INTEGRAL CALCULUS (MATH 106)

BEN AMIRA Aymen

abenamira@ksu.edu.sa

Department of Mathematics

College of Sciences

King Saud University

Chapter 2: Logarithmic and Exponential Functions

The student is expected to be able to:

1. Find the derivative and integrals natural exponential and logarithmic functions.
2. Find the derivative and integrals general exponential and logarithmic functions.

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The Natural Logarithmic Function

In The first chapter we said that:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad \text{where } n \neq -1, \;\; n \in \mathbb{Q}$$

In this chapter, we willl see what about the Value $n = -1$.
The problem that we should solve it is: what is the antiderivative of the function $\frac{1}{x} = x^{-1}$?

$$\int x^{-1} dx = \int \frac{1}{x} dx$$

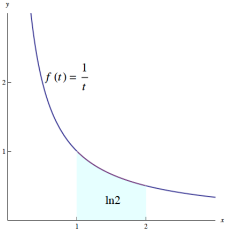
The Natural Logarithmic Function

Definition

For $x > 0$, the natural logarithmic function is defined by $\ln x = \int_1^x \frac{1}{t}\,dt$

Remark

The domain of the function ln x is the open interval $(0, \infty)$ $(x > 0)$



Example

Find the domain of the following functions:

1. $\ln(x-2)$.
2. $\ln(x+5)$
3. $\ln\frac{x}{4}$.

Solution

1. We shoold have $x-2>0 \Longrightarrow x>2 \Longrightarrow$ the domain is $(2,\infty)$.
2. We shoold have $x+5>0 \Longrightarrow x>-5 \Longrightarrow$ the domain is $(-5,\infty)$.
3. We shoold have $\frac{x}{4}>0 \Longrightarrow x>0 \Longrightarrow$ the domain is $(0,\infty)$.

Properties of natural logarithmic function

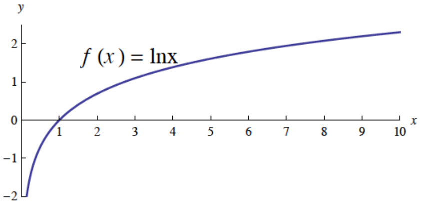
Notes

1. If $x > 1$ then $\ln x > 0$
2. $\ln 1 = 0$
3. If $0 < x < 1$ then $\ln x < 0$

Properties of natural logarithmic function

The graph of $\ln x$

1. First derivative test :
$$\frac{d}{dx}\ln x = \frac{d}{dx}\int_1^x \frac{1}{t}dt = \frac{1}{x} > 0 \text{ for every } x \in (0, \infty)$$
Hence $\ln x$ is an increasing function on $(0, \infty)$
2. Second derivative test :
$\frac{d^2}{dx^2}\ln x = \frac{d}{dx}\frac{1}{x} = -\frac{1}{x^2} < 0$ for every $x \in (0, \infty)$
Hence $\ln x$ is a convex function on $(0, \infty)$



Properties of natural logarithmic function

Notes

1. $\lim_{x \to \infty} \ln x = \infty$
2. $\lim_{x \to 0+} \ln x = -\infty$

Derivative of $\ln x$

1. $\frac{d}{dx} \ln |x| = \frac{1}{x}$
2. $\frac{d}{dx} \ln |f(x)| = \frac{f'(x)}{f(x)}$

Note

$\ln |x|$ is the antiderivative of $\frac{1}{x}$

Properties of natural logarithmic function

Integration

1. $\int \frac{1}{x} dx = \ln|x| + c$
2. $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$

Some properties of $\ln x$

1. $\ln(xy) = \ln x + \ln y$
2. $\ln(\frac{x}{y}) = \ln x - \ln y$
3. $\ln x^r = r \ln x$

Example

1. Simplify $\frac{1}{5}[2ln|x+1|+ln|x|-ln|x^2-2|]$.
$\frac{1}{5}[2ln|x+1|+ln|x|-ln|x^2-2|]=\frac{1}{5}[ln(x+1)^2+ln|x|-ln|x^2-2|]=$
$\frac{1}{5}[ln|x(x+1)^2|-ln|x^2-2|]=\frac{1}{5}ln\left|\frac{x(x+1)^2}{x^2-2}\right|$.

2. If $y=\sqrt{\frac{(x+1)^4(x+2)^3}{(x-1)^2}}$, Find y'.
$\ln y=\ln\sqrt{\frac{(x+1)^4(x+2)^3}{(x-1)^2}}=\frac{1}{2}[4\ln|x+1|+3\ln|x+2|-2\ln|x-1|]$.
Differentiate both sides: $\frac{y'}{y}=\frac{1}{2}\left[4\frac{1}{x+1}+3\frac{1}{x+2}-2\frac{1}{x-1}\right]$.
Hence $y'=\frac{1}{2}\sqrt{\frac{(x+1)^4(x+2)^3}{(x-1)^2}}\left[\frac{4}{x+1}+\frac{3}{x+2}-\frac{2}{x-1}\right]$

Exercise

If $f(x) = \dfrac{x^2(2x-1)^3}{(x+5)^2}$, then find $f'(x)$.

