Linear System Equations

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Introduction to Linear System Equations

Definition

A linear system of equations with m equations and n unknowns is defined as follows:

$$\begin{cases} a_{1,1}x_1 & + & a_{1,2}x_2 & + & \cdots & + & a_{1,n}x_n & = & b_1 \\ a_{2,1}x_1 & + & a_{2,2}x_2 & + & \cdots & + & a_{2,n}x_n & = & b_2 \\ \vdots & \vdots & & \ddots & + & \vdots & + & \vdots & \\ a_{m,1}x_1 & + & a_{m,2}x_2 & + & \cdots & + & a_{m,n}x_n & = & b_m. \end{cases}$$

where $b_1, \ldots, b_n$, $(a_{j,k})$ are real numbers with $(1 \le j \le m, 1 \le k \le n)$ called the data of the system and $x_1, \ldots, x_n$ the unknowns or the variables of the system.

This linear system can be represented in matrix form:
$AX = B$ where

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix}, \qquad B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}, \qquad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Example

The following linear system with two variables

$$\begin{cases} 4x & - & y & = & 5 \\ -7x & + & 2y & = & 3 \end{cases}$$

can be interpreted as the intersection in the plane of the straight lines of equations respectively $4x - y = 5$ and $-7x + 2y = 3$.

Example

The following linear system with three variables

$$\begin{cases} x & + & y & -3 & z = & 1 \\ 2 & + & y & - & z = & 3 \end{cases}$$

can be interpreted as the intersection in the space of the planes of equations respectively $x + y - 3z = 1$ and $2x + y - z = 3$.
The solution of this system is $\{(2,1,0)+z(-2,-5,1);\ z \in \mathbb{R}\}$. This is the equation of the line passing through the point A of coordinates $(2,1,0)$ and parallel of the vector v of coordinates $(-2,-5,1)$.

Example

Let the matrices $A = \begin{pmatrix} 1 & -1 \\ 1 & -2 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$ and we look for a matrix of order $(2,3)$ such that $AB = C$.

If $B = \begin{pmatrix} x & y & z \\ t & u & v \end{pmatrix}$ we find the following linear system:

$$\begin{cases} x - t & = & 0 \\ x - 2t & = & 1 \\ y - u & = & 1 \\ y - 2u & = & 2 \\ z - v & = & 2 \\ z - 2v & = & 3 \end{cases}$$

The solution of this system is $(-1, 0, 1, -1, -1, -1)$

$$x = t = -1, y = 0, u = -1, z = 1, v = -1.$$

Definition

1. We say that two linear systems are equivalent if they have the same set of solutions.
2. We say that a linear system is consistent if it has solutions and we call that it is inconsistent if it has no solutions.

Gauss And Gauss Jordan Method

The augmented matrix of the linear system $AX = B$ is the matrix $[A|B]$.
The elementary row operations on the augmented matrix of a system produce the augmented matrix of an equivalent system.

The Gauss-Jordan elimination method to solve a system of linear equations is described in the following steps.

1. Write the augmented matrix of the system.
2. Use elementary row operations to transform the augmented matrix in a reduced row echelon form.
3. Solve the obtained triangular system.

The Gauss and Gauss Jordan Method

• The Gauss Jordan method consists to take the reduced row echelon form of the augmented matrix $[A|B]$ and solve the obtained system.

Examples

Consider the following linear system

$$\begin{cases} x + 2y - z &= -4 \\ -x + y &= -2 \\ y - z &= -4 \end{cases}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -4 \\ -1 & 1 & 0 & -2 \\ 0 & 1 & -1 & -4 \end{array}\right]$$

and the matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -4 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

is a row echelon form of this matrix.

Using Gauss method, the system has a unique solution which is $x = 1, y = -1, z = 3$.
The reduced row echelon form of this matrix is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right].$$

Using Gauss Jordan method, the system has a unique solution which is $x = 1, y = -1, z = 3$.

