Chapter 2:

Inverse of a Matrix

DEFINITION 1 If A is a square matrix, and if a matrix B of the same size can be found such that $AB = BA = I$, then A is said to be ***invertible*** (or ***nonsingular***) and B is called an ***inverse*** of A. If no such matrix B can be found, then A is said to be ***singular***.

Remark The relationship $AB = BA = I$ is not changed by interchanging A and B, so if A is invertible and B is an inverse of A, then it is also true that B is invertible, and A is an inverse of B. Thus, when

$$AB = BA = I$$

we say that A and B are *inverses of one another*.

▶ EXAMPLE 5 An Invertible Matrix

Let

$$A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$$

Then

$$AB = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$BA = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Thus, A and B are invertible and each is an inverse of the other.

▶ EXAMPLE 6 A Class of Singular Matrices

A square matrix with a row or column of zeros is singular. To help understand why this is so, consider the matrix

$$A = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix}$$

To prove that A is singular we must show that there is no 3×3 matrix B such that $AB = BA = I$. For this purpose let $\mathbf{c}_1, \mathbf{c}_2, \mathbf{0}$ be the column vectors of A. Thus, for any 3×3 matrix B we can express the product BA as

$$BA = B[\mathbf{c}_1 \quad \mathbf{c}_2 \quad \mathbf{0}] = [B\mathbf{c}_1 \quad B\mathbf{c}_2 \quad \mathbf{0}] \quad \textbf{[Formula (6) of Section 1.3]}$$

The column of zeros shows that $BA \neq I$ and hence that A is singular. ◀

THEOREM 1.4.4 *If B and C are both inverses of the matrix A, then $B = C$.*

Proof Since B is an inverse of A, we have $BA = I$. Multiplying both sides on the right by C gives $(BA)C = IC = C$. But it is also true that $(BA)C = B(AC) = BI = B$, so $C = B$. ◀

THEOREM 1.4.5 *The matrix*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if $ad - bc \neq 0$, in which case the inverse is given by the formula

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{2}$$

We will omit the proof, because we will study a more general version of this theorem later. For now, you should at least confirm the validity of Formula (2) by showing that $AA^{-1} = A^{-1}A = I$.

Remark Figure 1.4.1 illustrates that the determinant of a 2×2 matrix A is the product of the entries on its main diagonal minus the product of the entries *off* its main diagonal.

▶ EXAMPLE 7 Calculating the Inverse of a 2 × 2 Matrix

In each part, determine whether the matrix is invertible. If so, find its inverse.

$$\text{(a) } A = \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix} \qquad \text{(b) } A = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}$$

Solution (a) The determinant of A is $\det(A) = (6)(2) - (1)(5) = 7$, which is nonzero. Thus, A is invertible, and its inverse is

$$A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & -1 \\ -5 & 6 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{5}{7} & \frac{6}{7} \end{bmatrix}$$

We leave it for you to confirm that $AA^{-1} = A^{-1}A = I$.

Solution (b) The matrix is not invertible since $\det(A) = (-1)(-6) - (2)(3) = 0$.

THEOREM 1.4.6 *If A and B are invertible matrices with the same size, then AB is invertible and*

$$(AB)^{-1} = B^{-1}A^{-1}$$

Proof We can establish the invertibility and obtain the stated formula at the same time by showing that

$$(AB)(B^{-1}A^{-1}) = (B^{-1}A^{-1})(AB) = I$$

But

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

and similarly, $(B^{-1}A^{-1})(AB) = I$. ◀

Although we will not prove it, this result can be extended to three or more factors:

A product of any number of invertible matrices is invertible, and the inverse of the product is the product of the inverses in the reverse order.

