Engineering Probability & Statistics (AGE 1150)
Chapter 2: Probability – Part 2

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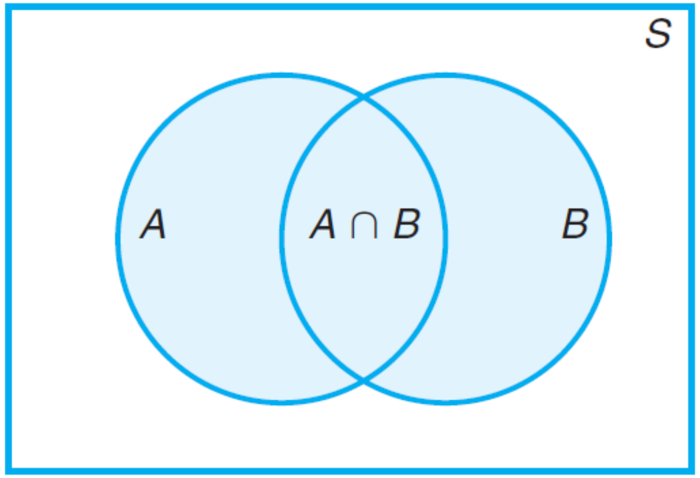
Additive Rules



Theorem 2.10:

- If A and B are any two events, then:

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Corollary 1:

- If A and B are mutually exclusive (disjoint) events (i.e. $P(A \cap B) = 0$, then:

$P(A \cup B) = P(A) + P(B)$

Corollary 2:

- If $A_1, A_2, \ldots, A_n$ are n mutually exclusive (disjoint) events, then:

$P(A_1 \cup A_2 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n)$

Example

John is going to graduate from an industrial engineering department in a university by the end of the semester. After being interviewed at two companies he likes, he assesses that his probability of getting an offer from company *A* is 0.8, and his probability of getting an offer from company *B* is 0.6. If he believes that the probability that he will get offers from both companies is 0.5, what is the probability that he will get at least one offer from these two companies?

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.8 + 0.6 - 0.5 = 0.9.$$

Example

What is the probability of getting a total of 7 or 11 when a pair of fair dice is tossed?

- Let *A* be the event that 7 occurs and *B* the event that 11 comes up. Now, a total of 7 occurs for 6 of the 36 sample points, and a total of 11 occurs for only 2 of the sample points. Since all sample points are equally likely, we have $P(A) = 1/6$ and $P(B) = 1/18$.
- The events *A* and *B* are mutually exclusive, since a total of 7 and 11 cannot both occur on the same toss. Therefore,

$$P(A \cup B) = P(A) + P(B) = \frac{1}{6} + \frac{1}{18} = \frac{2}{9}$$

- This result could also have been obtained by counting the total number of points for the event $A \cup B$, namely 8, and writing

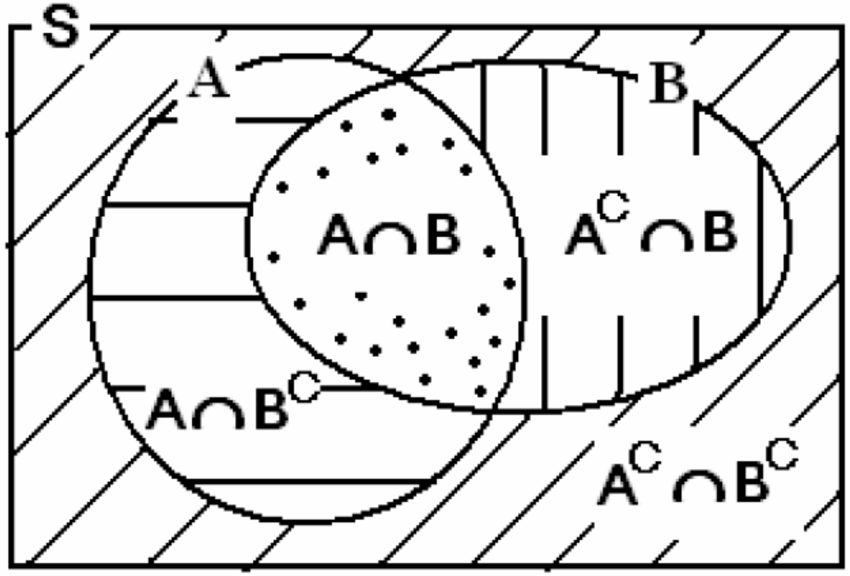
$$P(A \cup B) = \frac{n}{N} = \frac{8}{36} = \frac{2}{9}$$