More Basic Rules of Integration :

1. $\int \tan x \; dx = \ln|\sec x| + c$
2. $\int \cot x \; dx = \ln|\sin x| + c$
3. $\int \sec x \; dx = \ln|\sec x + \tan x| + c$
4. $\int \csc x \; dx = \ln|\csc x - \cot x| + c$

Examples

1. $$\int \frac{x^2+2x+3}{x^3+3x^2+9x}\,dx = \frac{1}{3}\int \frac{3x^2+6x+9}{x^3+3x^2+9x}\,dx = \frac{1}{3}\ln\left|x^3+3x^2+9x\right| + c$$

2. $$\int \frac{x^2+2x+3}{(x^3+3x^2+9x)^5}\,dx = \frac{1}{3}\int (3x^2+6x+9)^{-5}(x^3+3x^2+9x)\,dx = \frac{1}{3}(x^3+3x^2+9x)^{-4} + c$$

3. $$\int \frac{1}{x\sqrt{\ln x}}\,dx = \int (\ln x)^{-\frac{1}{2}}\frac{1}{x}\,dx = \frac{(\ln x)^{\frac{1}{2}}}{\frac{1}{2}} + c$$

Examples

Examples

4. $$\int \frac{1}{x\ln(\sqrt{x})}\,dx = \int \frac{1}{x\frac{1}{2}\ln x}\,dx = 2\int \frac{\frac{1}{x}}{\ln x}\,dx = \ln|\ln x| + c$$

5. $$\int \frac{x-1}{x+1}\,dx = \int \frac{(x+1)-2}{x+1}\,dx = \int \left(\frac{x+1}{x+1} - \frac{2}{x+1}\right)\,dx =$$
$$\int \left(1 - \frac{2}{x+1}\right)\,dx = \int 1dx - 2\int \frac{1}{x+1}\,dx = x - 2\ln|x+1| + c$$

6. Find $g(x)$ if $\int [\ln|x|]^2 g(x)dx = \frac{2}{3}[\ln|x|]^3 + c$.
$$[\ln|x|]^2 g(x) = \frac{d}{dx}\left(\frac{2}{3}[\ln|x|]^3 + c\right)$$
$$[\ln|x|]^2 g(x) = 2[\ln|x|]^2 \frac{1}{x}.$$
Hence $g(x) = \frac{2}{x}$.

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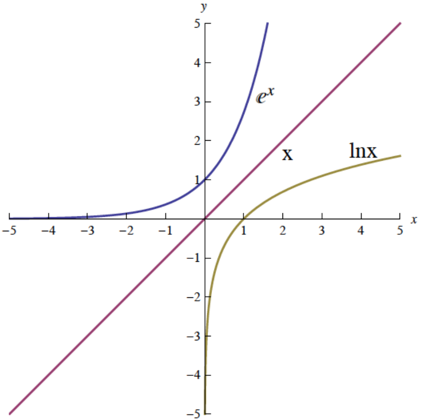
The natural exponential function

Definition

The natural exponential function is the inverse of the natural logarithmic function, and it is denoted by e^x



Properties of the natural exponential function

Notes

1. The domain of the function e^x is $\mathbb{R}$
2. The range of the function e^x is the open interval $(0, \infty)$
3. $e^x > 0$ for every $x \in \mathbb{R}$
4. $e^0 = 1$,
5. $e \approx 2.71828$ and $\ln e = 1$.
6. $\lim_{x \to \infty} e^x = \infty$
7. $\lim_{x \to -\infty} e^x = 0$
8. $\ln(e^x) = x$ and $e^{\ln x} = x$

Properties of the natural exponential function

Some properties of the natural exponential function

If $x, y \in \mathbb{R}$, then

1. $e^x e^y = e^{x+y}$
2. $\dfrac{e^x}{e^y} = e^{x-y}$
3. $(e^x)^y = e^{xy}$

Properties of the natural exponential function

Examples

1. Find the value of x that satisfies the equation $\ln \frac{1}{x} = 2$.

   Answer: $\ln \frac{1}{x} = 2 \Longrightarrow \ln(x^{-1}) = 2 \Longrightarrow -\ln x = 2 \Longrightarrow \ln x = -2$
   $\Longrightarrow e^{\ln x} = e^{-2} \Longrightarrow x = e^{-2} = \frac{1}{e^2}$

2. Find the value of x that satisfies the equation $e^{5x+3} = 4$.

   Answer:
   $e^{5x+3} = 4 \Longrightarrow \ln\left(e^{5x+3}\right) = \ln 4 \Longrightarrow 5x + 3 = \ln 4 \Longrightarrow x = \dfrac{-3 + \ln 4}{5}$

