Engineering Probability & Statistics (AGE 1150)
Chapter 2: Probability – Part 1

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Sample Space (S)

- **Experiment:** is some procedure (or process) that we do and it results in an outcome.

Definition 2.1:

- The set of all possible outcomes of a statistical experiment is called the **sample space** and is denoted by *S*.
- Each outcome (element or member) of the sample space *S* is called a **sample point**.
- Example: the sample space S, of possible outcomes when a coin is flipped, may be written

S = {H, T},
where *H* and *T* correspond to heads and tails, respectively

Events

Definition 2.2:

An event A is a subset of the sample space S. That is $A \subseteq S$.

- We say that an event A occurs if the outcome (the result) of the experiment is an element of A.
- $\phi \subseteq S$ is an event (ϕ is called the impossible event)
- $S \subseteq S$ is an event (S is called the sure event)

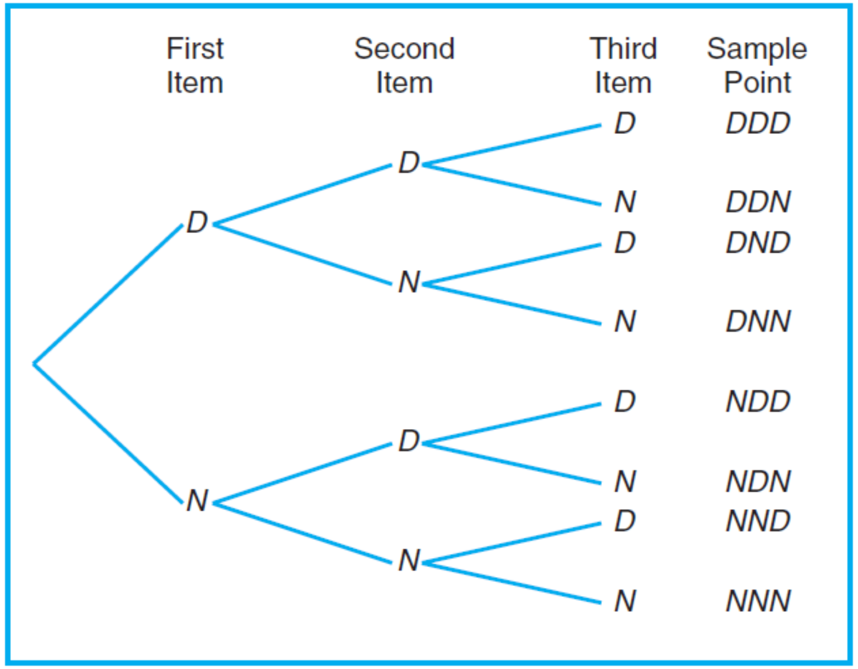
Example

- Experiment: Tossing a die, or Selecting a ball from a box containing 6 balls numbered 1,2,3,4,5 and 6.
- This experiment has 6 possible outcomes
- The sample space is S ={1,2,3,4,5,6}.
- Consider the following events:

  - E_1= getting an even number ={2,4,6}$\subseteq S$
  - E_2 = getting a number less than 4={1,2,3}$\subseteq S$
  - E_3 = getting 1 or 3={1,3}$\subseteq S$
  - E_4 = getting an odd number={1,3,5}$\subseteq S$
  - E_5 = getting a negative number={ }=φ $\subseteq S$
  - E_6 = getting a number less than 10 = {1,2,3,4,5,6}=$S\subseteq S$
- *n(S)= no. of outcomes (elements) in S. in the above example = 6*
- *n(E)= no. of outcomes (elements) in the event E. for* $E_1 = 3$, $E_5 = 0$, *and* $E_3 = 2$,

Example

- Experiment: Selecting 3 items from manufacturing process; each item is inspected and classified as defective (D) or non-defective (N).
- Draw tree diagram of the possible outcomes as



