Parametric Equations and Polar Coordinates

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Definition (Definition of a Plane Curve)

If f and g are continuous functions on an interval I, then the set of ordered pairs $(f(t), g(t))$ is called a plane curve $\mathcal{C}$.
The equations $x = f(t)$ and $y = g(t)$ are called parametric equations for $\mathcal{C}$, and t is called the parameter.

Remark

1. *If* $\mathcal{C} = \{(x = f(t), y = g(t));\ t \in I\}$ *is a curve an the function* $f : I \longrightarrow J$ *is bijective, then* $t = f^{-1}(x)$ *and the curve is represented by the equation* $y = g(t) = g \circ f^{-1}(x)$ *and the curve is the graph of the function* $y = g \circ f^{-1}(x)$, *for* $x \in J$.
2. *If* $\mathcal{C} = \{(x = f(t), y = g(t));\ t \in I\}$ *is a curve an the function* $g : I \longrightarrow J$ *is bijective, then* $t = g^{-1}(y)$ *and the curve is represented by the equation* $x = f(t) = f \circ g^{-1}(x)$ *and the curve is the graph of the function* $x = f \circ g^{-1}(y)$, *for* $y \in J$.

Definition

1. *A curve $\mathcal{C}$ represented by $\{(x = f(t), y = g(t));\ t \in I\}$ is called smooth if f' and g' are continuous on I and not simultaneously zero, except possibly at the endpoints of I.*
2. *If $y = f(x)$ for $x \in I$ is the equation of the plane curve and f' continuous on I, the curve $\mathcal{C}$ is smooth.*
3. *If $x = f(y)$ for $x \in I$ is the equation of the plane curve and f' continuous on I, the curve $\mathcal{C}$ is smooth.*
4. *The Orientation of the curve of parametric equations is the direction of movement along the graph relative to the order of the values of the parameter.*

Example 1 :

Consider the plane curve $\mathcal{C}$ given by the parametric equations

$$\begin{cases} x = 4t^2 - 5 \\ y = 2t + 3 \end{cases} \qquad t \in \mathbb{R}.$$

a) Find an equation in x and y whose graph contains the points of $\mathcal{C}$.

b) Sketch the graph of $\mathcal{C}$.

c) Indicate the orientation : the growing sense of t.

Solution.

a) $y = 2t + 3 \Rightarrow t = \frac{y-3}{2}$. As $x = 4t^2 - 5$, then

$$x = 4\left(\frac{y-3}{2}\right)^2 - 5 = (y-3)^2 - 5 = y^2 - 6y + 4.$$

The graph of the following equation

$$x = y^2 - 6y + 4, \quad y \in \mathbb{R}$$

contains the points of $\mathcal{C}$.

b) We remark that when the variable t described $\mathbb{R}$, the variable y described $\mathbb{R}$. Then, the graph of

$$x = y^2 - 6y + 4, \quad y \in \mathbb{R}$$

is exactly the set of the points of $\mathcal{C}$.

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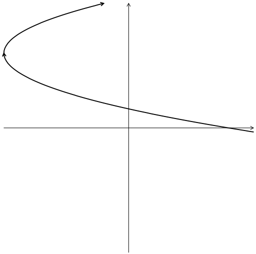

c) We have

$$x(0) = -5,\ y(0) = 3$$

$$x(1) = -1,\ y(1) = 5.$$

Then the orientation of t is from the point $A(-5, 3)$ to the point $B(-1, 5)$.

Theorem

If a smooth curve $\mathcal{C}$ is given by the parametric equations

$$\begin{array}{rcl} x & = & f(t) \\ y & = & g(t) \end{array}, \ t \in I.$$

Then the slope of tangent line to $\mathcal{C}$ at a point $P(x_0, y_0) = (f(t_0), g(t_0))$ is given by

$$m = \frac{\frac{dy}{dt}(t_0)}{\frac{dx}{dt}(t_0)} = \frac{g'(t_0)}{f'(t_0)}, \quad \text{if} \quad f'(t_0) \neq 0.$$

Example 2 :
Let $\mathcal{C}$ be the curve with the parametrization

$$\begin{array}{rcl} x & = & t^2 + t \\ y & = & 5t^2 - 3 \end{array}, \; t \in \mathbb{R}.$$

Find the points on $\mathcal{C}$ at which the slope m of the tangent line is given by $m = 4$.