▶ EXAMPLE 9 The Inverse of a Product

Consider the matrices

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$$

We leave it for you to show that

$$AB = \begin{bmatrix} 7 & 6 \\ 9 & 8 \end{bmatrix}, \quad (AB)^{-1} = \begin{bmatrix} 4 & -3 \\ -\frac{9}{2} & \frac{7}{2} \end{bmatrix}$$

and also that

$$A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & \frac{3}{2} \end{bmatrix}, \quad B^{-1}A^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & \frac{3}{2} \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -\frac{9}{2} & \frac{7}{2} \end{bmatrix}$$

Thus, $(AB)^{-1} = B^{-1}A^{-1}$ as guaranteed by Theorem 1.4.6. ◀

Powers of a Matrix

If A is a *square* matrix, then we define the nonnegative integer powers of A to be

$$A^0 = I \quad \text{and} \quad A^n = AA \cdots A \quad [n \text{ factors}]$$

and if A is invertible, then we define the negative integer powers of A to be

$$A^{-n} = (A^{-1})^n = A^{-1}A^{-1} \cdots A^{-1} \quad [n \text{ factors}]$$

THEOREM 1.4.7 *If A is invertible and n is a nonnegative integer, then:*

(a) *A^{-1} is invertible and $(A^{-1})^{-1} = A$.*

(b) *A^n is invertible and $(A^n)^{-1} = A^{-n} = (A^{-1})^n$.*

(c) *kA is invertible for any nonzero scalar k, and $(kA)^{-1} = k^{-1}A^{-1}$.*

▶ EXAMPLE 10 Properties of Exponents

Let A and A^{-1} be the matrices in Example 9; that is,

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

▶ EXAMPLE 10 Properties of Exponents

Let A and A^{-1} be the matrices in Example 9; that is,

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

Then

$$A^{-3} = (A^{-1})^3 = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix}$$

Also,

$$A^3 = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 30 \\ 15 & 41 \end{bmatrix}$$

so, as expected from Theorem 1.4.7(b),

$$(A^3)^{-1} = \frac{1}{(11)(41) - (30)(15)}\begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix} = \begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix} = (A^{-1})^3$$

REMARK:

$$(A + B)^2 = A^2 + AB + BA + B^2$$

It is only in the special case where A and B *commute* (i.e., $AB = BA$) that we can go a step further and write

$$(A + B)^2 = A^2 + 2AB + B^2 \quad ◀$$

Challenge: Can you give a subclass of matrices where the product is commutative?

▶ EXAMPLE 12 A Matrix Polynomial

Find $p(A)$ for

$$p(x) = x^2 - 2x - 3 \quad \text{and} \quad A = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}$$

Solution

$$\begin{aligned} p(A) &= A^2 - 2A - 3I \\ &= \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}^2 - 2\begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 4 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} -2 & 4 \\ 0 & 6 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

or more briefly, $p(A) = 0$. ◀

THEOREM 1.4.9 *If A is an invertible matrix, then A^T is also invertible and*

$$(A^T)^{-1} = (A^{-1})^T$$

Proof We can establish the invertibility and obtain the formula at the same time by showing that

$$A^T(A^{-1})^T = (A^{-1})^T A^T = I$$

But from part (*e*) of Theorem 1.4.8 and the fact that $I^T = I$, we have

$$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$$
$$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$$

which completes the proof. ◀

EXCERSICES:

(a) Give an example of two 2 × 2 matrices such that

$$(A + B)(A - B) \neq A^2 - B^2$$

(b) State a valid formula for multiplying out

$$(A + B)(A - B)$$

(c) What condition can you impose on A and B that will allo\
you to write $(A + B)(A - B) = A^2 - B^2$?

45. (a) Show that if A, B, and $A + B$ are invertible matrices with the same size, then

$$A(A^{-1} + B^{-1})B(A + B)^{-1} = I$$

(b) What does the result in part (a) tell you about the matrix $A^{-1} + B^{-1}$?

46. A square matrix A is said to be ***idempotent*** if $A^2 = A$.

(a) Show that if A is idempotent, then so is $I - A$.

(b) Show that if A is idempotent, then $2A - I$ is invertible and is its own inverse.

3.3.2 Method for finding Inverse of a matrix

To find the inverse of an invertible matrix, we must find a sequence of elementary row operations that reduces A to the identity and then perform this same sequence of operations on I_n to obtain A^{-1}.

$$[A \mid I] \text{ to } [I \mid A^{-1}]$$

Example:2. Find inverse of a matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix}$ by using Elementary matrix method.