Two Events Problems

- In Venn diagrams, consider the probability of an event *A* as the area of the region corresponding to the event *A*.
- Total area= $P(S) = 1$

Examples:

- $P(A) = P(A \cap B) + P(A \cap B^C)$
- $P(A \cup B) = P(A) + P(A^C \cap B)$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- $P(A \cap B^C) = P(A) - P(A \cap B)$
- $P(A^C \cap B^C) = 1 - P(A \cup B)$



Example

The probability that Paula passes Mathematics is 2/3, and the probability that she passes English is 4/9. If the probability that she passes both courses is 1/4, what is the probability that she will:

(a) pass at least one course?

(b) pass Mathematics and fail English?

(c) fail both courses?

- **Solution:**
- Define the events: M = {Paula passes Mathematics} , E = {Paula passes English}
- It is given that $P(M)=2/3$, $P(E)=4/9$, and $P(M\cap E)=1/4$.
- (a) Probability of passing at least one course is:
- $P(M\cup E)= P(M) + P(E) - P(M\cap E)$

$$= \frac{2}{3} + \frac{4}{9} - \frac{1}{4} = \frac{31}{36}$$

- (b) Probability of passing Mathematics and failing English is:
- $P(M\cap E^C)= P(M) - P(M\cap E)$

$$= \frac{2}{3} - \frac{1}{4} = \frac{5}{12}$$

- (c) Probability of failing both courses is:

P(MC∩EC)= 1 − P(M∪E)

$$= 1 - \frac{31}{36} = \frac{5}{36}$$

- **Theorem :**

If A and A^C are complementary events, then:

$P(A) + P(A^C) = 1 \Leftrightarrow P(A^C) = 1 - P(A)$

Example

If the probabilities that an automobile mechanic will service 3, 4, 5, 6, 7, or 8 or more cars on any given workday are, respectively, 0.12, 0.19, 0.28, 0.24, 0.10, and 0.07, what is the probability that he will service at least 5 cars on his next day at work?

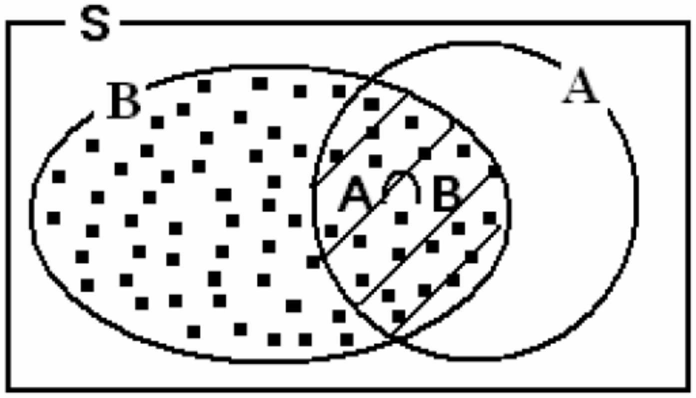
- Let E be the event that at least 5 cars are serviced. Now, $P(E) = 1 - P(E^c)$,

where E^c is the event that fewer than 5 cars are serviced. Since

- $P(E^c) = 0.12 + 0.19 = 0.31$,
- it follows from the above Theorem that:
- $P(E) = 1 - 0.31 = 0.69$.
- *You can also find* $P(E) = 0.28+0.24+0.1+0.07 = 0.69$

Conditional Probability

- The probability of occurring an event *A* when it is known that some event *B* has occurred is called the conditional probability of *A* given *B* and is denoted P(*A*|*B*).



