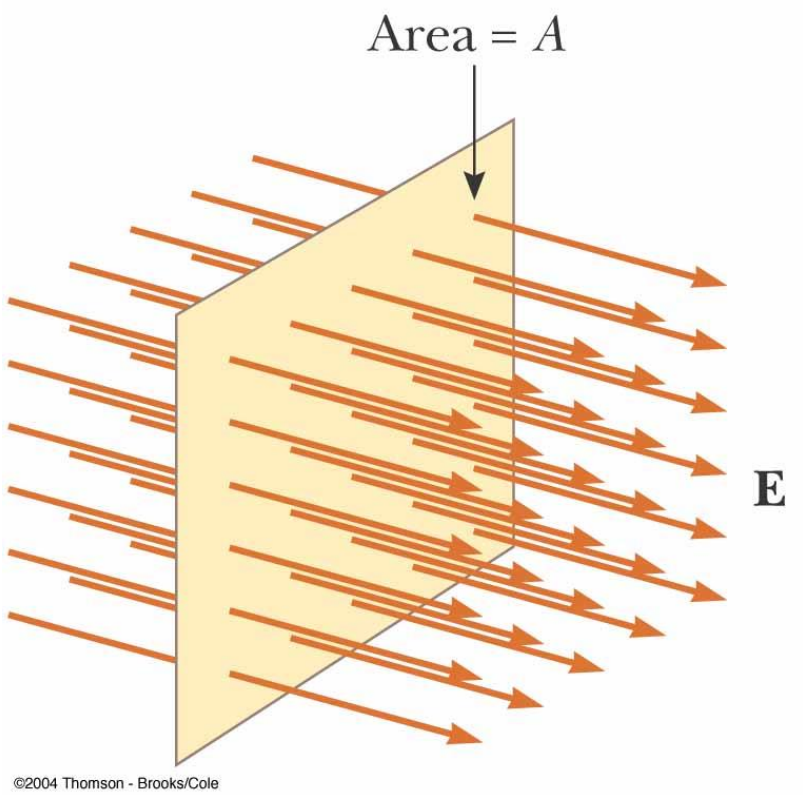
Chapter 24: Gauss’s Law

Electric Flux

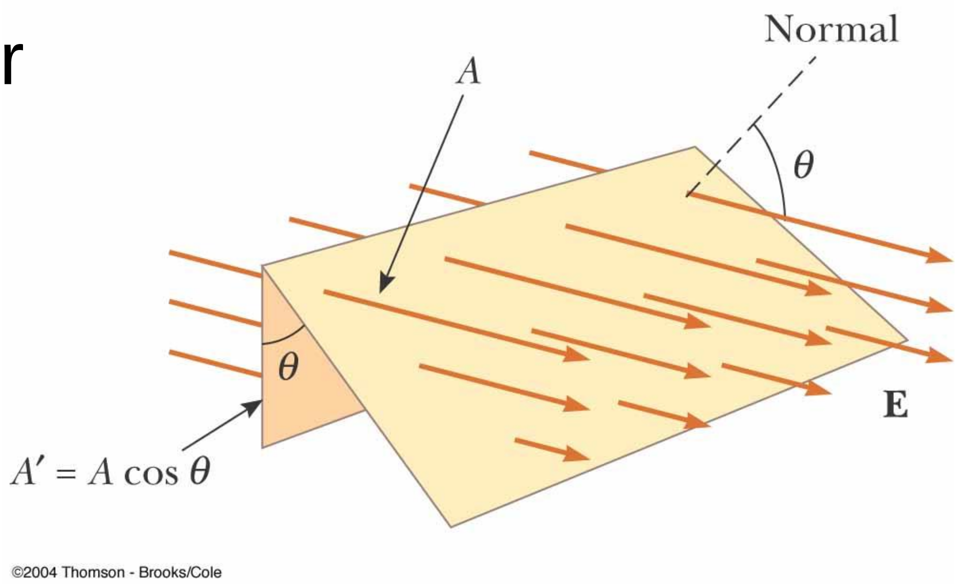
Electric Flux

- ***Electric flux*** is the product of the magnitude of the electric field and the surface area, *A*, perpendicular to the field
- $\Phi_E = EA$