Solution: $$[A|I] = \left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right]$$

$$\approx \left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right] \quad -2R_1 + R_2$$

$$\approx \left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & 1 & 2 & -1 \end{array}\right] \quad -R_2$$

$$\approx \left[\begin{array}{cc|cc} 1 & 0 & -7 & 4 \\ 0 & 1 & 2 & -1 \end{array}\right] \quad -4R_2 + R_1$$

$$= [I|A^{-1}]$$

$$A^{-1} = \begin{bmatrix} -7 & 4 \\ 2 & -1 \end{bmatrix}$$

Example:3. Use Elementary matrix method to find inverses of

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & 5 & -4 \end{bmatrix} \quad \text{if A is invertible.}$$

Solution:

$$[A|I] = \left[\begin{array}{ccc|ccc} 3 & 4 & -1 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{array}\right]$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 3 & 4 & -1 & 1 & 0 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{array}\right] R_1 \leftrightarrow R_2$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 4 & -10 & 1 & -3 & 0 \\ 0 & 5 & -10 & 0 & -2 & 1 \end{array}\right] -3R_1 + R_2, -2R_1 + R_3$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 4 & -10 & 1 & -3 & 0 \\ 0 & 1 & 0 & -1 & -2 & 1 \end{array}\right] -R_2 + R_3$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & \frac{1}{2} & \frac{-7}{10} & \frac{-2}{5} \end{array}\right] \quad R_2 \leftrightarrow R_3, \frac{(-4R_3 + R_2)}{10}$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{2} & \frac{-11}{10} & \frac{-6}{5} \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & \frac{-1}{2} & \frac{7}{10} & \frac{2}{5} \end{array}\right] \quad -3R_3 + R_1, -R_3$$

$$\approx \left[I \middle| A^{-1} \right]$$

$$A^{-1} = \begin{bmatrix} \frac{3}{2} & \frac{-11}{10} & \frac{-6}{5} \\ -1 & 1 & 1 \\ \frac{-1}{2} & \frac{7}{10} & \frac{2}{5} \end{bmatrix}.$$

4.1 Determinant of a matrix

The determinant is a useful value that can be computed from the elements of a square matrix. The **determinant** of a matrix A is denoted det(A), det A, or | A|.

4.2 Evaluation of determinant of Matrix

1.The determinant of a (1×1) matrix $A = [\ a\ ]$ is just det $A = a$.

2. The determinant of 2x2 matrix is defined as

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$|A| = \det A = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - cb$$

Example:1. Find determinant of matrix

$$A = \begin{bmatrix} 4 & 5 \\ 3 & 6 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 4 & 5 \\ 3 & 6 \end{bmatrix}$$

$$\begin{aligned} \det A &= 4x6 - 3x5 \\ &= 24 - 15 \\ &= 9 \end{aligned}$$

4.3 The determinant of 3x3 matrix is defined as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\det A = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

DEFINITION 1 If A is a square matrix, then the ***minor of entry a_{ij}*** is denoted by M_{ij} and is defined to be the determinant of the submatrix that remains after the ith row and jth column are deleted from A. The number $(-1)^{i+j}M_{ij}$ is denoted by C_{ij} and is called the ***cofactor of entry a_{ij}***.

▶ EXAMPLE 1 Finding Minors and Cofactors

Let

$$A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$$

The minor of entry a_{11} is

$$M_{11} = \begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 5 & 6 \\ 4 & 8 \end{vmatrix} = 16$$

The cofactor of a_{11} is

$$C_{11} = (-1)^{1+1}M_{11} = M_{11} = 16$$

Similarly, the minor of entry a_{32} is

$$M_{32} = \begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 3 & -4 \\ 2 & 6 \end{vmatrix} = 26$$

The cofactor of a_{32} is

$$C_{32} = (-1)^{3+2}M_{32} = -M_{32} = -26 \blacktriangleleft$$

Remark Note that a minor M_{ij} and its corresponding cofactor C_{ij} are either the same or negatives of each other and that the relating sign $(-1)^{i+j}$ is either $+1$ or -1 in accordance with the pattern in the "checkerboard" array

$$\begin{bmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

For example,

$$C_{11} = M_{11}, \quad C_{21} = -M_{21}, \quad C_{22} = M_{22}$$

DEFINITION 2 If A is an $n \times n$ matrix, then the number obtained by multiplying the entries in any row or column of A by the corresponding cofactors and adding the resulting products is called the ***determinant of A***, and the sums themselves are called ***cofactor expansions of A***. That is,

