Chapter 3: Methods of solving linear system of equations:

2.1.1 Elementary Row operations:

Elementary row operations are steps for solving the linear system of equations:

I. Interchange two rows.
II. Multiply a row with non zero real number.
III. Add a multiple of one row to another row.

Note: *Elementary row operations produce same results when operated either on a system or on its augmented matrix form.*

1. Gaussian Elimination Method

STEP 1. by using elementary row operations

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & A_{12} & A_{13} & B_1 \\ 0 & 1 & A_{23} & B_2 \\ 0 & 0 & 1 & B_3 \end{bmatrix}$$

STEP 2. Find solution by back – substitutions.

Example:3. Solve the system of linear equations by Gaussian- elimination method

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 8 \\ -x_1 - 2x_2 + 3x_3 &= 1 \\ 3x_1 - 7x_2 + 4x_3 &= 10 \end{aligned}$$

Solution: **Augmented matrix is**

$$[A \vdots b] = \begin{bmatrix} 1 & 1 & 2 & \mathbf{8} \\ -1 & -2 & 3 & \mathbf{1} \\ 3 & -7 & 4 & \mathbf{10} \end{bmatrix}$$

$$\approx \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{bmatrix} \qquad R_1+R_2, \quad -3R_1+R_3$$

$$\approx \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{bmatrix} -R_2, \quad 10R_2+R_3$$

$$\approx \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{bmatrix} -R_3/52$$

Notice here that the number of variables = the number of equations. Therefore, there are a unique solution, we can get it as follows:

. **Back Substitution** $\quad x_3 = 2$

$x_2 = 5x_3 - 9 = 10 - 9 = 1$

$x_1 = -x_2 - 2x_3 + 8 = -1 - 4 + 8 = 3$

Solution is $\quad x_1 = 3, \quad x_2 = 1, \quad x_3 = 2.$

REMARK:

1- Don't forget that any linear system of finite equation has got 3 possibilities: Unique solution, infinite many solutions or no solution.
2- After doing the elimination method for the augmented matrix of the system, we will get one of the following forms:
   a- The last row of coefficients are zeros opposites non zero number of the constant which means there is no solution
   b- The last row of coefficients is zeros opposites zero number of the constant which means that the number of equations is less than the number of variables, and therefore there are infinite many solutions (i.e. the solution is written by parameters) (number of parameters= number of variables – number of equations)
   c- Similar to the last example which means there is a unique solution.

3- The homogenous linear system AX=0 is always consistent (i.e. has got a solution) because X=0 is a guaranteed solution. So, our task is to determine this solution is unique or there are infinite many solutions.

2.2.1 <u>Gauss – Jordan Elimination Method</u>

Matrix equation form

$$AX=b$$

$$\textbf{[A:b]} \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & B_1 \\ 0 & 1 & 0 & B_2 \\ 0 & 0 & 1 & B_3 \end{bmatrix}$$

By using elementary row operations we reduced the given system of equation with "1" as diagonal entries and all other entries of Matrix A are " 0 "

2.2.2 <u>Row Echelon Form</u>

A form of a matrix, which satisfies following conditions, is row echelon form

i. '1' (leading entry) must be in the beginning of each row,
ii. '1' must be on the right of the above leading entry,
iii. Below the leading entry all values must be zero,
iv. A row containing all zero values must be in the bottom.

Examples:

(i) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (iii) $\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Note: 1. Gaussian Elimination method is reducing the given Augmented matrix to Row echelon form and backward substitution.
Note: 2. Gauss- Jordan Elimination method is reducing the given Augmented matrix to Reduced Row echelon form.

Example:5. Use Gauss – Jorden method to solve the system of linear system

$$\begin{aligned} x - y + 2z - w &= -1 \\ 2x + y - 2z - 2w &= -2 \\ -x + 2y - 4z + w &= 1 \\ 3x \qquad\qquad - 3w &= -3 \end{aligned}$$

Solution: Gauss-Jorden method is same as to reduce the augmented matrix to reduced row echelon from.
Augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -1 \\ 2 & 1 & -2 & -2 & -2 \\ -1 & 2 & -4 & 1 & 1 \\ 3 & 0 & 0 & -3 & -3 \end{array}\right]$$

$$\approx \left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{array}\right] \quad (-2R_1+R_2)/3,\ R_1+R_3,\ -3R_1+R_4$$

$$\approx \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad R_2+R_1,\ -R_2+R_3,\ -3R_2+R_4$$

is reduced row echelon form
Equivalent matrix form is

$x - w = -1 \Rightarrow x = -1 + w$

$y - 2z = 0 \Rightarrow y = 2z$

(if $w = t \Rightarrow x = -1 + t$)
(if $z = s \Rightarrow y = 2s$)