Electric Flux, General Area

- The field lines may make some angle θ with the perpendicular to the surface
- Then $\Phi_E = EA \cos \theta$



Electric Flux, Interpreting the Equation

- The flux is a maximum when the surface is perpendicular to the field
- The flux is zero when the surface is parallel to the field
- If the field varies over the surface, $\Phi = EA \cos \theta$ is valid for only a small element of the area

Example

EXAMPLE 24.1 **Flux Through a Sphere**

What is the electric flux through a sphere that has a radius of 1.00 m and carries a charge of $+1.00\ \mu C$ at its center?

Solution The magnitude of the electric field 1.00 m from this charge is given by Equation 23.4,

$$E = k_e \frac{q}{r^2} = (8.99 \times 10^9\ \text{N}\cdot\text{m}^2/\text{C}^2)\ \frac{1.00 \times 10^{-6}\ \text{C}}{(1.00\ \text{m})^2}$$

$$= 8.99 \times 10^3\ \text{N/C}$$

Example

$$\Phi_E = EA = (8.99 \times 10^3\ \mathrm{N/C})(12.6\ \mathrm{m^2})$$

$$= 1.13 \times 10^5\ \mathrm{N \cdot m^2/C}$$

Exercise What would be the (a) electric field and (b) flux through the sphere if it had a radius of 0.500 m?

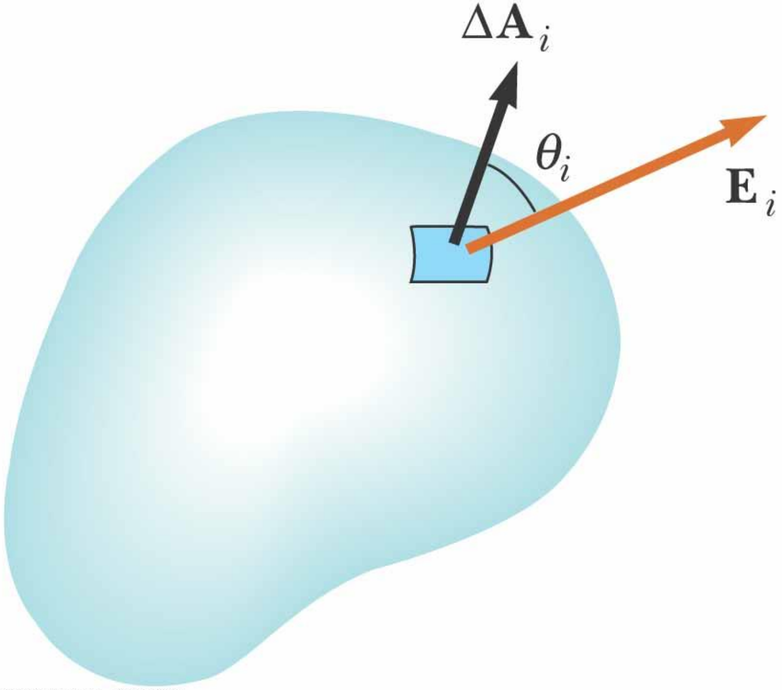
Answer (a) 3.60×10^4 N/C; (b) $1.13 \times 10^5\ \mathrm{N \cdot m^2/C}$.