- This experiment has 8 possible outcomes

S={DDD,DDN,DND,DNN,NDD,NDN,NND,NNN}

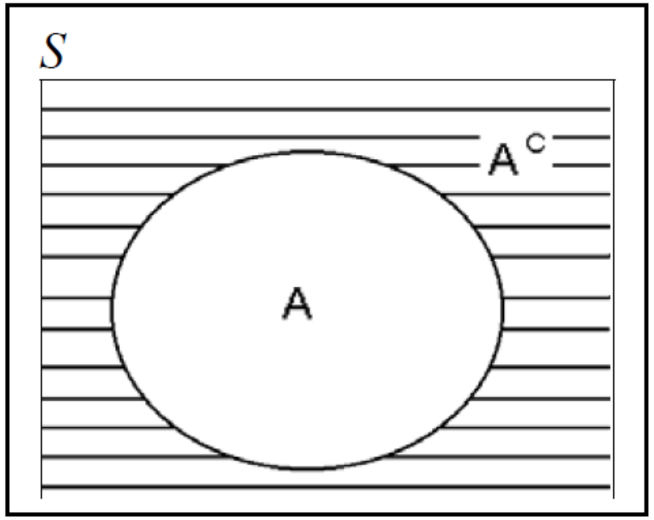
- Consider the following events:
  - A={at least 2 defectives}= {DDD,DDN,DND,NDD}⊆S
  - B={at most one defective}={DNN,NDN,NND,NNN}⊆S
  - C={3 defectives}={DDD}⊆S

Some Operations on Events 1

- Let A and B be two events defined on the sample space S.

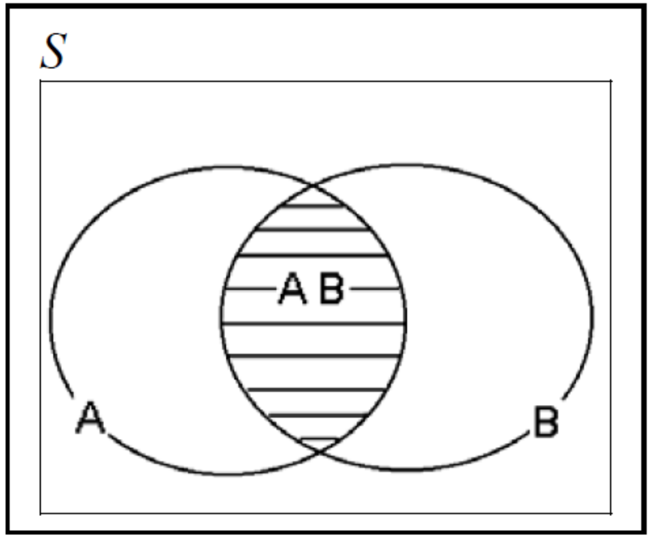
Definition 2.3: **<u>Complement of The Event A</u>**:

- A^c or A' or $\overline{A}$
- $A^c = \{x \in S: x \notin A\}$
- A^c consists of all points of S that are not in A.
- A^c occurs if A does not.



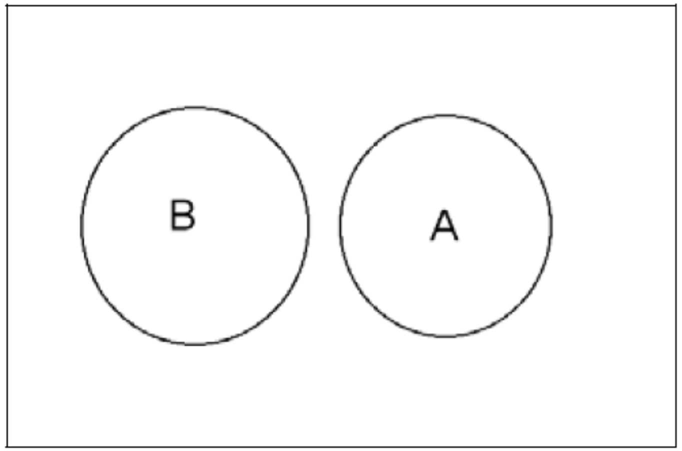
Some Operations on Events 2

- Definition 2.4: **Intersection**:
- $A \cap B = AB = \{x \in S: x \in A \text{ and } x \in B\}$
- A∩B Consists of all points in both A and B.
- A∩B Occurs if both A and B occur together.