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} \quad (7)$$

[cofactor expansion along the jth column]

and

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} \quad (8)$$

[cofactor expansion along the ith row]

Example:2. Find determinant of matrix

$$A = \begin{bmatrix} 2 & 4 & 5 \\ 3 & 6 & 8 \\ 4 & 5 & 9 \end{bmatrix}$$

Solution:

Expanding along the top row and noting alternating signs $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$

$$\begin{aligned} \det A &= +2x\begin{vmatrix} 6 & 8 \\ 5 & 9 \end{vmatrix} - 4x\begin{vmatrix} 3 & 8 \\ 4 & 9 \end{vmatrix} + 5x\begin{vmatrix} 3 & 6 \\ 4 & 5 \end{vmatrix} \\ &= 2x(54-40) - 4x(27-32) + 5x(15-24) \\ &= 2x(14) - 4x(-5) + 5x(-9) \\ &= 28 + 20 - 45 = 48 - 45 = 3 \end{aligned}$$

Note: we can write determinant of a matrix as

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ or } \begin{vmatrix} a & b \\ c & d \end{vmatrix} \text{ or } \det A \text{ or } |A|$$

Example:3.

Find the determinant of $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

Solution:

$$\det A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$\det A = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = +1\times\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\times\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\times\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= 1(45-48) - 2(36-42) + 3(32-35) = -3+12-9 = 0$$

Example4. Find determinant of matrix of order 4x4

$$A = \begin{bmatrix} 0 & 1 & 2 & 5 \\ 2 & -1 & 2 & 3 \\ 3 & 2 & 1 & 5 \\ 1 & 0 & 4 & 0 \end{bmatrix}$$

Solution:

Two entries in 4th row are zero, so determinant is calculated by opening from 4th row.

$$\begin{aligned} \det A &= a_{41}c_{41} + a_{42}c_{42} + a_{43}c_{43} + a_{44}c_{44} \\ &= (1)c_{41} + (0)c_{42} + (4)c_{43} + (0)c_{44} \\ &= c_{41} + (4)c_{43} \end{aligned}$$

$$\det A = c_{41} + (4)c_{43} = -\begin{vmatrix} 1 & 2 & 5 \\ -1 & 2 & 3 \\ 2 & 1 & 5 \end{vmatrix} - 4\begin{vmatrix} 0 & 1 & 5 \\ 2 & -1 & 2 \\ 3 & 2 & 1 \end{vmatrix}$$

Finding values of cofactors c_{41} and c_{43}

$$\begin{aligned} \det A &= -(4) - 4(34) \\ &= -4 - 136 \\ &= -140 \end{aligned}$$

Example:5.

Solving matrix equation

Find all values of λ for which det (A) = 0 for matrix

$$A = \begin{bmatrix} \lambda - 4 & 0 & 0 \\ 0 & \lambda & 2 \\ 0 & 3 & \lambda - 1 \end{bmatrix}$$

Solution: Two entries of 1st row are zero, we open it from first row

$$\det\ A = (\lambda - 4)\begin{vmatrix} \lambda & 2 \\ 3 & \lambda - 1 \end{vmatrix}$$

$$= (\lambda - 4)[\lambda(\lambda - 1) - 6]$$

$$= (\lambda - 4)[\lambda^2 - \lambda - 6]$$

$$= (\lambda - 4)(\lambda - 3)(\lambda + 2)$$

We need to find the value of λ, when det A = 0

$$\Rightarrow\ (\lambda - 4)(\lambda - 3)(\lambda + 2) = 0$$

$\Rightarrow$ λ=4, λ=3 and λ= -2.