Definition:

- The conditional probability of the event *A*

given the event *B* is defined by

$$P(A\mid B)=\frac{P(A\cap B)}{P(B)} \qquad ;\ P(B)>0$$

Notes:

1. $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)} =$

$$= \frac{n(A \cap B)/n(S)}{n(B)/n(S)} = \frac{n(A \cap B)}{n(B)}; \textit{ for equally likely outcomes case}$$

2. $P(B \mid A) = \dfrac{P(A \cap B)}{P(A)}$

3. $P(A \cap B) = P(A)\,P(B \mid A)$ (Multiplicative Rule=Theorem 2.13)

$$= P(B)\,P(A \mid B)$$

Example

- 339 physicians are classified as given in the table below. A physician is to be selected at random.
- (1) Find the probability that:

  (a) the selected physician is aged 40 – 49

  (b) the selected physician smokes occasionally

  (c) the selected physician is aged 40 – 49 and smokes occasionally.
- (2) Find the probability that the selected physician is aged 40 – 49 given that the physician smokes occasionally.

		Smoking Habit			
		Daily (B_1)	Occasionally (B_2)	Not at all (B_3)	Total
Age	20 - 29 (A_1)	31	9	7	47
	30 - 39 (A_2)	110	30	49	189
	40 - 49 (A_3)	29	21	29	79
	50+ (A_4)	6	0	18	24
	Total	176	60	103	339

- .$n(S)$ = 339 equally likely outcomes.
- Define the following events:
- A_3 = the selected physician is aged 40 – 49
- B_2 = the selected physician smokes occasionally
- $A_3 \cap B_2$ = the selected physician is aged 40 – 49 and smokes occasionally

(1) (a) A_3 = the selected physician is aged 40 – 49

$$P(A_3) = \frac{n(A_3)}{n(S)} = \frac{79}{339} = 0.2330$$

(b) B_2 = the selected physician smokes occasionally

$$P(B_2) = \frac{n(B_2)}{n(S)} = \frac{60}{339} = 0.1770$$

(c) $A_3 \cap B_2$ = the selected physician is aged 40 – 49 and smokes occasionally.

$$P(A_3 \cap B_2) = \frac{n(A_3 \cap B_2)}{n(S)} = \frac{21}{339} = 0.06195$$

(2) $A_3 \mid B_2$ = the selected physician is aged 40 – 49 given that the physician smokes occasionally.

(i) $P(A_3 \mid B_2) = \dfrac{P(A_3 \cap B_2)}{P(B_2)} = \dfrac{0.06195}{0.1770} = 0.35$

(ii) $P(A_3 \mid B_2) = \dfrac{n(A_3 \cap B_2)}{n(B_2)} = \dfrac{21}{60} = 0.35$

(iii) We can use the restricted table directly: $P(A_3 \mid B_2) = \dfrac{21}{60} = 0.35$

Notice that $P(A_3|B_2)=0.35 > P(A_3)=0.233$.
The conditional probability does not equal unconditional probability; i.e., $P(A_3|B_2) \neq P(A_3)$! What does this mean?

Note:

- $P(A|B) = P(A)$ means that knowing B has no effect on the probability of occurrence of A. In this case A is independent of B.
- $P(A|B)>P(A)$ means that knowing B increases the probability of occurrence of A.
- $P(A|B)<P(A)$ means that knowing B decreases the probability of occurrence of A.

Independent Events:

- **Definition**

Two events A and B are independent if and only if $P(A|B)=P(A)$ and $P(B|A)=P(B)$. Otherwise A and B are dependent.

In the previous example, we found that $P(A_3|B_2) \neq P(A_3)$. Therefore, the events A_3 and B_2 are dependent, i.e., they are not independent. Also, we can verify that $P(B_2|A_3) \neq P(B_2)$.

