Techniques of Integration

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Techniques of Integration

The product rule of differentiation yields an integration technique known as integration by parts. Let us begin with the product rule:

$$\frac{d}{dx}(u(x)v(x)) = u(x)\frac{dv(x)}{dx} + \frac{du(x)}{dx}v(x).$$

On integrating each term with respect to x from $x = a$ to $x = b$, we get

$$\int_a^b \frac{d}{dx}(u(x)v(x))dx = \int_a^b u(x)\left(\frac{dv(x)}{dx}\right)dx + \int_a^b v(x)\left(\frac{du(x)}{dx}\right)dx.$$

By using the differential notation and the fundamental theorem of calculus, we get

The standard form of this integration by parts formula is written as

$$\int_a^b u(x)v'(x)dx = [u(x)v(x)]_a^b - \int_a^b v(x)u'(x)dx.$$

and

$$\int udv = uv - \int vdu.$$

We state this result in the following theorem

Theorem

(Integration by Parts)
If u and v are two continuously differentiable functions on the interval $[a, b]$, then we have

$$\int_a^b u(x)v'(x)dx = [u(x)v(x)]_a^b - \int_a^b v(x)u'(x)dx$$

and for the indefinite integrals

$$\int udv = uv - \int vdu.$$

Example 1 :
Evaluate the following integrals:

1. $\displaystyle\int \sinh^{-1}(x)dx$,
2. $\displaystyle\int \cosh^{-1}(x)dx$,
3. $\displaystyle\int \tanh^{-1}(x)dx$,
4. $\displaystyle\int \sin^{-1}(x)dx$,
5. $\displaystyle\int \tan^{-1}(x)dx$.

Solution

1. By parts $\int \sinh^{-1}(x)dx = x\sinh^{-1}(x) - \int \frac{x}{\sqrt{1+x^2}}dx =$ $x\sinh^{-1}(x) - \sqrt{1+x^2} + c,$
2. $\int \cosh^{-1}(x)dx = x\cosh^{-1}(x) - \int \frac{x}{\sqrt{x^2-1}}dx =$ $x\cosh^{-1}(x) - \sqrt{x^2-1},$
3. $\int \tanh^{-1}(x)dx = x\tanh^{-1}(x) - \int \frac{x}{1-x^2}dx =$ $x\tanh^{-1}(x) + \frac{1}{2}\ln(1-x^2) + c,$
4. $\int \sin^{-1}(x)dx = x\sin^{-1}(x) - \int \frac{x}{\sqrt{1-x^2}}dx =$ $x\sin^{-1}(x) + \sqrt{1-x^2} + c,$

Example 2 :

Evaluate the following integrals:

1. $\displaystyle\int \ln(x)dx,$
2. $\displaystyle\int \ln^2(x)dx,$
3. $\displaystyle\int x\cos(x)dx,$
4. $\displaystyle\int xe^x dx,$
5. $\displaystyle\int e^x \sin(x)dx,$
6. $\displaystyle\int \cosh(x)\cos(x)dx.$

Solution

1. By parts $\displaystyle\int \ln(x)dx = x\ln(x) - x + c$,
2. $\displaystyle\int \ln^2(x)dx = x\ln^2(x) - 2(x\ln(x) - x) + c$,
3. $\displaystyle\int x\cos(x)dx = x\sin(x) + \cos(x) + c$,
4. $\displaystyle\int xe^x dx = xe^x - e^x + c$,
5. $\displaystyle\int e^x\sin(x)dx = \frac{e^x}{2}(\sin(x) - \cos(x)) + c$,
6. $\displaystyle\int \cosh(x)\cos(x)dx = \frac{1}{2}(\sin(x)\cosh(x) + \cos(x)\sinh(x)) + c$.

Trigonometric Integrals

Some Important Trigometric Formulas:

$$\cos^2(x) + \sin^2(x) = 1,$$

$$\cos(2x) = \cos^2(x) - \sin^2(x),$$
$$\cos^2(x) = \frac{1 + \cos(2x)}{2},$$
$$\sin^2(x) = \frac{1 - \cos(2x)}{2},$$

$$\sec^2(x) = 1 + \tan^2(x),$$
$$\csc^2(x) = 1 + \cot^2(x),$$

$$\frac{d}{dx}\sin(x) = \cos(x),$$

$$\frac{d}{dx}\cos(x) = -\sin(x),$$

$$\frac{d}{dx}\tan(x) = \sec^2(x),$$

$$\frac{d}{dx}\sec(x) = \sec(x)\tan(x),$$

$$\frac{d}{dx}\cot(x) = -\csc^2(x),$$

$$\frac{d}{dx}\csc(x) = -\csc(x)\cot(x).$$

$$\begin{aligned}
\sin(x+y) &= \sin(x)\cos(y) + \cos(x)\sin(y),\\
\sin(x-y) &= \sin(x)\cos(y) - \cos(x)\sin(y),\\
\cos(x+y) &= \cos(x)\cos(y) - \sin(x)\sin(y),\\
\cos(x-y) &= \cos(x)\cos(y) + \sin(x)\sin(y),
\end{aligned}$$

$$\begin{aligned}
\sin(x)\cos(y) &= \frac{1}{2}(\sin(x+y) + \sin(x-y)), && (1)\\
\sin(x)\sin(y) &= \frac{1}{2}(\cos(x-y) - \cos(x+y)), && (2)\\
\cos(x)\cos(y) &= \frac{1}{2}(\cos(x+y) + \cos(x-y)), && (3)
\end{aligned}$$

Integrals of Type $K_{n,m} = \int \cos^n(x)\sin^m(x)dx,\ m,n \in \mathbb{N}$.