3. Simplify $\ln\left((e^x)^2\right)$

   Answer: $\ln (e^x)^2 = \ln\left(e^{2x}\right) = 2x$

Properties of the natural exponential function

Derivative of the natural exponential function

1. $\dfrac{d}{dx}e^x = e^x,$
2. $\dfrac{d}{dx}e^{f(x)} = e^{f(x)}f'(x)$

Integration

1. $\displaystyle\int e^x dx = e^x + c$
2. $\displaystyle\int e^{f(x)}f'(x)dx = e^{f(x)} + c$

Properties of the natural exponential function

Examples

1. Find $f'(x)$ If $f(x) = e^{5x} + \frac{1}{e^x}$
   $f'(x) = 5e^{5x} - e^{-x}$
2. $\displaystyle\int \frac{e^{-x}}{(1-e^{-x})^2}dx = \int (1-e^{-x})^{-2}e^{-x}dx = \frac{(1-e^{-x})^{-1}}{-1} + c$
3. $\displaystyle\int \frac{e^{\frac{3}{x}}}{x^2}dx = -\frac{1}{3}\int e^{\frac{3}{x}}\frac{-3}{x^2}dx = -\frac{1}{3}e^{\frac{3}{x}} + c.$
4. $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}} = 2\int e^{\sqrt{x}}\frac{1}{2\sqrt{x}} = 2e^{\sqrt{x}} + c$
5. $\displaystyle\int_1^e \frac{\sqrt[3]{\ln x}}{x}dx = \int_1^e (\ln x)^{\frac{1}{3}}\frac{1}{x}dx = \left[\frac{(\ln x)^{\frac{4}{3}}}{\frac{4}{3}}\right]_1^e$
   $\displaystyle= \frac{3}{4}(\ln e)^{\frac{4}{3}} - \frac{3}{4}(\ln 1)^{\frac{4}{3}} = \frac{3}{4}$

Properties of the natural exponential function

Examples

6. $\displaystyle\int \frac{e^{\sin x}}{\sec x}dx = \int e^{\sin x}\cos x dx = e^{\sin x} + c$
7. Find $g(x)$ if $\displaystyle\int e^{3x^2} g(x)dx = -e^{3x^2} + c$

   $e^{3x^2} g(x) = \dfrac{d}{dx}\left[-e^{3x^2} + c\right] \Longrightarrow e^{3x^2} g(x) = -e^{3x^2}(6x)$

   $\Longrightarrow e^{3x^2} g(x) = -6xe^{3x^2}$.

   Hence $g(x) = -6x$.
8. $\displaystyle\int e^{(x^2+\ln x)}dx = \int e^{x^2} e^{\ln x}dx = \int e^{x^2} x dx = \frac{1}{2}\int e^{x^2} 2x dx = \frac{1}{2}e^{x^2} + c$

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The general exponential function

Definition(The general exponential function)

It has the form a^x where $a > 0$ and $a \neq 1$.
Note: $a^x = e^{x \ln a}$

Derivative of the general exponential function

1. $\frac{d}{dx} a^x = a^x \ln a,$
2. $\frac{d}{dx} a^{f(x)} = a^{f(x)} f'(x) \ln a$

Integration of the general exponential function

1. $\displaystyle\int a^x dx = \frac{a^x}{\ln a} + c,$
2. $\displaystyle\int a^{f(x)} f'(x) dx = \frac{a^{f(x)}}{\ln a} + c.$

The general logarithmic function

Definition(The general logarithmic function)

The general logarithmic function of base a where $a > 0$ and $a \neq 1$ is denoted by $\log_a x$ and it is the inverse function of the general exponential function a^x

Note

1. $\log_a x = y \Leftrightarrow a^y = x$
2. $\log_a x = \dfrac{\ln x}{\ln a}$

Notations

1. $\log x = \log_{10} x$
2. $\ln x = \log_e x$

The general logarithmic function

Derivative of the general logarithmic function

1. $\dfrac{d}{dx} \log_a |x| = \dfrac{1}{x} \dfrac{1}{\ln a}$
2. $\dfrac{d}{dx} \log_a |f(x)| = \dfrac{f'(x)}{f(x)} \dfrac{1}{\ln a}$

The general logarithmic function

Examples

1. Find the value of x if $\log_2 x = 3$?
   $\log_2 x = 3 \Leftrightarrow x = 2^3 = 8.$
2. Find y' if $y = (\sin x)^x$
   $y = (\sin x)^x \Rightarrow \ln y = \ln(\sin x)^x = x \ln|\sin x|$
   Differentiate both sides : $\dfrac{y'}{y} = \ln|\sin x| + x\dfrac{\cos x}{\sin x} = \ln|\sin x| + x \cot x$
3. $\displaystyle\int x^2 6^{x^3} dx = \frac{1}{3}\int 6^{x^3}(3x^2)\ dx = \frac{6^{x^3}}{3\ln 6} + c$

The general logarithmic function

Exercises

1. Find $f'(x)$ if $f(x) = (x^2 + 1)^x$
2. Evaluate $\int \frac{3^{\sqrt{x}}}{\sqrt{x}}$?