Multiplicative (or Product) Rule

Theorem:

- If $P(A) \neq 0$ and $P(B) \neq 0$, then:

$$P(A \cap B) = P(A)\,P(B|A)$$
$$= P(B)\,P(A|B)$$

Example

Suppose we have a fuse box containing 20 fuses of which 5 are defective (D) and 15 are non-defective (N). If 2 fuses are selected at random and removed from the box in succession without replacing the first, what is the probability that both fuses are defective?

Solution:

- Define the following events:
- A = {the first fuse is defective}
- B = {the second fuse is defective}
- $A \cap B$={the first fuse is defective and the second fuse is defective}

  = {both fuses are defective}

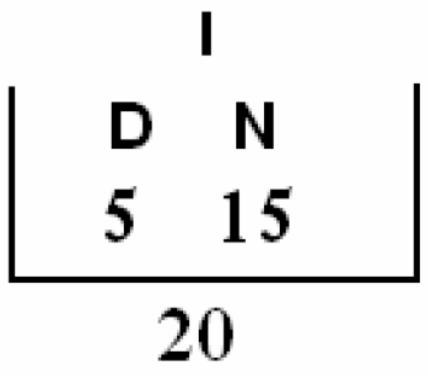
We need to calculate P($A \cap B$).

$$P(A) = \frac{5}{20}$$

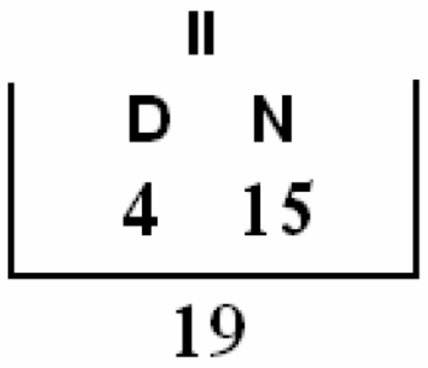
$$P(B|A) = \frac{4}{19}$$

$$P(A \cap B) = P(A)\, P(B|A)$$

$$= \frac{5}{20} \times \frac{4}{19} = 0.052632$$



First Selection



Second Selection: given that the first is defective (D)

- **Theorem:**
- Two events A and B are independent if and only if
- $P(A \cap B) = P(A)\, P(B)$

*(Multiplicative Rule for independent events)

- **Note:**
- Two events A and B are independent if one of the following conditions is satisfied:

(i) $P(A|B)=P(A)$

$\Leftrightarrow$ (ii) $P(B|A)=P(B)$

$\Leftrightarrow$ (iii) $P(A \cap B) = P(A)\, P(B)$

- **Theorem:** (k=3)
- If A_1, A_2, A_3 are 3 events, then:

$P(A_1 \cap A_2 \cap A_3) = P(A_1)\, P(A_2 \mid A_1)\, P(A_3 \mid A_1 \cap A_2)$

- If A_1, A_2, A_3 are 3 independent events, then:
- $P(A_1 \cap A_2 \cap A_3) = P(A_1)\, P(A_2)\, P(A_3)$

Example

Three cards are drawn in succession, without replacement, from an ordinary deck of playing cards (Number of cards = 52).

Find P($A_1 \cap A_2 \cap A_3$), where the events A_1, A_2, and A_3 are defined as follows:

- A_1 = {the 1-st card is a red ace}
- A_2 = {the 2-nd card is a 10 or a jack}
- A_3 = {the 3-rd card is a number greater than 3 but less than 7}

$$P(A_1) = 2/52$$

$$P(A_2 \mid A_1) = 8/51$$

$$P(A_3 \mid A_1 \cap A_2) = 12/50$$

$$\begin{aligned} P(A_1 \cap A_2 \cap A_3) &= P(A_1)\, P(A_2 \mid A_1)\, P(A_3 \mid A_1 \cap A_2) \\ &= \frac{2}{52} \times \frac{8}{51} \times \frac{12}{50} \\ &= \frac{192}{132600} \\ &= 0.0014479 \end{aligned}$$