- If $m = 2q+1$, we set $u = \cos x$, then $du = -\sin(x)dx$ and

$$
\begin{aligned}
K_{n,m} &= \int \cos^n(x)\sin^{2q+1}(x)dx \\
&= \int \cos^n(x)\sin^{2q}(x).\sin(x)dx \\
&= \int \cos^n(x)\Big(\sin^2(x)\Big)^q.\sin(x)dx \\
&= -\int \cos^n(x)\Big(1-\cos^2(x)\Big)^q.(-\sin(x))dx \\
&= -\int u^n\Big(1-u^2\Big)^q du.
\end{aligned}
$$

- If $n = 2p + 1$, we set $u = \sin(x)$, then $du = \cos(x)dx$ and

$$\begin{aligned} K_{n,m} &= \int \cos^{2p+1}(x)(x)\sin^m(x)dx \\ &= \int \cos^{2p}(x)\sin^m(x).\cos(x)dx \\ &= \int \left(\cos^2(x)\right)^p \sin^m(x).\cos(x)dx \\ &= \int \left(1 - \sin^2(x)\right)^p \sin^m(x).\cos(x))dx \\ &= \int \left(1 - u^2\right)^p u^m du. \end{aligned}$$

- If $n = 2p$ and $m = 2q$,

$$\begin{aligned} K_{n,m} &= \int \cos^{2p}(x)(x)\sin^{2q}(x)dx \\ &= \int \cos^{2p}(x)(1-\cos^2(x))^q dx. \end{aligned}$$

We compute the integral $J_n = \int \cos^{2n}(x)dx$ by induction and by parts: We set $u = \cos^{2n-1}(x)$ and $v' = \cos x$, then

$$\begin{aligned} J_n &= \sin(x)\cos^{2n-1}(x) + (2n-1)\int \cos^{2n-2}(x)\sin^2(x)dx \\ &= \sin(x)\cos^{2n-1}(x) + (2n-1)J_{n-1} - (2n-1)J_n. \end{aligned}$$

Thus $J_n = \frac{1}{2n}\sin(x)\cos^{2n-1}(x) + \frac{2n-1}{2n}J_{n-1}.$

In particular

$$J_1 = \int \cos^2(x)dx = \frac{\sin(x)\cos(x)}{2} + \frac{x}{2} + c = \frac{\sin(2x)}{4} + \frac{x}{2} + c.$$

$$J_2 = \int \cos^4(x)dx = \frac{\sin(x)\cos^3(x)}{4} + \frac{3\sin(x)\cos(x)}{8} + \frac{3x}{8} + c.$$

Examples 1 :

1.
$$\int \sin^2(x)\cos^2(x)dx = \int (1-\cos^2(x))\cos^2(x)dx = \frac{\sin(x)\cos(x)}{8} + \frac{x}{8} - \frac{\sin(x)\cos^3(x)}{4} + c.$$

We can also

$$\int \sin^2(x)\cos^2(x)dx = \frac{1}{4}\int \sin^2(2x)dx = \frac{1}{8}\int (1-\cos(4x))dx = \frac{x}{8} - \frac{\sin(4x)}{32} + c.$$

2. $\int \sin^2(x)dx = \int 1-\cos^2(x)dx = \frac{x}{2} - \frac{\sin(2x)}{4} + c,$

3.

$$\int \sin^3(x)dx \overset{u=\cos(x)}{=} -\int (1-u^2)du = -\cos(x) + \frac{1}{3}\cos^3(x) + c,$$

4.

$$\int \sin^4(x)dx = \int (1-\cos^2(x))^2 dx = -\frac{5\sin(2x)}{16} + \frac{3}{4}x + \frac{\sin(x)\cos^3(x)}{4} + c,$$

5. $\int \cos^3(x)dx \overset{u=\sin(x)}{=} \int (1-u^2)du = \sin(x) - \frac{1}{3}\sin^3(x) + c.,$
6. $\int \sin^5(x)\cos^4(x)dx \overset{u=\cos(x)}{=} -\int u^4(1-u^2)^2 du =$ $-\frac{\cos^5(x)}{5} - \frac{\cos^9(x)}{9} + \frac{2\cos^7(x)}{7} + c,$
7. $\int \sin^4(x)\cos^3(x)dx \overset{u=\sin(x)}{=} \int u^4(1-u^2)du =$ $\frac{\sin^5(x)}{5} - \frac{\sin^7(x)}{7} + c.$

Integrals of Type $L_{n,m} = \int \sec^m(x)\tan^n(x)dx,\ m, n \in \mathbb{N}$.

- If $m = 2q$ and $q \neq 0$, we set $u = \tan x$, then $du = \sec^2(x)dx$ and

$$\begin{aligned} L_{n,2q} &= \int \sec^{2q}(x)\tan^n(x)dx \\ &= \int u^n(1+u^2)^{q-1}du. \end{aligned}$$

Examples 2 :

1. $$\int \sec^4(x)\tan^7(x)dx = \int u^7(1+u^2)du = \frac{\tan^8(x)}{8}+\frac{\tan^{10}(x)}{10}+c.$$
2. $\displaystyle\int \sec^2(x)dx = \tan(x) + c.$
3. $$\begin{aligned}\int \sec^4(x)dx &= \int \sec^2(x)\sec^2(x)dx \\ &= \int (1+\tan^2(x))\sec^2(x)dx \\ &\overset{u=\tan(x)}{=} \int (1+u^2)du\end{aligned}$$

4.

$$\begin{aligned}
\int \sec^{2p+2}(x)dx &= \int \sec^{2p}(x)\sec^2(x)dx \\
&= \int (1+\tan^2(x))^p \sec^2(x)dx \\
&\overset{u=\tan(x)}{=} \int (1+u^2)^p du.
\end{aligned}$$

