1-x^2}$, where $|x| > 1$
- $\dfrac{d}{dx}\tanh^{-1} f(x) = \dfrac{f'(x)}{1-(f(x))^2}$, where$|f(x)| > 1$

Derivatives of the inverse hyperbolic functions

1. $\dfrac{d}{dx}coth^{-1}x = \dfrac{-1}{1-x^2}$ where $|x| > 1$
2. $\dfrac{d}{dx}coth^{-1}f(x) = \dfrac{-f'(x)}{1-(f(x))^2}$ where $|f(x)| > 1$
3. $\dfrac{d}{dx}sech^{-1}x = \dfrac{-1}{x\sqrt{1-x^2}}$ where $0 < x < 1$
4. $\dfrac{d}{dx}sech^{-1}f(x) = \dfrac{-f'(x)}{f(x)\sqrt{1-(f(x))^2}}$ where $0 < f(x) < 1$
5. $\dfrac{d}{dx}csch^{-1}x = \dfrac{-1}{|x|\sqrt{1+x^2}}$, where $x \neq 0$
6. $\dfrac{d}{dx}csch^{-1}f(x) = \dfrac{-f'(x)}{|f(x)|\sqrt{1+(f(x))^2}}$, where $f(x) \neq 0$

The Inverse Hyperbolic Functions

Examples

1. Find $f'(x)$ if $f(x) = \tanh^{-1} 3x$?
   $$f'(x) = \frac{3}{1-(3x)^2} = \frac{3}{1-9x^2}$$
2. Find $f'(x)$ if $f(x) = \sinh^{-1} \sqrt{x}$?
   $$f'(x) = \frac{\frac{1}{2\sqrt{x}}}{\sqrt{1+(\sqrt{x})^2}} = \frac{1}{2\sqrt{x}\sqrt{1+x}}$$
3. Find $f'(x)$ if $f(x) = sech^{-1}(\cos 2x)$?
   $$f'(x) = \frac{-(-2\sin 2x)}{\cos 2x\sqrt{1-(\cos 2x)^2}} = \frac{2\sin 2x}{\cos 2x\sqrt{1-\cos^2 2x}}$$

Integration of the inverse hyperbolic functions

- $\int \frac{1}{\sqrt{a^2+x^2}}dx = \sinh^{-1}(\frac{x}{a}) + c$
- $\int \frac{f'(x)}{\sqrt{a^2+(f(x))^2}}dx = \sinh^{-1}(\frac{f(x)}{a}) + c$
- $\int \frac{1}{\sqrt{x^2-a^2}}dx = cosh^{-1}(\frac{x}{a}) + c, (x > a)$
- $\int \frac{f'(x)}{\sqrt{(f(x))^2-a^2}}dx = cosh^{-1}(\frac{f(x)}{a}) + c, \ (f(x) > a)$
- $\int \frac{1}{a^2-x^2}dx = \frac{1}{a}tanh^{-1}(\frac{x}{a}) + c(|x| < a)$
- $\int \frac{f'(x)}{a^2-(f(x))^2}dx = \frac{1}{a}tanh^{-1}(\frac{f(x)}{a}) + c, \ (|f(x)| < a)$

The Inverse Hyperbolic Functions

Integration of the inverse hyperbolic functions

- $\int \frac{1}{x\sqrt{a^2-x^2}}dx = -\frac{1}{a}sech^{-1}(\frac{x}{a}) + c, \ (0 < x < a)$
- $\int \frac{f'(x)}{f(x)\sqrt{a^2-(f(x))^2}}dx = -\frac{1}{a}sech^{-1}(\frac{f(x)}{a}) + c,$ $(0 < f(x) < a)$
- $\int \frac{1}{x\sqrt{x^2+a^2}}dx = -\frac{1}{a}csch^{-1}(\frac{x}{a}) + c, \ (x \neq 0)$
- $\int \frac{f'(x)}{x\sqrt{(f(x))^2+a^2}}dx = -\frac{1}{a}csch^{-1}(\frac{f(x)}{a}) + c, \ (f(x) \neq 0)$

Examples

1. $$\int \frac{e^x}{1-e^{2x}}dx = \int \frac{e^x}{(1)^2-(e^x)^2}dx = \tanh^{-1}(e^x)+c$$
2. $$\int \frac{1}{\sqrt{x}\sqrt{4+x}}dx = 2\int \frac{\frac{1}{2\sqrt{x}}}{\sqrt{(2)^2+(\sqrt{x})^2}}dx = 2\sinh^{-1}(\frac{\sqrt{x}}{2})+c$$
3. $$\int \frac{1}{\sqrt{1+e^{2x}}}dx = \int \frac{e^x}{e^x\sqrt{1+e^{2x}}}dx = -csch^{-1}(e^x)+c$$