Solution:

$$\boxed{\begin{aligned} x &= -1 + w = -1 + t \\ y &= 2z \quad\; = 2s \\ z &= s \\ w &= t \end{aligned}}$$

Example: 6 . Solve the system of linear equations

$$
\begin{aligned}
x - 2y + z - 4u &= 1\\
x + 3y + 7z + 2u &= 2\\
x - 12y - 11z - 16u &= 5
\end{aligned}
$$

Solution:

Augmented matrix is:

$$
\left[\begin{array}{cccc|c}
1 & -2 & 1 & -4 & 1\\
1 & 3 & 7 & 2 & 2\\
1 & -12 & -11 & -16 & 5
\end{array}\right]
$$

Reducing it to row echelon form (using Gaussian - elimination method)

$$
\approx \left[\begin{array}{cccc|c}
1 & -2 & 1 & -4 & 1\\
0 & 5 & 6 & 6 & 1\\
0 & -10 & -12 & -12 & 4
\end{array}\right] \quad R_2 - R_1,\ R_3 - R_1
$$

$$
\approx \left[\begin{array}{cccc|c}
1 & -2 & 1 & -4 & 1\\
0 & 5 & 6 & 6 & 1\\
0 & 0 & 0 & 0 & 6
\end{array}\right] \quad -R_3 + 2R_2 \qquad \approx \left[\begin{array}{cccc|c}
1 & -2 & 1 & -4 & 1\\
0 & 1 & 6/5 & 6/5 & 1/5\\
0 & 0 & 0 & 0 & 6
\end{array}\right] \ \frac{1}{5}R_2
$$

Last equation is

$$
\Rightarrow \quad 0x + 0y + 0z + 0u = 6
$$

but $0 \neq 6$ (impossible case)

Therefore, there is no solutions.

Example:7. For which values of 'a' will be following system

$$
\begin{aligned}
x + 2y - 3z &= 4\\
3x - y + 5z &= 2\\
4x + y + (a^2 - 14)z &= a + 2
\end{aligned}
$$

(i) infinitely many solutions?

(ii) No solution?

(iii) Exactly one solution?

Solution:

Augmented matrix is

$$
\left[\begin{array}{ccc|c}
1 & 2 & -3 & 4\\
3 & -1 & 5 & 2\\
4 & 1 & a^2 - 14 & a + 2
\end{array}\right]
$$

Reducing it to row echelon form

$$\approx\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & -14 & -10 \\ 0 & -7 & a^2-2 & a-14 \end{array}\right] \quad R_2-3R_1,\ R_3-4R_1$$

$$\approx\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & \frac{10}{7} \\ 0 & 0 & a^2-16 & a-4 \end{array}\right] \quad -\tfrac{1}{7}R_2,\ R_3-R_2$$

We have

$$(a+4)(a-4)z = a-4$$

$\boxed{\text{CASE I.}}$

$\boxed{a=4 \quad \Rightarrow 0z=0}$ then we have 2 equations

$$\begin{cases} x+2y-3z = 4 \\ \quad y-2z = \frac{10}{7} \end{cases}$$

as number of equations are less than number of unknowns, hence the system has infinite many solutions,

let $z = t$

$y = \frac{10}{7} + 2t$

$x = 4 + 3t - 4t - \frac{20}{7} = -t + \frac{8}{7}$

where 't' is any real number.

$\boxed{\text{CASE II}}$

Impossible case.

$\boxed{a=-4} \quad \Rightarrow \quad 0z = -8$, but $0 \neq -8$, hence, there is no solution.

$\boxed{\text{CASE III}}$

$\boxed{a \neq 4, a \neq -4,}$ let $a = 1$ For example to ease

Equatins .3. $\Rightarrow$ $(1-4)(1+4)z = 1-4$

$$-15z = -3$$

$$z = \tfrac{1}{5}$$

$$y = \tfrac{10}{7} + \tfrac{2}{5} = \tfrac{64}{35}$$

$$x = 4 + \tfrac{3}{5} - 2(\tfrac{64}{35}) = \tfrac{47}{35}$$

the system will have unique solution when $a \neq 4$ and $a \neq -4$ and for $a=1$ the solution is

In brief:

(i) $a=-4$, no solution,
(ii) $a=4$, infinite many solutions and
(iii) $a \neq 4$, $a \neq -4$, exactly one solution .