- If $m = 0$,

$$\int \tan(x)dx = \ln|\sec(x)| + c.$$

$$\int \tan^2(x)dx = \int (\sec^2(x) - 1)dx = \tan(x) - x + c.$$

For $n \geq 3$,

$$\begin{aligned} L_n &= \int \tan^n(x)dx = \int \tan^{n-2}(x)\tan^2(x)dx \\ &= \int \tan^{n-2}(x)\sec^2(x)dx - L_{n-2} \\ &= \frac{\tan^{n-1}(x)}{n-1} - L_{n-2}. \end{aligned}$$

Example 3 :

$L_5 = \int \tan^5(x)dx.$

$$\begin{aligned} L_5 &= \frac{\tan^4(x)}{4} - L_3 = \frac{\tan^4(x)}{4} - \frac{\tan^2(x)}{2} + L_1 \\ &= \frac{\tan^4(x)}{4} - \frac{\tan^2(x)}{2} + \ln|\sec(x)| + c. \end{aligned}$$

• If $m = 2q + 1$ and $n = 2p + 1$, we set $u = \sec(x)$, then $du = \sec(x)\tan(x)$.

$$L_{m,n} = \int \sec^{2q+1}(x)\tan^{2p+1}(x)dx = \int u^{2q}(1 - u^2)^p du.$$

Example 4 :

$$\int \sec^5(x)\tan^3(x)dx = \frac{\sec^5(x)}{5} - \frac{\sec^7(x)}{7} + c.$$

• If $m = 2q + 1$ and $n = 2p$. The result is obtained by integration by parts and induction.

Example 5 :

1. $\int \sec(x)dx = \ln|\sec(x) + \tan(x)| + c$.
2. $\int \sec^3(x)dx = \int \sec(x)\sec^2(x)dx$. Using integration by parts, with $u(x) = \sec(x)$ and $v'(x) = \sec^2(x)$, we have

$$\begin{aligned}\int \sec^3(x)dx &= \int \sec(x)\sec^2(x)dx \\ &= \sec(x)\tan(x) - \int \sec(x)\tan^2(x)dx\end{aligned}$$

Therefore

$$\int \sec^3(x)dx = \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\ln|\sec(x)+\tan(x)| + c.$$

$$\begin{aligned}\int \sec^5(x)dx &= \int \sec^3(x)\sec^2(x)dx \\ &= \sec^3(x)\tan(x) - 3\int \sec^3(x)\tan^2(x)dx \\ &= \sec^3(x)\tan(x) - 3\int \sec^5(x)dx + 3\int \sec^3(x)dx.\end{aligned}$$

Then

$$\begin{aligned}\int \sec^5(x)dx &= \frac{1}{4}\sec^3(x)\tan(x) + \frac{3}{8}\sec(x)\tan(x) \\ &\quad + \frac{3}{8}\ln|\sec(x)+\tan(x)| + c.\end{aligned}$$

Integrals of Type $\int \sin(ax)\sin(bx)dx$, $\int \sin(ax)\cos(bx)dx$ and $\int \cos(ax)\cos(bx)dx$

In use the formulas (1), we have

$$\int \sin(ax)\sin(bx)dx = \frac{1}{2}\int \cos((a-b)x) - \cos((a+b)x)dx,$$

$$\int \sin(ax)\cos(bx)dx = \frac{1}{2}\int \sin((a+b)x) + \sin((a-b)x)dx,$$

$$\int \cos(ax)\cos(bx)dx = \frac{1}{2}\int \cos((a+b)x) + \cos((a-b)x)dx,$$

Examples 3 :

- $$\begin{aligned} \int \sin(5x)\sin(3x)dx &= \frac{1}{2}\int \cos(2x) - \cos(8x)dx \\ &= \frac{1}{2}\left(\frac{\sin(2x)}{2} - \frac{\sin(8x)}{8}\right) + c \\ &= \frac{\sin(2x)}{4} - \frac{\sin(8x)}{16} + c. \end{aligned}$$

- $$\begin{aligned} \int \sin(4x)\cos(3x)dx &= \frac{1}{2}\int \sin(7x) + \sin(x)dx \\ &= \frac{1}{2}\left(-\cos(x) - \frac{\cos(7x)}{7}\right) + c \\ &= -\frac{\cos(x)}{\;} - \frac{\cos(7x)}{\;} + c \end{aligned}$$

Trigonometric Substitutions

Let $a > 0$.

- If in an integral we have $\sqrt{a^2 - x^2}$, $(-a \le x \le a)$, we set

$$x = a\sin(\theta),\ \ \theta \in [-\frac{\pi}{2}; \frac{\pi}{2}].$$

Then

$$dx = a\cos(\theta)d\theta$$
$$\sqrt{a^2 - x^2} = a\cos(\theta).$$

Also, if we put

$$x = a\cos(\theta), \;\; \theta \in [0;\pi].$$

Then

$$dx = -a\sin(\theta)d\theta$$
$$\sqrt{a^2 - x^2} = a\sin(\theta).$$

- If in an integral we have $\sqrt{a^2 + x^2}$, $(x \in \mathbb{R})$, we set

$$x = a\tan(\theta),\ \ \theta \in (-\frac{\pi}{2}; \frac{\pi}{2}).$$

Then

$$dx = a\sec^2(\theta)d\theta$$
$$\sqrt{a^2 + x^2} = a\sec(\theta).$$

- If in an integral we have $\sqrt{x^2 - a^2}$, $(x > a)$, we set

$$x = a\sec(\theta),\ \ \theta \in [0; \frac{\pi}{2}).$$

Then

$$dx = a\sec(\theta)\tan(\theta)d\theta$$
$$\sqrt{x^2 - a^2} = a\tan(\theta).$$

Examples 4 :

1.