Electric Flux, General

- In the more general case, look at a small area element

$$\Delta\Phi_E = E_i \Delta A_i \cos\theta_i = \mathbf{E}_i \cdot \Delta\mathbf{A}_i$$

- In general, this becomes



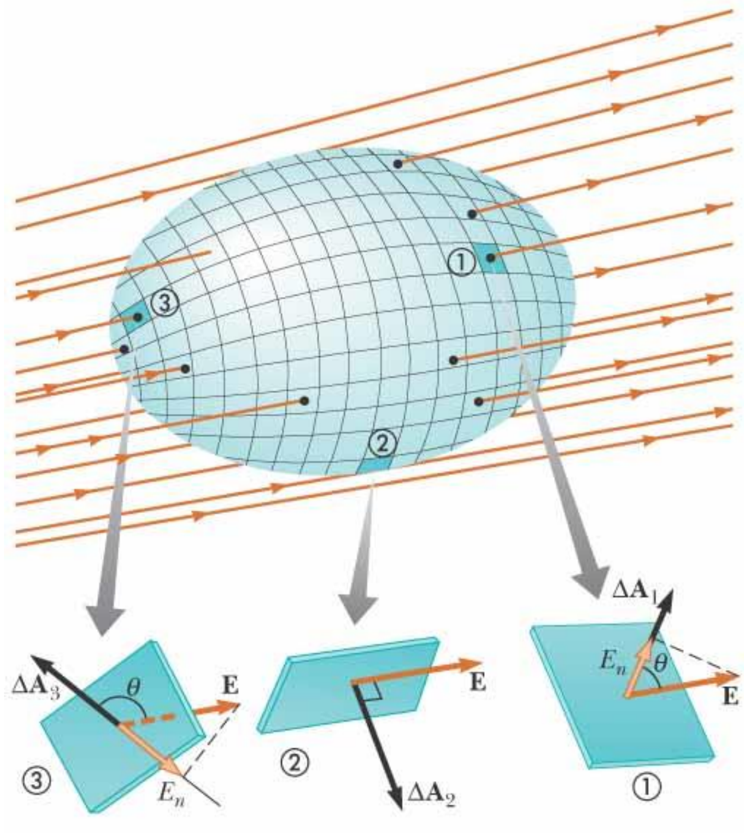
$$\Phi_E = \lim_{\Delta A_i \to 0} \sum E_i \cdot \Delta A_i = \int_{\text{surface}} \mathbf{E} \cdot d\mathbf{A}$$

Electric Flux, final

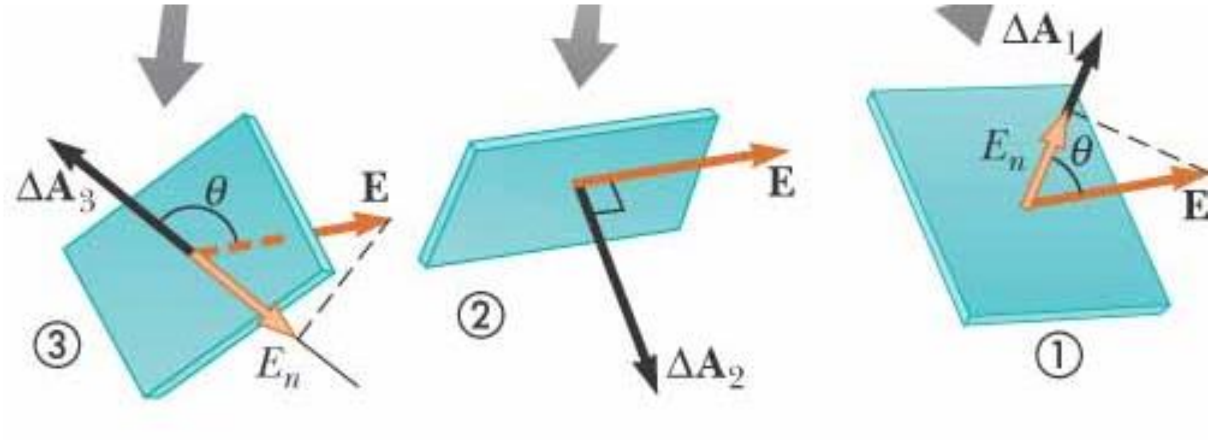
- The surface integral means the integral must be evaluated over the surface in question
- In general, the value of the flux will depend both on the field pattern and on the surface
- The units of electric flux will be $N \cdot m^2/C^2$

Electric Flux, Closed Surface

- Assume a closed surface
- The vectors $\Delta \boldsymbol{A}_i$ point in different directions
  - At each point, they are perpendicular to the surface
  - By convention, they point outward



Flux Through Closed Surface, cont.