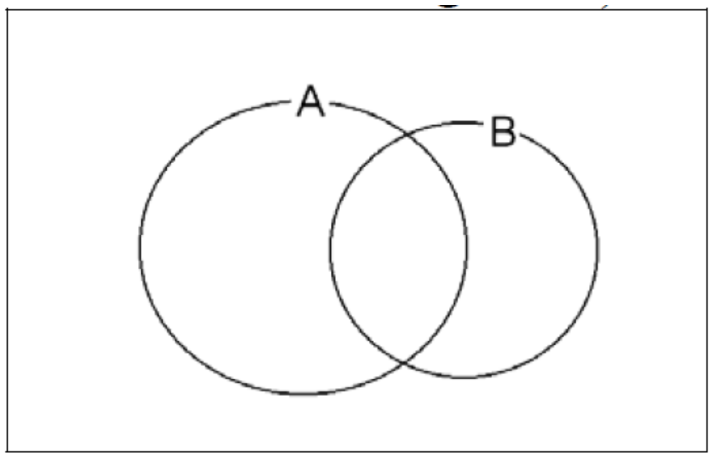
Some Operations on Events 3

- **Definition 2.5: Mutually Exclusive (Disjoint) Events:**
- Two events *A* and *B* are mutually exclusive (or disjoint) if and only if $A \cap B = \phi$; that is, *A* and *B* have no common elements (they do not occur together).



$A \cap B = \phi$

A and *B* are mutually exclusive (disjoint)

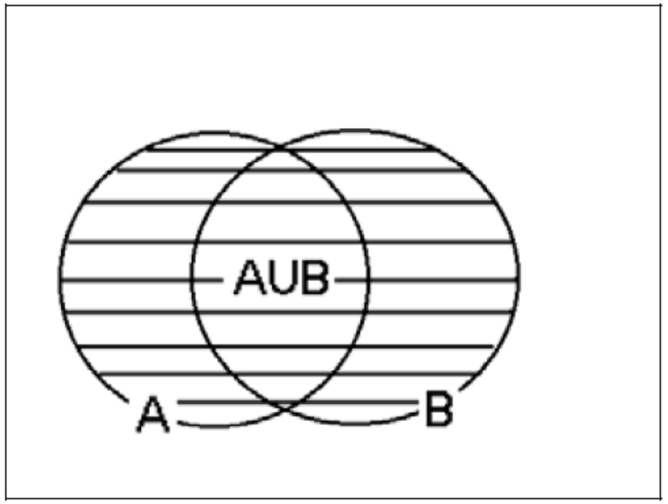


$A \cap B \neq \phi$

A and *B* are not mutually exclusive

Some Operations on Events 4

- **Definition 2.6: <u>Union</u>:**
- $A \cup B = \{x \in S: x \in A \text{ or } x \in B\}$
- $A \cup B$ Consists of all outcomes in *A* **or** in *B* **or** in both *A* and *B*.
- $A \cup B$ Occurs if *A* occurs, **or** *B* occurs, **or** both *A* and *B* occur.
- That is $A \cup B$ Occurs if at least one of *A* and *B* occurs.



Counting Sample Points

- There are many counting techniques which can be used to count the number points in the sample space (or in some events) without listing each element.
- In many cases, we can compute the probability of an event by using the counting techniques.
- **multiplication rule:** If an operation can be performed in n_1 ways, and if for each of these ways a second operation can be performed in n_2 ways, then the two operations can be performed together in n_1n_2 ways.

Counting Sample Points

Example: How many sample points are there in the sample space when a pair of dice is thrown once?

- ***Solution* :** The first die can land face-up in any one of $n_1 = 6$ ways. For each of these 6 ways, the second die can also land face-up in $n_2 = 6$ ways. Therefore, the pair of dice can land in $n_1 n_2 = (6)(6) = 36$ possible ways.

If an operation can be performed in n_1 ways, and if for each of these a second operation can be performed in n_2 ways, and for each of the first two a third operation can be performed in n_3 ways, and so forth, then the sequence of k operations can be performed in $\mathbf{n_1 n_2 \cdots n_k}$ ways.