$$\int \frac{dx}{x\sqrt{4-x^2}} \overset{x=2\sin(\theta)}{=} \int \frac{d\theta}{2\sin(\theta)} = -\frac{1}{2}\ln(\csc(\theta)+\cot(\theta))+c = -\frac{1}{2}\ln(\frac{2+\sqrt{4-x^2}}{x})+c,$$

We know that

$$\int \frac{dx}{x\sqrt{4-x^2}} = -\frac{1}{2}\mathrm{sech}^{-1}(\frac{x}{2})+c = -\frac{1}{2}\ln(\frac{2+\sqrt{4-x^2}}{x})+c.$$

2.

$$\begin{aligned}
\int \frac{dx}{x^2\sqrt{x^2+9}} &\overset{x=3\tan(\theta)}{=} \int \frac{\sec(\theta)}{9\tan^2(\theta)}d\theta \\
&= \frac{1}{9}\int \frac{\cos(\theta)}{\sin^2(\theta)}d\theta \\
&= -\frac{\csc(\theta)}{9} + c \\
&= -\frac{\sqrt{9+x^2}}{9X} + c,
\end{aligned}$$

3.

$$\begin{aligned}
\int \frac{dx}{x^2\sqrt{x^2-25}} &\overset{x=5\sec(\theta)}{=} \int \frac{d\theta}{25\sec(\theta)} \\
&= \frac{1}{25}\int \cos(\theta)d\theta \\
&= \frac{1}{25}\sin(\theta) + c \\
&= \frac{\sqrt{x^2-25}}{25x} + c.
\end{aligned}$$

Integrals of Rational Functions

In this section, we study the integrals of the form

$$\int F(x)dx,$$

where

$$F(x) = \frac{P(x)}{Q(x)},\ P,\ Q \in \mathbb{R}[X].$$

We shall describe a method for computing this type of integrals. The method is to decompose a given rational function into a sum of simpler fractions (called partial fractions) which is easier to integrate.

Definition

The irreducible polynomials in $\mathbb{R}[X]$ are
The irreducible linear polynomials are the polynomials of the form

$$R(x) = x - \alpha, \quad \alpha \in \mathbb{R}.$$

The irreducible quadratic polynomials are the polynomials of the form

$$R(x) = ax^2 + bx + c, \quad a,\, b,\, c \in \mathbb{R}: \; b^2 - 4ac < 0.$$

If the rational functions $\dfrac{P}{Q}$ in given which the degree P is not less than that of Q, we can express $\dfrac{P}{Q}$ as the sum of a polynomial and a proper rational function, that is

$$\frac{P}{Q} = R + \frac{S}{Q},$$

where R and S are polynomials and the degree S is less than that of Q. For example,

$$\frac{x^4 + 5x^2 + 3}{x^3 - x} = x + \frac{6x^2 + 3}{x^3 - x}.$$

Example 1 :

Evaluation of the following integral $\int \frac{x^4 + 5x^2 + 3}{x^3 - x} dx$.

A general theorem in algebra states that there is three real numbers a, b, c such that

$$\frac{6x^2 + 3}{x(x-1)(x+1)} = \frac{a}{x} + \frac{b}{x-1} + \frac{c}{x+1}.$$

Then

$$\begin{aligned} \frac{6x^2 + 3}{x(x-1)(x+1)} &= \frac{a(x-1)(x+1) + bx(x+1) + cx(x-1)}{x(x-1)(x+1)} \\ &= \frac{a(x^2-1) + b(x^2+x) + c(x^2-x)}{x(x-1)(x+1)} \\ &= \frac{ax^2 - a + bx^2 + bx + cx^2 - cx}{x(x-1)(x+1)} \end{aligned}$$

By identification we have $a = -3, b = \frac{9}{2} = c$. So

$$\frac{6x^2+3}{x(x-2)(x+2)} = -3.\frac{1}{x} + \frac{9}{2}.\frac{1}{x-1} + \frac{9}{2}.\frac{1}{x+1}.$$

Then

$$\begin{aligned}\int \frac{x^4+5x^2+3}{x^3-x}dx &= \int \left(x - \frac{3}{x} + \frac{9}{2(x-1)} + \frac{9}{2(x+1)}\right) dx \\ &= \frac{x^2}{2} - 3\ln|x| + \frac{9}{2}\ln|x-1| + \frac{9}{2}\ln|x+1| + c.\end{aligned}$$

Example 2 :

Evaluation of the following integral

$$\int \frac{x+16}{x^2+2x-8}dx.$$

We have $x^2+2x-8=(x-2)(x+4)$ and

$$\frac{x+16}{(x-2)(x+4)}=\frac{3}{x-2}-\frac{2}{x+4}.$$

Then

$$\int \frac{3}{x-2}-\frac{2}{x+4}dx=3\ln|x-2|-2\ln|x+2|+c.$$

Case of repeated linear factors of denominator:

Example 3 : Evaluation of the following integral

$$\int \frac{2x^2 - 25x - 33}{(x+1)^2(x-5)}dx.$$

A general theorem in algebra states that there is three real numbers a, b, c such that

$$\frac{2x^2 - 25x - 33}{(x+1)^2(x-5)} = \frac{a}{x+1} + \frac{b}{(x+1)^2} + \frac{c}{x-5}.$$

We find

$$\frac{2x^2 - 25x - 33}{(x+1)^2(x-5)} = \frac{5}{x+1} + \frac{1}{(x+1)^2} - \frac{3}{x-5}.$$

Then

$$I = 5\ln|x+1| - \frac{1}{\quad} - 3\ln|x-5| + c$$

Case of irreducible quadratic factor of denominator:

Example 4 :

Evaluate $\int \frac{x^2+3x+1}{x^4+5x^2+4}dx$.