Solution.

$$x(t) = t^2 + t \implies \frac{dx}{dt}(t) = 2t + 1$$
$$y(t) = 5t^2 - 3 \implies \frac{dy}{dt}(t) = 10t.$$

Then

$$m = 4 \implies \frac{\frac{dy}{dt}(t)}{\frac{dx}{dt}(t)} = 4 \implies t = 2.$$

We check $\frac{dx}{dt}(2) = 5 \neq 0$. Then the required point is $P(x(2), y(2)) = (6, 17)$.

Remark

If m is the slope of the tangent line of a curve, then

$$\text{The tangent line is horizontal} \iff m = 0.$$

$$\text{The tangent line is vertical} \iff \frac{1}{m} = 0.$$

Theorem

Let $\mathcal{C}$ be a smooth curve given by the parametric equations $(x(t), y(t)), \qquad t \in [\alpha; \beta]$. We assume that the function $t \longmapsto (x(t), y(t))$ is an injection from $[\alpha; \beta]$ to $\mathbb{R}^2$. Then

a) The length of the arc $\mathcal{C}$ is given by

$$L_\alpha^\beta = \int_\alpha^\beta \sqrt{x'^2(t) + y'^2(t)} dt.$$

b) The area of the surface generated by revolving the curve $\mathcal{C}$ around the x-axis is

$$SA = 2\pi \int_\alpha^\beta |y(t)| \sqrt{x'^2(t) + y'^2(t)} dt.$$

c) The area of the surface generated by revolving the curve $\mathcal{C}$ around the y-axis is

$$SA = 2\pi \int_{\alpha}^{\beta} |x(t)| \sqrt{x'^2(t) + y'^2(t)} dt.$$

Example 3 :
Find the area of the surface generated by revolving the curve $\mathcal{C}$ given by $(t^2, t - \frac{1}{3}t^3)$, $\qquad t \in [0, 1]$ around the x-axis.
Solution. The area of given surface is

$$\begin{aligned} SA &= 2\pi \int_0^1 (t - \frac{1}{3}t^3)\sqrt{4t^2 + 1 - 2t^2 + t^4}dt \\ &= 2\pi \int_0^1 (t - \frac{1}{3}t^3)(1 + t^2)\, dt, \\ &= 2\pi(\frac{1}{2} + \frac{1}{6} - \frac{1}{18}), \\ &= \frac{11\pi}{9}. \end{aligned}$$

The Polar Coordinates System

Definition

If M is a point in the plane $\mathbb{R}^2$ *and* (x, y) *its cartesian coordinates. A polar coordinates of M is a couple* $(r, \theta) \in \mathbb{R}^2$ *such that*

$$\begin{cases} x &= r\cos(\theta) \\ y &= r\sin(\theta). \end{cases}$$

Clearly $r = \pm\sqrt{x^2 + y^2}$. And θ is not unique because we can replace θ by $\theta + 2n\pi$ for any $n \in \mathbb{Z}$.

Example 4 :
The cartesian coordinates of the point $A(r,\theta) = (2, \frac{\pi}{4})$ are $(\sqrt{2}, \sqrt{2})$.
The cartesian coordinates of the point $B(r,\theta) = (-2, \frac{5\pi}{4})$ are $(\sqrt{2}, \sqrt{2})$.
The cartesian coordinates of the point $C(r,\theta) = (1, \frac{\pi}{2})$ are $(0, 1)$.
The cartesian coordinates of the point $D(r,\theta) = (0, \theta)$ are $(0, 0)$.

Definition

A polar equation is an equation of type

$$r = f(\theta),\ \theta \in I,$$

where I is an interval.

Example 5 :

Sketch the curves of the following polar equations

$$(E_1) : \qquad r = 2\cos(\theta),$$

$$(E_2) : \qquad r = \theta,$$

$$(E_3) : \qquad r = a,$$

$$(E_4) : \qquad r = 2 + 2\cos(\theta),$$

$$(E_5) : \qquad r = 2 + 4\cos(\theta),$$

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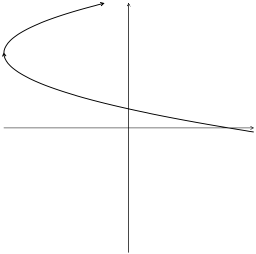

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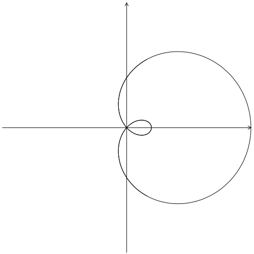

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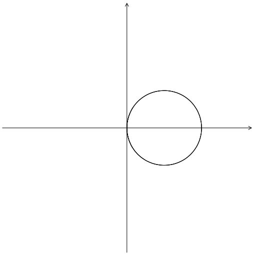

Example 6 :

a) Sketch the curves of the following equations

$$\theta = \frac{\pi}{4}.$$

a) Let $\alpha \in \mathbb{R}$. Transforms the following equation in Cartesian coordinates equations

$$\theta = \alpha.$$

And deduce her nature.