Consider the following linear system

$$\begin{cases} x+2y-z+t &= 1 \\ 3x-y+5z-t &= 2 \\ 5x+3y+3z+t &= m \end{cases} \quad ; \quad m \in \mathbb{R}.$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 1 \\ 0 & 1 & -\frac{8}{7} & \frac{4}{7} & \frac{1}{7} \\ 0 & 0 & 0 & 0 & m-4 \end{array}\right]$$

is a row echelon form of this matrix .
If $m \neq 4$, the system, is inconsistent.
If $m = 4$, the system has infinite solutions :

$$\{(\frac{5}{7} - \frac{9}{7}z + \frac{1}{7}t, \frac{1}{7} + \frac{8}{7}z - \frac{4}{7}t, z, t) \in \mathbb{R}^4\}.$$

Consider the following linear system $\begin{cases} -2y+3z &= 0 \\ 2x-4y+2z &= 1 \\ -x-2y+5z &= 0 \\ x-2y &= 1 \end{cases}$,

The augmented matrix of the system is:

$$\left[\begin{array}{ccc|c} 0 & -2 & 3 & 0 \\ 2 & -4 & 2 & 1 \\ -1 & -2 & 5 & 0 \\ 1 & -2 & 0 & 1 \end{array}\right]$$

A row echelon form of the augmented matrix is $\left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & -2 & 3 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 3 \end{array}\right]$

The system is inconsistent.

Give the relations between the numbers a, b and c such that the following linear system is consistent.

$$\begin{cases} x + y + 2z &= a \\ x + z &= b \\ 2x + y + 3z &= c \end{cases}$$

The augmented matrix of the system is: $\left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 1 & 0 & 1 & b \\ 2 & 1 & 3 & c \end{array}\right]$.

A row echelon form of the augmented matrix is $\left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 0 & 1 & 1 & a-b \\ 0 & 0 & 0 & c-a-b \end{array}\right]$.

The system is consistent if and only if $c - a - b = 0$.

Consider the following linear system

$$\begin{cases} x + my + (m-1)z & = & m+1 \\ 3x + 2y + mz & = & 3 \\ (m-1)x + my + (m+1)z & = & m-1 \end{cases}$$

The determinant of the system is $m^2(m-4)$.

If $m = 0$, the augmented matrix of the system is:

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 3 & 2 & 0 & 3 \\ -1 & 0 & 1 & -1 \end{array}\right]$$

A row echelon form of the augmented matrix is $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$.

The system has an infinity of solutions $\{(1+z, -\frac{3}{2}z, z);\ z \in \mathbb{R}\}$.

If $m = 4$, a row echelon form of the augmented matrix is $\left[\begin{array}{ccc|c} 1 & 4 & 3 & 5 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 12 \end{array}\right]$.

The system is inconsistent.

Homogeneous Linear Systems

Definition

We say that a linear system $AX = B$ is homogeneous if $B = 0$.

Remarks

1. Any homogeneous linear system is consistent. 0 is a solution of the system.
2. If X_1 and X_2 are solutions of the homogeneous system $AX = 0$, then $X_1 + \lambda X_2$ is also a solution of the linear system for all $\lambda \in \mathbb{R}$.
3. If the homogeneous linear system $AX = 0$ has a non zero solution, it has an infinite number of solutions.

Theorem

If X_0 is a solution of the linear system $AX = B$, then any solution X of the system is in the following form: $X = X_0 + X_1$ with X_1 is a solution of the homogeneous system. $AX = 0$.

Conclusion Any consistent linear system can has only one solution or an infinite number of solutions.

Crammer Method

Theorem

If A is a square matrix of order n and has an inverse, then the linear system $AX = B$ has the following unique solution

$$x_1 = \frac{\det A_1}{\det A}, \dots, x_n = \frac{\det A_n}{\det A}.$$

with A_j is the matrix obtained by replace the j^{th} column in the matrix A by the column matrix B.

Example

Use Crammer method to solve the following system:

$$\begin{cases} 3x - 2z = 2 \\ -2x + 3y - 2z = 3 \\ -5x + 4y - z = 1 \end{cases}$$

$$\begin{vmatrix} 3 & 0 & -2 \\ -2 & 3 & -2 \\ -5 & 4 & -1 \end{vmatrix} = 1, \qquad \begin{vmatrix} 2 & 0 & -2 \\ 3 & 3 & -2 \\ 1 & 4 & -1 \end{vmatrix} = -8 = x,$$

$$\begin{vmatrix} 3 & 2 & -2 \\ -2 & 3 & -2 \\ -5 & 1 & -1 \end{vmatrix} = -13 = y, \qquad \begin{vmatrix} 3 & 0 & 2 \\ -2 & 3 & 3 \\ -5 & 4 & 1 \end{vmatrix} = -13 = z.$$