We have

$$x^4+5x^2+4=(x^2+1)(x^2+4).$$

The polynomials x^2+1 and x^2+4 are irreducible.
A general theorem in algebra states that there is three real numbers a, b, c, d such that

$$\frac{x^2+3x+1}{(x^2+1)(x^2+4)}=\frac{ax+b}{x^2+1}+\frac{cx+d}{x^2+4}.$$

We have

$$\frac{x^2+3x+1}{(x^2+1)(x^2+4)} = \frac{x}{x^2+1} + \frac{-x+1}{x^2+4}$$

and

$$\begin{aligned}\int \frac{x^2+3x+1}{x^4+5x^2+4}dx &= \int \frac{x}{x^2+1} + \frac{-x+1}{x^2+4}dx \\ &= \int \frac{1}{2}\frac{2x}{x^2+1} - \frac{1}{2}\frac{2x}{x^2+4} + \frac{1}{x^2+2^2}dx \\ &= \frac{1}{2}\ln(x^2+1) - 12\ln(x^2+4) + \frac{1}{2}\tan^{-1}(\frac{x}{2}) + c.\end{aligned}$$

Examples 5 :

1. $$\int \frac{dx}{x^2+9} = \frac{1}{3}\tan^{-1}(\frac{x}{3}) + c.$$

2. $$\frac{1}{2}\int \frac{2x}{x^2+9}dx = \frac{1}{2}\ln(x^2+9) + c.$$

3. $$\begin{aligned} \int \frac{x^2}{x^2+9}dx &= \int \frac{(x^2+9)-9}{x^2+9}dx = \int 1 - \frac{9}{x^2+3^2}dx \\ &= x - 3\tan^{-1}(x/3) + c. \end{aligned}$$

4.

$$\begin{aligned} \int \frac{x+1}{x^2+2x-3}dx &= \frac{1}{2}\int \frac{dx}{x-1} + \frac{1}{2}\int \frac{dx}{x+3} \\ &= \frac{1}{2}\ln|x-1| + \frac{1}{2}\ln|x+3| + c. \end{aligned}$$

5.

$$\begin{aligned} \int \frac{x^2+3x-1}{x^3+x^2-2x}dx &= \frac{1}{2}\int \frac{dx}{x} + \int \frac{dx}{x-1} - \frac{1}{2}\int \frac{dx}{x+2} \\ &= \frac{1}{2}\ln|x| + \ln|x-1| - \frac{1}{2}\ln|x+2| + c. \end{aligned}$$

6.

$$\begin{aligned}\int \frac{x^2+2x+3}{(x-1)(x+1)^2}dx &= \frac{3}{2}\int \frac{dx}{x-1} - \frac{1}{2}\int \frac{dx}{x+1} - \int \frac{dx}{(x+1)^2} \\ &= \frac{3}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + \frac{1}{x+1} + c.\end{aligned}$$

7.

$$\begin{aligned}\int \frac{2x+5}{x^2+x+1}dx &= \int \frac{2x+1}{x^2+x+1}dx + \int \frac{4}{x^2+x+1}dx \\ &= \ln(x^2+x+1) + 2\int \frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}} \\ &= \ln(x^2+x+1) + \frac{8}{\sqrt{3}}\tan^{-1}(\frac{2x+1}{\sqrt{3}}) + c.\end{aligned}$$

8.

$$\begin{aligned}\int \frac{2x^2 - x + 2}{x(x^2+1)^2}dx &= \int \frac{2dx}{x} - \int \frac{2xdx}{x^2+1} - \int \frac{dx}{(x^2+1)^2} \\ &= \ln x^2 - \ln(x^2+1) + \frac{1}{2}\tan^{-1}(x) + \frac{x}{2(x^2+1)}\end{aligned}$$

Integrals Involving Quadratic Expressions

Example 1 : Evaluation of the following integral

$$\int \frac{2x-5}{x^2-6x+13}\,dx.$$

Using the method of completing square, we have

$x^2-6x+13=(x-3)^2+4.$

Then

$$\begin{aligned}
\int \frac{2x-5}{x^2-6x+13}\,dx &= \int \frac{2x-5}{(x-3)^2+4}\,dx \\
&\overset{2u=x-3}{=} \frac{1}{2}\int \frac{2(2u+3)-5}{u^2+1}\,du \\
&= \frac{1}{2}\int \frac{4u}{u^2+1}+\frac{1}{u^2+1}\,du \\
&= \ln(u^2+1)+\frac{1}{2}\tan^{-1}(u)+c \\
&= \ln((x-3)^2+4)+\frac{1}{2}\tan^{-1}\left(\frac{x-3}{2}\right)+c.
\end{aligned}$$

Example 2 : Complete the square in the following cases:

$$x^2 + 2x + 5,\ x^2 + x + 1,\ x^2 - x + 1,\ x^2 + 5x.$$

$$x^2 + 2x + 5 = (x^2 + 2x + 1) - 1 + 5 = (x + 1)^2 + 4.$$

$$x^2 + x + 1 = (x^2 + 2.\frac{1}{2}.x + (\frac{1}{2})^2) - (\frac{1}{2})^2 + 1, = (x + \frac{1}{2})^2 + \frac{3}{4}.$$

$$x^2 - x + 1 = (x^2 - 2.\frac{1}{2}.x + (-\frac{1}{2})^2) - (-\frac{1}{2})^2 + 1 = (x - \frac{1}{2})^2 + \frac{3}{4}.$$

$$x^2 + 5x = (x^2 + 2.\frac{5}{2}.x + (\frac{5}{2})^2) - (\frac{5}{2})^2 = (x + \frac{5}{2})^2 - \frac{25}{4}.$$

Example 3 :

Evaluation of the following integral $\int \frac{dx}{x^2 - 2x + 2}$.

$$\int \frac{dx}{x^2 - 2x + 2} \overset{u=x-1}{=} \int \frac{du}{u^2 + 1} = \tan^{-1}(u) + c = \tan^{-1}(x-1) + c.$$

Example 4 :

Evaluation of the following integral $\int \frac{1}{\sqrt{7+6x-x^2}}dx$.