Example 7 :
Let $a \in \mathbb{R} \setminus \{0\}$. Prove that the curve of the following polar equation

$$(\mathcal{C}) \quad r = a\cos(\theta),$$

is the equation of a circle.
Sketch the graph of $(\mathcal{C})$ if $a = 2$ and $a = -3$.

Example 8 :
Let $a \in \mathbb{R} \setminus \{0\}$. Prove that the curve of the following polar equation

$$(\mathcal{C}) \quad r = a\sin(\theta),$$

is the equation of a circle.
Sketch the graph of $(\mathcal{C})$ if $a = 2$ and $a = -3$.

Definition

Let $\mathcal{C}$ be a curve whose polar equation is

$$r = f(\theta),\ \theta \in I,$$

where I is an interval.
We assume that f is smooth, then the slope of the tangent line at the point $(r_0 = f(\theta_0), \theta_0)$ is given by

$$m = \frac{\frac{d(f(\theta)\sin(\theta))}{d\theta}(\theta_0)}{\frac{d(f(\theta)\cos(\theta))}{d\theta}(\theta_0)},$$

provided $\frac{d(f(\theta)\cos(\theta))}{d\theta}(\theta_0) \neq 0.$

Particularly $\mathcal{C}$ has an horizontal tangent line if and only if

$$\frac{d(f(\theta)\sin(\theta))}{d\theta}(\theta_0) = 0 \text{ and } \frac{d(f(\theta)\cos(\theta))}{d\theta}(\theta_0) \neq 0.$$

$\mathcal{C}$ has a vertical tangent line if and only if

$$\frac{d(f(\theta)\cos(\theta))}{d\theta}(\theta_0) = 0 \text{ and } \frac{d(f(\theta)\sin(\theta))}{d\theta}(\theta_0) \neq 0.$$

Theorem

Let $f : [\alpha, \beta] \to \mathbb{R}^+$ *be a continuous function, where* $0 \leq \alpha < \beta \leq 2\pi$*(generally* $0 < \beta - \alpha \leq 2\pi$*). Then the area of the region bounded by the graphs of*

$$r = f(\theta),\ \theta = \alpha,\ \theta = \beta,$$

is equal to

$$A = \int_{\alpha}^{\beta} \frac{1}{2}\left[f(\theta)\right]^2 d\theta.$$

Example 9 :

The area of region bounded by the graph of the polar equation

$$r = 6 - 6\sin(\theta).$$

$r = 6(1 - \sin(\theta)) \geq 0$ then the required interval is $[0, 2\pi]$.
Therefore the area of this region is

$$\begin{aligned} A &= \int_0^{2\pi} \frac{1}{2}\big[f(\theta)\big]^2 d\theta \\ &= 18\int_0^{2\pi} (1 - \sin(\theta))^2 d\theta \\ &= 18\int_0^{2\pi} 1 - 2\sin(\theta) + \frac{1 - \cos(2\theta)}{2} d\theta \\ &= 54\pi. \end{aligned}$$

Example 10 :
The area of region that is outside the graph of $r = 2$ and inside the graph of $r = 4\cos(\theta)$.
We denote $f_1(\theta) = 2$ and $f_2(\theta) = 4\cos(\theta)$.
$r = 2$ is the equation of the circle with center is $(0, 0)$ and radius 2.
$r = 4\cos(\theta)$ is the equation of the circle with center $(2, 0)$ and radius 2.
The points of intersection
$2 = 4\cos(\theta) \iff \cos(\theta) = \frac{1}{2} \iff \theta = \pm\frac{\pi}{3} + 2n\pi,\ n \in \mathbb{Z}$.