Example:8. What conditions must a, b, and c satisfy in order for the system of equations

$$x + y + 2z = a$$
$$x \qquad + z = b$$
$$2x + y + 3z = c$$

to be consistent.

Solution: The augmented matrix is

$$\begin{bmatrix} 1 & 1 & 2 & a \\ 1 & 0 & 1 & b \\ 2 & 1 & 3 & c \end{bmatrix}$$ reducing it to reduced row echelon form

$$\approx \begin{bmatrix} 1 & 1 & 2 & a \\ 0 & -1 & -1 & b-a \\ 0 & -1 & -1 & c-2a \end{bmatrix} \quad R_2\text{-}R_1,\ R_3\text{-}2R_1$$

$$\approx \begin{bmatrix} 1 & 1 & 2 & a \\ 0 & -1 & -1 & b-a \\ 0 & 0 & 0 & c-a-b \end{bmatrix} \quad R_3\text{-}R_1$$

The system will be consistent if only if $c - a - b = 0$

Solving Linear system by Inverse Matrix

Let a given linear system of equations is

$$AX = B$$

Find A^{-1}

Multiply with A^{-1} from left

$$A^{-1}AX = A^{-1}B$$

$$IX = A^{-1}B$$

$$X = A^{-1}B \text{ is a solution.}$$

Example1.

Write the system of equations in a matrix form, find A^{-1}, use A^{-1} to solve the system

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 4 \\ 2x_1 + 2x_2 + x_3 &= -1 \\ 2x_1 + 3x_2 + x_3 &= 3 \end{aligned}$$

Solution: 1. Matrix Form is:

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & 2 & 1 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} \quad \text{is in form of AX = B}$$

2. Find A^{-1}

$$A^{-1} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 2 & 3 & -4 \end{bmatrix}$$

$$X = A^{-1}B = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 2 & 3 & -4 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ -7 \end{bmatrix}$$

Solution set is $x_1 = -1, x_2 = 4, x_3 = -7.$

2- If $AX = 0$ is the matrix form of (Homogeneous system) then

Case 1 A has inverse (i.e. $|A| \neq 0$) $\Rightarrow A^{-1}(AX) = A^{-1} \cdot 0$
$\Rightarrow X = 0$ (has only one solution which is zero solution)

Case 2 A ~~have~~ doesn't have inverse (i.e. $|A| = 0$) then the system has ∞ many solutions, one of them is zero solution.

$\boxed{\text{Every Homogeneous System is Consistent (i.e. has at least one solution).}}$

Question:1 . **Let**

$$x - y - z = 0$$
$$2x + y + z = 3$$
$$x + 2y + z = 0$$

(a) Write the above system of linear equations in the form $AX=B$,
(b) Find A^{-1} ,if exists ,by using elementary matrix method, and
(c) Use A^{-1} to solve the above system of equations.

Question: 2 . (a)Evaluate det(A) by using row reduction, where

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 2 & 0 & -1 & 3 \\ 0 & 2 & 1 & 4 \\ -2 & -1 & 0 & 1 \end{bmatrix}$$

(b) Find all values of x for which matrix

$$A = \begin{bmatrix} 1 & -1 \\ 1 & x^2 - 2 \end{bmatrix} \text{ is invertible.}$$

Question: 3 . Solve the linear system by using Crammer's Rule

$$3x_1 + 5x_2 \qquad = 7$$
$$6x_1 + 2x_2 + 4x_3 = 10$$
$$-x_1 + 4x_2 - 3x_3 = 0$$

Cramer's Rule

Using determinants to solve a system of linear equations.

Theorem:

If A is $n \times n$ matrix with det(A) ≠ 0, then the linear system $AX = B$ has a *__unique__ solution $X = (x_j)$ given by*

$$x_j = \frac{\det(A_j)}{\det(A)} \quad , j = 1,2,\dots,n$$

Where A_j is the matrix obtained by replacing the jth column of A by B.