Using the method of completing square, we get

$$\begin{aligned}\int \frac{1}{\sqrt{7+6x-x^2}}dx &= \int \frac{1}{\sqrt{16-(x-3)^2}}dx \\ &\overset{u=x-3}{=} \int \frac{1}{\sqrt{4^2-u^2}}du \\ &= \sin^{-1}(\frac{u}{4})+c = \sin^{-1}(\frac{x-3}{4})+c.\end{aligned}$$

Example 5 :

Evaluation of the following integral $\int \frac{1}{(x^2+6x+13)^{\frac{3}{2}}}dx$.

Using the method of completing square , we get

$$\begin{aligned}\int \frac{1}{(x^2+6x+13)^{\frac{3}{2}}}dx &= \int \frac{1}{((x+3)^2+4)^{\frac{3}{2}}}dx \\ &\overset{u=x+3}{=} \int \frac{dx}{((x+3)^2+4)^{\frac{3}{2}}} \\ &= \int \frac{du}{(u^2+2^2)^{\frac{3}{2}}}.\end{aligned}$$

Put $u = 2\tan(\theta), \theta \in (-\frac{\pi}{2}; \frac{\pi}{2})$, then

$$\begin{aligned}\int \frac{1}{(x^2+6x+13)^{\frac{3}{2}}}dx &= \int \frac{2\sec^2(\theta)}{8\sec^3(\theta)}d\theta = \frac{1}{4}\int \frac{d\theta}{\sec(\theta)} \\ &= \frac{1}{4}\int \cos(\theta)d\theta = \frac{1}{4}\sin(\theta)+c.\end{aligned}$$

We have $\sqrt{u^2+2^2} = 2\sec(\theta)$ and $\sin(\theta) = \frac{u}{\sqrt{u^2+2^2}}$. Therefore

$$\int \frac{1}{(x^2+6x+13)^{\frac{3}{2}}}dx = \frac{1}{4}\frac{u}{\sqrt{u^2+4}}+c = \frac{1}{4}\frac{x+3}{\sqrt{(x+3)^2+4}}+c.$$

Example 6 :

Evaluation of the following integral $\int \sqrt{x^2+10x}\,dx$, $x>0$.

Using the method of completing square, we get

$$\begin{aligned}\int \sqrt{x^2+10x}dx &= \int \sqrt{(x+5)^2-5^2}dx \\ &\overset{u=x+5}{=} \int \sqrt{u^2-5^2}du \\ &\overset{u=5\sec(\theta)}{=} \int 5\tan(\theta)5\sec(\theta)\tan(\theta)d\theta,\end{aligned}$$

where $\theta \in [0; \frac{\pi}{2})$. Then

$$\begin{aligned}
I &= 25\int \sec(\theta)\tan^2(\theta)d\theta \\
&= 25\int \sec(\theta)(\sec^2(\theta)-1)d\theta \\
&= 25\int \sec^3(\theta)-\sec(\theta)d\theta \\
&= 25\int \sec^3(\theta)d\theta - 25\int \sec(\theta)d\theta \\
&= 25\Big(\frac{1}{2}\sec(\theta)\tan(\theta)+\frac{1}{2}\ln|\sec(\theta)+\tan(\theta)|\Big) \\
&\quad -25\ln|\sec(\theta)+\tan(\theta)|+c \\
&= \frac{25}{2}\sec(\theta)\tan(\theta)-\frac{25}{2}\ln|\sec(\theta)+\tan(\theta)|+c.
\end{aligned}$$

As

$$\sec(\theta) = \frac{u}{5} \quad \text{and} \quad \tan(\theta) = \frac{\sqrt{u^2 - 25}}{5}$$

then

$$\int \sqrt{x^2 + 10x}dx = \frac{1}{2}u\sqrt{u^2 - 25} - \frac{25}{2}\ln\left|\frac{u}{5} + \frac{\sqrt{u^2 - 25}}{5}\right| + c.$$

Using the fact $u = x + 5$, we get

$$I = \frac{1}{2}(x + 5)\sqrt{x^2 + 10x} - \frac{25}{2}\ln\left|\frac{x + 5}{5} + \frac{\sqrt{x^2 + 10x}}{5}\right| + c.$$

Example 7 :

$$\begin{aligned}
\int \frac{x+3}{\sqrt{4-x^2}}dx &= \int \frac{x}{\sqrt{4-x^2}} + \frac{3}{\sqrt{4-x^2}}dx \\
&= \int -(\frac{1}{2})(-2x)(4-x^2)^{-1/2} + 3\frac{1}{\sqrt{2^2-x^2}}dx \\
&= -(4-x^2)^{1/2} + 3\sin^{-1}(x/2) + c.
\end{aligned}$$

Integrals of Rational Functions of $\sin(x)$ and $\cos(x)$

In this section, we treat the integrals of the following form

$$\int \frac{P\big(\cos(x); \sin(x)\big)}{Q\big(\cos(x); \sin(x)\big)} dx$$

where $P(X;Y)$ and $Q(X;Y)$ are two polynomial functions in X, Y.