▶ **EXAMPLE 5 Smart Choice of Row or Column**

If A is the 4×4 matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$

then to find $\det(A)$ it will be easiest to use cofactor expansion along the second column, since it has the most zeros:

$$\det(A) = 1 \cdot \begin{vmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ 2 & 0 & 1 \end{vmatrix}$$

For the 3×3 determinant, it will be easiest to use cofactor expansion along its second column, since it has the most zeros:

$$\det(A) = 1 \cdot -2 \cdot \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}$$

$$= -2(1 + 2)$$

$$= -6$$

EXAMPLE 7 A Technique for Evaluating 2 × 2 and 3 × 3 Determinants

$$\begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} = \begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} = (3)(-2) - (1)(4) = -10$$

$$\begin{vmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{vmatrix} = \left|\begin{matrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{matrix}\right|\begin{matrix} 1 & 2 \\ -4 & 5 \\ 7 & -8 \end{matrix}$$

$$= [45 + 84 + 96] - [105 - 48 - 72] = 240$$

Upper triangular matrix

In upper triangular matrix all the entries below the diagonal are zero.

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix}$$

Lower triangular matrix

In lower triangular matrix all the entries above the diagonal are zero.

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 8 & 4 & 0 \\ 4 & 7 & 3 \end{bmatrix}$$

Note: Determinant of triangular matrix is product of diagonal elements.

$$\text{Det } A = (1)(4)(5) = 20$$

$$\text{Det } B = (1)(4)(3) = 12$$

EXAMPLE:

$$A = \begin{bmatrix} 2 & 4 & 5 & 3 \\ 0 & 5 & 3 & -1 \\ 0 & 0 & 3 & 9 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

$$\det A = (2)(4)(5)(3) = 120$$

Example:7. Determinant of Diagonal matrix

Find determinant of matrix $B = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Solution:

$$\det B = (5)(4)(3) = 60$$

Example:8.

Evaluate $\det C = \begin{vmatrix} 3 & 0 & 0 & 0 & 0 \\ -4 & 2 & 0 & 0 & 0 \\ 67 & e & 4 & 0 & 0 \\ 0 & 1 & -47 & 2 & 0 \\ \pi & -3 & 6 & -\sqrt{2} & -1 \end{vmatrix}$

Matrix C is lower triangular $\Rightarrow$ $\det C = 3\times2\times4\times2\times(-1) = -48$

Example:9.

Evaluate $\det D = \begin{vmatrix} 2 & -1 & 1 & 1 \\ -3 & 2 & -4 & -3 \\ 4 & 2 & 7 & 4 \\ 2 & 3 & 11 & 2 \end{vmatrix}$

Columns 1 and 4 of matrix D are identical $\Rightarrow$ $\det D = 0$.

2.2 Evaluating Determinants by Row Reduction

THEOREM 2.2.1 *Let A be a square matrix. If A has a row of zeros or a column of zeros, then* $\det(A) = 0$.

THEOREM 2.2.2 *Let A be a square matrix. Then* $\det(A) = \det(A^T)$.

Relationship	Operation
$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $\det(B) = k\det(A)$	In the matrix B the first row of A was multiplied by k.
$\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $\det(B) = -\det(A)$	In the matrix B the first and second rows of A were interchanged.
$\begin{vmatrix} a_{11}+ka_{21} & a_{12}+ka_{22} & a_{13}+ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $\det(B) = \det(A)$	In the matrix B a multiple of the second row of A was added to the first row.

▶ EXAMPLE 3 Using Row Reduction to Evaluate a Determinant

Evaluate $\det(A)$ where

$$A = \begin{bmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{bmatrix}$$

Solution We will reduce A to row echelon form (which is upper triangular) and then apply Theorem 2.1.2.

$$\det(A) = \begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix} = - \begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$$

← The first and second rows of A were interchanged.

$$= -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$$

← A common factor of 3 from the first row was taken through the determinant sign.

$$= -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 10 & -5 \end{vmatrix}$$

← -2 times the first row was added to the third row.

$$= -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -55 \end{vmatrix}$$

← -10 times the second row was added to the third row.

$$= (-3)(-55) \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{vmatrix}$$

← A common factor of -55 from the last row was taken through the determinant sign.