__NOTE:__ If A is 3x3 matrix , then the solution of the system AX = B is

$$x = \frac{\det(A_1)}{\det(A)}, \quad y = \frac{\det(A_2)}{\det(A)}, \quad z = \frac{\det(A_3)}{\det(A)}$$

Example 4.
Use Cramer's Rule to solve

$$\begin{aligned} 4x + 5y \quad &= 2 \\ 11x + y + 2z &= 3 \\ x + 5y + 2z &= 1 \end{aligned}$$

Solution:

$$\mathbf{A} = \begin{bmatrix} 4 & 5 & 0 \\ 11 & 1 & 2 \\ 1 & 5 & 2 \end{bmatrix}, \quad \mathbf{A_1} = \begin{bmatrix} 2 & 5 & 0 \\ 3 & 1 & 2 \\ 1 & 5 & 2 \end{bmatrix}, \quad \mathbf{A_2} = \begin{bmatrix} 4 & 2 & 0 \\ 11 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix}, \quad \mathbf{A_3} = \begin{bmatrix} 4 & 5 & 2 \\ 11 & 1 & 3 \\ 1 & 5 & 1 \end{bmatrix}$$

det(A) = -132, det(A_1) =-36, det(A_2) = -24, det(A_3) = 12

$$x = \frac{\det(A_1)}{\det(A)} = \frac{-36}{-132} = \frac{3}{11},$$
$$y = \frac{\det(A_2)}{\det(A)} = \frac{-24}{-132} = \frac{2}{11},$$
$$z = \frac{\det(A_{31})}{\det(A)} = \frac{12}{-132} = \frac{-1}{11}$$

NOTE: If det(A) = 0, then there does not exist any solution of the system.

$$ Some questions from previous exams$$

Question: 1 (a) Let

$$A = \begin{bmatrix} x+3y+z & 2x+3y+4z \\ 4x+3y+5z & 8 \end{bmatrix} \text{ and } B = \begin{bmatrix} -y-z & 8-x-y-2z \\ -x-y-z & 3x+4y+6z \end{bmatrix}$$

Use Gauss Jordan method to find x, y *and* z **such that A and B are equal.** **[**

Solution: **As $A = B$,**

we can write the system of equations by using equality of matrices

$$x+3y+z=-y-z$$
$$2x+3y+4z=8-x-y-2z$$
$$4x+3y+5z=-x-y-z$$
$$8=3x+4y+6z$$

rearranging both the sides

$$x+4y+2z=0$$
$$3x+4y+6z=8$$
$$5x+4y+6z=0$$
$$3x+4y+6z=8$$

Equation 2 and equation 4 are same so the sytem is reduced to three equations

$$x+4y+2z=0$$
$$3x+4y+6z=8$$
$$5x+4y+6z=0$$

Solving system by Gauss Jorden method, rewritting the system in Augmented form and reducing the augmented matrix to reduced row echlon form

$$\begin{bmatrix} 1 & 4 & 2 & 0 \\ 3 & 4 & 6 & 8 \\ 5 & 4 & 6 & 0 \end{bmatrix} \xrightarrow{\text{performing row operations}} \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

Solution of the system is $x=-4, y=-1$ and $z=4$.

1 (b) Find the values of λ for which the system of linear equations

$$2x+y=5$$
$$x-3y=-1 \qquad \text{will have solutions and find the solutions.}$$
$$3x+4y=\lambda$$

Solution:

$$2x+y=5$$
$$x-3y=-1$$
$$3x+4y=\lambda$$

Augmented form of the system is

$$\begin{bmatrix} 2 & 1 & 5 \\ 1 & -3 & -1 \\ 3 & 4 & \lambda \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & -3 & -1 \\ 2 & 1 & 5 \\ 3 & 4 & \lambda \end{bmatrix} \xrightarrow{R_2-2R_1, R_3-3R_1} \begin{bmatrix} 1 & -3 & -1 \\ 0 & 7 & 7 \\ 0 & 13 & \lambda+3 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{7}R_2, R_3-13R_2, R_1+3R_2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & \lambda-10 \end{bmatrix}$$

The system will have solution if and only if $\lambda\text{-}10=0$ or $\lambda=10$

Sytem is reduced to the form

$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$, hence the solutionis $x=2$ *and* $y=1$

2(b) Let $A=\begin{bmatrix} a^2 & 1 \\ 4a & 16 \end{bmatrix}$ **, find all values of a for which A is not invertible.**

[4]

Solution:

If matrix A is not invertible then detA=0

detA= $16a^2-4a=0$

$$4a(4a-1)=0 \Rightarrow \quad a=\ 0 \quad or \quad a\ =\frac{1}{4}$$