Method: Generally we use the following substitution

$$u = \tan(\frac{x}{2}) \Rightarrow du = \frac{1}{2}\sec^2(\frac{x}{2})dx = \frac{1}{2}\big(1+\tan^2(\frac{x}{2})\big)dx \Rightarrow dx = \frac{2du}{1+u^2}.$$

We have

$$\sin(x) = \frac{2u}{1+u^2}, \qquad \cos(x) = \frac{1-u^2}{1+u^2}.$$

Indeed:

$$\begin{aligned}\sin(x) &= \sin(2.\frac{x}{2}) = 2\sin(\frac{x}{2})\cos(\frac{x}{2}) \\ &= \frac{\tan(\frac{x}{2})}{\sec^2(\frac{x}{2})} = \frac{2u}{1+u^2}.\end{aligned}$$

and

$$\begin{aligned}\cos(x) &= \cos^2(\frac{x}{2}) - \sin^2(\frac{x}{2}) = \cos^2(\frac{x}{2})(1-\tan^2(\frac{x}{2})) \\ &= \frac{1-\tan^2(\frac{x}{2})}{1+\tan^2(\frac{x}{2})} = \frac{1-u^2}{1+u^2}.\end{aligned}$$

Example 1 : Evaluation of the following integral $\displaystyle\int \frac{dx}{2+\sin(x)}$.

We set $u = \tan(\frac{x}{2})$, then

$$\begin{aligned}
\int \frac{dx}{2+\sin(x)} &= \int \frac{1}{2+\dfrac{2u}{1+u^2}} \cdot \frac{2}{1+u^2} du \\
&= \int \frac{du}{u^2+u+1} \\
&= \int \frac{du}{(u+\frac{1}{2})^2+\frac{3}{4}} \\
&= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2\tan(\frac{x}{2})+1}{\sqrt{3}}\right) + c.
\end{aligned}$$

Miscellaneous Substitutions

Example 1 :

Evaluation of the integral $\int \frac{\sqrt{x}}{1+\sqrt[3]{x}}dx$.

We set $x = u^6$,

$$\int \frac{\sqrt{x}}{1+\sqrt[3]{x}}dx = \int \frac{u^3}{1+u^2}6u^5du = 6\int \frac{u^8}{1+u^2}du.$$

Therefore

$$\begin{aligned}\int \frac{\sqrt{x}}{1+\sqrt[3]{x}}dx &= 6\int u^6 - u^4 + u^2 - 1 + \frac{1}{1+u^2}du \\ &= \frac{6}{7}x^{\frac{7}{6}} - \frac{6}{5}x^{\frac{5}{6}} + 2x^{\frac{1}{2}} - 6x^{\frac{1}{6}} + 6\tan^{-1}(x^{\frac{1}{6}}) + c.\end{aligned}$$

Example 2 :

Evaluation of the integral $\displaystyle\int_0^4 \frac{2x+3}{\sqrt{1+2x}}dx$.

First compute the indefinite integral $\displaystyle\int \frac{2x+3}{\sqrt{1+2x}}dx$. For this set $u = 1+2x$ then

$$\begin{aligned}
\int \frac{2x+3}{\sqrt{1+2x}}dx &\overset{x=\frac{u-1}{2}}{=} \frac{1}{2}\int \frac{2(\frac{u-1}{2})+3}{u^{\frac{1}{2}}}du \\
&= \frac{1}{2}\int \frac{u+2}{u^{\frac{1}{2}}}du \\
&= \frac{1}{3}(1+2x)^{\frac{3}{2}} + (1+2x)^{\frac{1}{2}} + c.
\end{aligned}$$

Therefore

Improper Integrals

Definition

Let f be a piecewise continuous function on the interval $[a, b[$, where $a \in \mathbb{R}$, $b \in \mathbb{R} \cup \{+\infty\}$.
We say that the integral of f on the interval $[a, b[$ is convergent if the function $F(x) = \int_a^x f(t)dt$ defined on $[a, b[$ has a finite limit when x tends to b ($x < b$). This limit is called the improper integral of f on $[a, b[$ and will be denoted by: $\int_a^b f(x)dx$.

Example 1 :

1. $\displaystyle\int_0^1 \frac{e^{\sin^{-1}(x)}}{\sqrt{1-x^2}}dx \overset{u=\sin^{-1}(x)}{=} \int_0^{\frac{\pi}{2}} e^u du = e^{\frac{\pi}{2}} - 1$. This integral is convergent.
2. $\displaystyle\int_0^{+\infty} x^n e^{-x} dx = n!$. This integral is convergent.
3. $\displaystyle\int_0^{+\infty} \frac{x}{1+x^2}dx = \frac{1}{2}\left[\ln(1+x^2)\right]_0^{+\infty} = +\infty$. This integral is divergent.

Definition

Let f a piecewise continuous function on the interval $]a, b]$, where $a \in \mathbb{R} \cup \{-\infty\}$, $b \in \mathbb{R}$.
We say that the integral of f on the interval $]a, b]$ is convergent if the function $G(x) = \int_x^b f(t)dt$ defined on $]a, b]$ has a finite limit when x tends to a ($x > a$). This limit is called the improper integral of f on $]a, b]$ and will be denoted by: $\int_a^b f(x)dx$.

Example 2 :

1. $\int_0^1 \ln(x)dx = [x\ln(x) - x]_0^1 = -1$. This integral is convergent.
2. $\int_{-\infty}^0 \frac{1}{(x-3)^2}dx = \left[\frac{-1}{x-3}\right]_{-\infty}^0 = \frac{1}{3}$. This integral is convergent.
3. $\int_0^1 \frac{dx}{x\ln(x)} = [\ln(\ln(x))]_0^1 = -\infty$. This integral is divergent.

Definition

Let f be a piecewise continuous function on the interval $]a, b[$, where $a \in \mathbb{R} \cup \{-\infty\}$, $b \in \mathbb{R} \cup \{+\infty\}$.
We say that the integral of f on the interval $]a, b[$ is convergent if the integral of f is convergent on $]a, c]$ and on $[c, b[$ for any c in $]a, b[$.