Counting Sample Points

Combinations:

- In many problems, we are interested in the number of ways of selecting *r* objects from *n* objects without regard to order. These selections are called combinations.
- Notation:
- *n* factorial is denoted by $n!$ and is defined by:

$$n! = n \times (n-1) \times (n-2) \times \cdots \times (2) \times (1) \qquad \textit{for} \quad n = 1, 2, \cdots$$
$$0! = 1$$

Example: $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

Theorem 2.8:

- The number of combinations of *n* distinct objects taken *r* at a time is denoted by $\binom{n}{r}$ and is given by:

$$\binom{n}{r} = \frac{n!}{r!\ (n-r)!}; \qquad r = 0, 1, 2, \ldots, n$$

- And it is read as “ *n* ” choose “ *r* ”.
- The number of different ways of selecting *r* objects from *n* distinct objects.
- The number of different ways of dividing n distinct objects into two subsets; one subset contains r objects and the other contains the rest (n−r) objects.

Example

- If we have 10 equal–priority operations and only 4 operating rooms are available, in how many ways can we choose the 4 patients to be operated on first?

n = 10 r = 4

The number of different ways for selecting 4 patients from 10 patients is

$$\binom{10}{4} = \frac{10!}{4!\,(10-4)!} = \frac{10!}{4! \times 6!} = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{(4 \times 3 \times 2 \times 1) \times (6 \times 5 \times 4 \times 3 \times 2 \times 1)}$$

$$= 210 \qquad (different\ ways)$$

Probability of an Event

- To every point (outcome) in the sample space of an experiment *S*, we assign a weight (or probability), ranging from 0 to 1, such that the sum of all weights (probabilities) equals 1.
- The weight (or probability) of an outcome measures its likelihood (chance) of occurrence.
- To find the probability of an event *A*, we sum all probabilities of the sample points in *A*. This sum is called the probability of the event *A* and is denoted by P(*A*).
- **Definition 2.8:**
- The probability of an event *A* is the sum of the weights (probabilities) of all sample points in *A*. Therefore,

1. $0 \leq P(A) \leq 1$
2. $P(S) = 1$
3. $P(\phi) = 0$

Example

A balanced coin is tossed twice. What is the probability that at least one head occurs?

Solution: H = Head, T = Tail.

- S = {HH, HT, TH, TT}
- A = {at least one head occurs}= {HH, HT, TH}
- Since the coin is balanced, the outcomes are equally likely; i.e., all outcomes have the same weight or probability.

Outcome	Weight (Probability)
HH	P(HH) = w
HT	P(HT) = w
TH	P(TH) = w
TT	P(TT) = w
sum	4w=1

$4w = 1 \Leftrightarrow w = 1/4 = 0.25$

P(HH)=P(HT)=P(TH)=P(TT)=0.25

The probability that at least one head occurs is:

P(A) = P({at least one head occurs})=P({HH, HT, TH})

= P(HH) + P(HT) + P(TH)

= 0.25+0.25+0.25

= 0.75

Theorem 2.9:

- If an experiment has $n(S)=N$ equally likely different outcomes, then the probability of the event A is:

$$P(A)=\frac{n(A)}{n(S)}=\frac{n(A)}{N}=\frac{no.\ of\ outcomes\ \ in\ \ A}{no.\ of\ outcomes\ \ in\ \ S}$$

Example

A mixture of candies consists of 6 mints, 4 toffees, and 3 chocolates. If a person makes a random selection of one of these candies, find the probability of getting:

(a) a mint

(b) a toffee or chocolate.

- Define the following events:
- M = {getting a mint}
- T = {getting a toffee}
- C = {getting a chocolate}
- Experiment: selecting a candy at random from 13 candies,

$n(S)$ = no. of outcomes of the experiment of selecting a candy.

= no. of different ways of selecting a candy from 13 candies.

$= \binom{13}{1} = 13$

Cont.

- The outcomes of the experiment are equally likely because the selection is made at random.

(a) M = {getting a mint}

$n(M)$ = no. of different ways of selecting a mint candy from 6 mint candies = $\binom{6}{1}$ = 6

P(M)= P({getting a mint})= $\frac{n(M)}{n(S)} = \frac{6}{13}$

(b) $T \cup C$ = {getting a toffee or chocolate}

$n(T \cup C)$ = no. of different ways of selecting a toffee **or** a chocolate candy

= no. of different ways of selecting a toffee candy **+** no. of different ways of selecting a chocolate candy

$= \binom{4}{1} + \binom{3}{1} = 4 + 3 = 7$ ⇒ $P(T \cup C) = P(\{\text{getting a toffee or chocolate}\}) = \frac{n(T \cup C)}{n(S)} = \frac{7}{13}$