We can choose the interval $[-\pi, \pi]$, then the solutions are $\theta = \pm\frac{\pi}{3}$ and the points of intersection are

$$(r, \theta) = (2, \frac{\pi}{3}) \text{ and } (r, \theta) = (2, -\frac{\pi}{3}).$$

Then

$$\begin{aligned} A &= \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left[f_2(\theta)\right]^2 - \left[f_1(\theta)\right]^2 d\theta \\ &= 4\int_0^{\frac{\pi}{3}} (2\cos(2\theta) + 1)\theta \\ &= 2 + 2\sqrt{3}. \end{aligned}$$

Theorem

The arc length of a continuously differentiable polar curve

$$r = f(\theta) \geq 0$$

from the point (r_1, θ_1) *to the point* (r_2, θ_2)*, such that* $0 \leq \theta_2 - \theta_1 \leq 2\pi$*, is given by*

$$\text{Arc length} = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + (\frac{dr}{d\theta})^2}\, d\theta = \int_{\theta_1}^{\theta_2} \sqrt{(f(\theta))^2 + (f'(\theta))^2}\, d\theta.$$

Example 11 :
Find the length of the curve $r = 2 - 2\cos(\theta)$.
We have

$$r = f(\theta) = 2 - 2\cos(\theta) = 2(1 - \cos(\theta)) \geq 0 \Longrightarrow [\theta_1, \theta_2] = [0, 2\pi].$$

Therefore

$$\begin{aligned} Arc\ length &= \int_0^{2\pi} \sqrt{(2 - 2\cos(\theta))^2 + (2\sin(\theta))^2}\, d\theta \\ &= \int_0^{2\pi} \sqrt{8 - 8\cos(\theta)}\, d\theta \\ &= 4\int_0^{2\pi} \sin(\frac{\theta}{2})\, d\theta \\ &= 16. \end{aligned}$$

Theorem

Let $\mathcal{C}$ be the graph of a continuously differentiable polar equation

$$r = f(\theta) \geq 0, \ \theta \in [\alpha, \beta] \subset [0, \pi] \text{ or } \subset [-\pi, 0].$$

Then the surface of revolution generated by revolving $\mathcal{C}$ around the x-axis is

$$(SA)_1 = 2\pi \int_{\alpha}^{\beta} f(\theta)|\sin(\theta)|\sqrt{(f(\theta))^2 + (f'(\theta))^2}\, d\theta.$$

Theorem

Let $\mathcal{C}$ be the graph of a continuously differentiable polar equation

$$r = f(\theta) \geq 0,\ \theta \in [\alpha, \beta] \subset [-\frac{\pi}{2}, \frac{\pi}{2}] \text{ or } \subset [\frac{\pi}{2}, 3\frac{\pi}{2}].$$

Then the surface of revolution generated by revolving $\mathcal{C}$ around the y-axis(or $\theta = \frac{\pi}{2}$) is

$$(SA)_2 = 2\pi \int_\alpha^\beta f(\theta)|\cos(\theta)|\sqrt{(f(\theta))^2 + (f'(\theta))^2}\, d\theta.$$

Remark

Polar axis : $|\sin(\theta)| = \sin(\theta) \Longrightarrow \theta \in [0, \pi]$.
Line $\theta = \frac{\pi}{2}$ *:* $|\cos(\theta)| = \cos(\theta) \Longrightarrow \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Example 12 :

Find the area of the surface of revolution generated by revolving the graph of

$$r = 2\sin(\theta),\ \theta \in [0, \frac{\pi}{2}]$$

around the

a) polar axis.

b) line $\theta = \frac{\pi}{2}$.

a) The required area is

$$
\begin{aligned}
(SA) &= 2\pi \int_0^{\frac{\pi}{2}} 2\sin(\theta).\sin(\theta)\sqrt{(2\sin(\theta))^2+(2\cos(\theta))^2}\, d\theta \\
&= 8\pi \int_0^{\frac{\pi}{2}} \sin^2(\theta)\, d\theta \\
&= 4\pi \int_0^{\frac{\pi}{2}} 1-\cos(2\theta)\, d\theta \\
&= 4\pi \Big[\theta - \frac{\sin(2\theta)}{2}\Big]_0^{\frac{\pi}{2}} \\
&= 2\pi^2.
\end{aligned}
$$

b) The required area is

$$\begin{aligned}(SA) &= 2\pi \int_0^{\frac{\pi}{2}} 2\sin(\theta).\cos(\theta)\sqrt{(2\sin(\theta))^2 + (2\cos(\theta))^2}\; d\theta \\ &= 4\pi \int_0^{\frac{\pi}{2}} 2\sin(\theta)\cos(\theta)\; d\theta \\ &= 4\pi \int_0^{\frac{\pi}{2}} \sin(2\theta)\; d\theta \\ &= 4\pi \Big[-\frac{\cos(2\theta)}{2}\Big]_0^{\frac{\pi}{2}} \\ &= 4\pi.\end{aligned}$$