$$= (-3)(-55)(1) = 165$$

EXAMPLE 5 Row Operations and Cofactor Expansion

Evaluate det(A) where

$$A = \begin{bmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{bmatrix}$$

Solution By adding suitable multiples of the second row to the remaining rows, we obtain

$$\det(A) = \begin{vmatrix} 0 & -1 & 1 & 3 \\ 1 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 1 & 8 & 0 \end{vmatrix}$$

$$= -\begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 1 & 8 & 0 \end{vmatrix}$$

← Cofactor expansion along the first column

$$= -\begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 0 & 9 & 3 \end{vmatrix}$$

← We added the first row to the third row.

$$= -(-1)\begin{vmatrix} 3 & 3 \\ 9 & 3 \end{vmatrix}$$

← Cofactor expansion along the first column

$$= -18$$

24. Verify the formulas in parts (a) and (b) and then make a conjecture about a general result of which these results are special cases.

(a) $\det\begin{bmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = -a_{13}a_{22}a_{31}$

(b) $\det\begin{bmatrix} 0 & 0 & 0 & a_{14} \\ 0 & 0 & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = a_{14}a_{23}a_{32}a_{41}$

▶ In Exercises **25–28**, confirm the identities without evaluating the determinants directly. ◀

25. $\begin{vmatrix} a_1 & b_1 & a_1+b_1+c_1 \\ a_2 & b_2 & a_2+b_2+c_2 \\ a_3 & b_3 & a_3+b_3+c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

26. $\begin{vmatrix} a_1+b_1t & a_2+b_2t & a_3+b_3t \\ a_1t+b_1 & a_2t+b_2 & a_3t+b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (1-t^2)\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

27. $\begin{vmatrix} a_1+b_1 & a_1-b_1 & c_1 \\ a_2+b_2 & a_2-b_2 & c_2 \\ a_3+b_3 & a_3-b_3 & c_3 \end{vmatrix} = -2\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

28. $\begin{vmatrix} a_1 & b_1+ta_1 & c_1+rb_1+sa_1 \\ a_2 & b_2+ta_2 & c_2+rb_2+sa_2 \\ a_3 & b_3+ta_3 & c_3+rb_3+sa_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Questions were raised in exams:

2. Compute the following determinant $A = \begin{vmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{vmatrix}$.

(b) **The matrix A satisfies $A^3+4A^2-2A+2I=\underline{0}$. Show that A is invertible.**

Question 1 : [7pts]

1. Let A, B, C and D be matrices of order 3 such that $AB + AC - D = 0$, $|D| = 6$, $B = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ -1 & 1 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -2 & -1 \\ 1 & -1 & 3 \end{pmatrix}$.

   Find $|A|$.

2. Let R and S be matrices of order 3 such that $RS + R - 2I = 0$.

   Find R^{-1} if $S = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 3 & 4 \\ 0 & 2 & 5 \end{pmatrix}$.

I) Choose the correct answer:

(a) If A, B, C are square matrices of the same size, then

$$(A - B)(C - A) + (C - B)(A - C) + (C - A)^2$$

equals

A-B	0	B-C	C-B

(b) If A and B are 3×3 invertible square matrices and

$$\det[2A^{-1}] = \det[A^3(B^{-1})^T] = -4,$$

then

det(A)=4 det(B)=4	det(A)=-4 det(B)=4	det(A)=2 det(B)=-2	det(A)=-2 det(B)=2

(c) If $A^3 - 2B^T = \begin{bmatrix} 18 & -2 \\ -6 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -5 & 3 \\ 1 & 0 \end{bmatrix}$, then the matrix A is

$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -4 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$

II) Decide if the following statements are true (T) or false (F). Justify your answer.

(a) If A and B are two matrices, such that $A \cdot B = O$, then either $A = O$ or $B = O$.

(b) If A and B are square matrices of the same size, such that $A + B$ is symmetric, then both A and B are symmetric.

(c) If A is a $n \times n$ square matrix, $n > 1$ and $k \in \mathbb{R}$, $k \neq 0$, $k \neq \pm 1$, then $\det[kA] = k \cdot \det[A]$.