Example 3 :

1. $\displaystyle\int_{-\infty}^{+\infty} \frac{e^{\tan^{-1}(x)}}{1+x^2}dx \overset{u=\tan^{-1}(x)}{=} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^u du = 2\sinh(\frac{\pi}{2})$. This integral is convergent.
2. $\displaystyle\int_0^1 \frac{\ln x}{(1-x)^{\frac{3}{2}}}dx \overset{x=1-t^2}{=} 2\int_0^1 \frac{\ln(1-t^2)}{t^2}dt = 2-2\ln 2$. This integral is convergent.
3. $\displaystyle\int_{-\infty}^{+\infty} e^x dx = [e^x]_{-\infty}^{+\infty} = +\infty$. This integral is divergent.

Definition

Let f be a piecewise continuous function on an interval I. The function is called integrable on I (or the integral is absolutely convergent) if the integral of $|f|$ on the interval I is convergent.

Examples 6 :

1. $\displaystyle\int_0^{+\infty} \frac{dx}{1+x} = +\infty.$ This integral is divergent.
2. $\displaystyle\int_0^{+\infty} \frac{dx}{1+x^2} = \frac{\pi}{2}.$ This integral is divergent.
3. $\displaystyle\int_0^{1} \frac{dx}{\sqrt{x}} = 2.$ This integral is divergent.
4. Let $\alpha \in \mathbb{R}$ and $a \in \mathbb{R}_+^*$. The integral $\int_a^{+\infty} \frac{dx}{x^\alpha}$ is convergent if and only if $\alpha > 1$ and the integral $\displaystyle\int_0^{a} \frac{dx}{x^\alpha}$ is convergent if and only if $\alpha < 1$.

5. The integral $\int_0^{+\infty} \sin(x)$ is divergent since $\int_0^x \sin(t)dt = 1 - \cos(x)$ dont have a limit when x tends to $+\infty$.
6. The integral $\int_0^1 \frac{\sin t}{t}$ is convergent since the function $\frac{\sin t}{t}$ can be considered continuous on the interval $[0, 1]$.
7. The integral $\int_0^1 \sin(\frac{1}{t})$ is convergent since the function $\sin(\frac{1}{t})$ is continuous on the interval $]0, 1]$ and bounded.

8. $\displaystyle\int_0^1 \frac{x\ln x}{(1-x^2)^{3/2}}dx$, by integration by parts

$$\int_0^1 \frac{x\ln x}{(1-x^2)^{3/2}}dx \overset{x^2=1-t^2}{=} \frac{1}{2}\int_0^1 \frac{\ln(1-t^2)}{t^2}dt = -\ln 2.$$

9. $\displaystyle\int_0^{\frac{\pi}{2}} \sin 2x \ln(\tan x)dx$, by integration by parts

$$\int_0^{\frac{\pi}{2}} \sin 2x \ln(\tan x)dx = \sin^2 \ln(\tan x) + \ln(\cos x)\Big]_0^{\frac{\pi}{2}} = 0.$$

10. $\int_0^{\frac{\pi}{2}} \cos x \ln(\tan x) dx =$
$\int_0^{\frac{\pi}{4}} \cos x \ln(\tan x) dx - \int_0^{\frac{\pi}{4}} \sin x \ln(\tan x) dx.$
By integration by parts, we have
$\int_0^{\frac{\pi}{4}} \cos x \ln(\tan x) dx = -\int_0^{\frac{\pi}{4}} \frac{dx}{\cos x} = -\ln(1+\sqrt{2}).$

$$\begin{aligned} -\int_0^{\frac{\pi}{4}} \sin x \ln(\tan x) dx &= \left[(\cos x - 1)\ln(\sin x) + \ln(1+\cos x)\right]_0^{\frac{\pi}{4}} \\ &= -\frac{\sqrt{2}}{4}\ln 2 - \ln 2 + \ln(1+\sqrt{2}). \end{aligned}$$

Then $\int_0^{\frac{\pi}{2}} \cos x \ln(\tan x) dx = -\frac{\sqrt{2}}{4}\ln 2 - \ln 2.$

11. $$\int_1^{+\infty} \frac{x^4+1}{x^3(x+1)(1+x^2)}dx = \int_1^{+\infty}\left(\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3} - \frac{1}{x+1} + 2\frac{x+1}{1+x^2}\right)dx = \frac{\pi}{2} - \frac{1}{2}.$$

12. $$\int_1^{+\infty} \frac{dx}{x\sqrt[4]{1+x^2}} \overset{t^4=1+x^2}{=} \int_{2^{\frac{1}{4}}}^{+\infty} \frac{2t^2dt}{(t^4-1)} = \int_{2^{\frac{1}{4}}}^{+\infty}\left(\frac{1}{2(t-1)} - \frac{1}{2(t+1)} + \frac{1}{1+t^2}\right)dt = \frac{1}{2}\ln\left(\frac{2^{\frac{1}{4}}+1}{2^{\frac{1}{4}}-1}\right) + \frac{\pi}{2} - \tan^{-1}(2^{\frac{1}{4}}).$$

13. $$\int_{-1}^{0} \frac{dx}{\sqrt{4-x^2}} = \left[\sin^{-1}\left(\frac{x}{2}\right)\right]_{-1}^{0} = \frac{\pi}{2}.$$

14. $$\int_{-2}^{0} \frac{dx}{\sqrt[3]{x+1}} = \int_{-2}^{-1} \frac{dx}{\sqrt[3]{x+1}} + \int_{-1}^{0} \frac{dx}{\sqrt[3]{x+1}} = \frac{3}{2}\left[(x+1)^{\frac{2}{3}}\right]_{-2}^{0} = 0.$$

15. $$\int_{-3}^{1} \frac{dx}{x^2} = \int_{-3}^{0} \frac{dx}{x^2} + \int_{0}^{1} \frac{dx}{x^2} = +\infty.$$