I) Choose the correct answer (write it on the table above):

1) If $A^3 - 2B^T = \begin{bmatrix} 18 & -2 \\ -6 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -5 & 3 \\ 1 & 0 \end{bmatrix}$, then the matrix A is

(A) $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$	(B) $A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$	(C) $A = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$	(D) None

2) If $A^T = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ and $p(x) = x^2 - x + 3$, then $p(A)$ equals

(A) $\begin{bmatrix} 5 & 3 \\ 6 & 11 \end{bmatrix}$	(B) $\begin{bmatrix} 5 & 11 \\ 3 & 6 \end{bmatrix}$	(C) $\begin{bmatrix} 5 & 6 \\ 3 & 11 \end{bmatrix}$	(D) None

3) The values of x and y for which the matrix $\begin{bmatrix} x^2 & 0 & x^2 - 4 \\ -1 & 3 & 2y - 6 \\ 1 & 7 & 2x - 5y \end{bmatrix}$ is lower triangular are

(A) $x = 2, y = 3$	(B) $x = \pm 2, y = 3$	(C) $x = \pm 2, y = \pm 3$	(D) None

Lecture 6.2 : Inverse by method of Cofactors

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \det A \neq 0.$$

Step:1. Find Matrix of cofactors

$$\mathbf{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

Step : 2. Find Adjoint of matrix A , adj(A)

$$\mathbf{Adj(A)} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

Step: 3.

If A is an invertible matrix, det(A) $\neq$ 0, then

$$A^{-1} = \frac{1}{\det A}[adj(A)]$$

REMARK:

(1) $A.adj(A) = adj(A).A = |A|I_n$ where, **A** is a **square matrix**, **I** is an **identity matrix** of same order as of **A** and $|A|$ represents **determinant** of **matrix A**.

(2) $|adjA| = |A|^{n-1}$ **determinant** of **adjoint A** is equal to **determinant** of **A** power n-1 where **A** is **invertible n x n square matrix**.

(3) $adj(adjA) = |A|^{n-2}.A$ {**A** is **n x n invertible square matrix**}

(4) $adj(AB) = adj(B).adj(A)$

Example: 3 . Find A^{-1} of matrix A

$$A = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ -2 & 0 & -4 \end{bmatrix}$$ by the method of cofactors.

Solution: Cofactors of the matrix A are

$$C_{11} = \begin{vmatrix} 3 & 2 \\ 0 & -4 \end{vmatrix} = -12, C_{12} = -\begin{vmatrix} 0 & 2 \\ -2 & -4 \end{vmatrix} = -4, C_{13} = \begin{vmatrix} 0 & 3 \\ -2 & 0 \end{vmatrix} = 6$$

$$C_{21} = -\begin{vmatrix} 0 & 3 \\ 0 & 4 \end{vmatrix} = 0, \quad C_{22} = \begin{vmatrix} 2 & 3 \\ -2 & -4 \end{vmatrix} = -2, \ C_{23} = -\begin{vmatrix} 2 & 0 \\ -2 & 0 \end{vmatrix} = 0,$$

$$C_{31} = \begin{vmatrix} 0 & 3 \\ 3 & 2 \end{vmatrix} = -9, \quad C_{32} = -\begin{vmatrix} 2 & 3 \\ 0 & 2 \end{vmatrix} = -4, \quad C_{33} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6$$

Matrix of cofactors, $C = \begin{bmatrix} -12 & -4 & 6 \\ 0 & -2 & 0 \\ -9 & -4 & 6 \end{bmatrix}$

$$\text{Adjoint of matrix A, adj(A)} = \begin{bmatrix} -12 & 0 & -9 \\ -4 & -2 & -4 \\ 6 & 0 & 6 \end{bmatrix}$$

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$
$$= 2(-12)+0(-4)+3(6)$$
$$=-24 +18 = - 6 \neq 0$$

Inverse of the matrix A is

$$A^{-1} = \frac{1}{\det A}[adj(A)] = \frac{1}{-6}\begin{bmatrix} -12 & 0 & -9 \\ -4 & -2 & -4 \\ 6 & 0 & 6 \end{bmatrix}$$

Exercise:

(c) Find $\left|3(adjA)^{-1} + A\right|$ where A is a matrix of size 4×4 such that $|A| = 3$.