- At (1), the field lines are crossing the surface from the inside to the outside; $\theta < 90^o$, Φ is positive
- At (2), the field lines graze surface; $\theta = 90^o$, $\Phi = 0$
- At (3), the field lines are crossing the surface from the outside to the inside; $180^o > \theta > 90^o$, Φ is negative

Flux Through Closed Surface, final

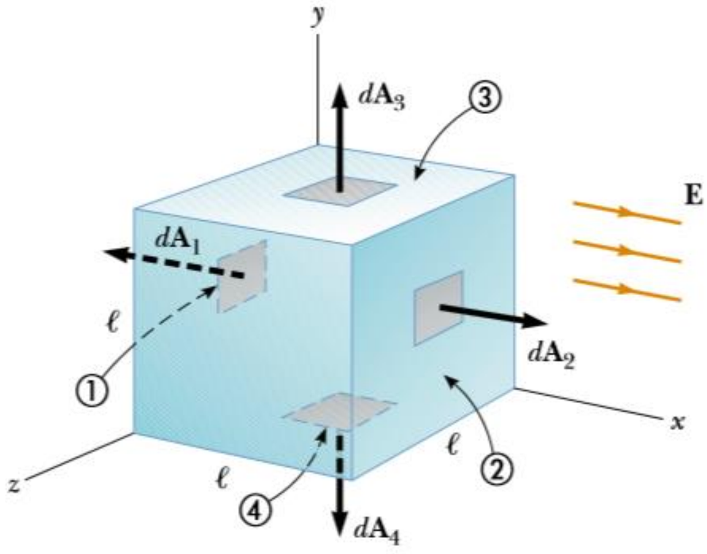
- The ***net*** flux through the surface is proportional to the net number of lines leaving the surface
  - This net number of lines is the number of lines leaving the surface minus the number entering the surface
- If E_n is the component of **E** perpendicular to the surface, then

$$\Phi_E = \oint \mathbf{E} \cdot d\mathbf{A} = \oint E_n \, dA$$

Example 24.2

EXAMPLE 24.2 **Flux Through a Cube**

Consider a uniform electric field $\mathbf{E}$ oriented in the x direction. Find the net electric flux through the surface of a cube of edges ℓ, oriented as shown in Figure 24.5.



Solution of Example 24.2

For ①, **E** is constant and directed inward but $d\mathbf{A}_1$ is directed outward ($\theta = 180°$); thus, the flux through this face is

$$\int_1 \mathbf{E} \cdot d\mathbf{A} = \int_1 E(\cos 180°)\, dA = -E \int_1 dA = -EA = -E\ell^2$$

because the area of each face is $A = \ell^2$.

For ②, **E** is constant and outward and in the same direction as $d\mathbf{A}_2 (\theta = 0°)$; hence, the flux through this face is

$$\int_2 \mathbf{E} \cdot d\mathbf{A} = \int_2 E(\cos 0°)\, dA = E \int_2 dA = +EA = E\ell^2$$

Therefore, the net flux over all six faces is

$$\Phi_E = -E\ell^2 + E\ell^2 + 0 + 0 + 0 + 0 = \boxed{0}$$

Gauss's Law

- Gauss's law states $\Phi_E = \oint \mathbf{E} \cdot d\mathbf{A} = \frac{q_{\text{in}}}{\varepsilon_o}$
- q_{in} is the net charge inside the surface
- **E** represents the electric field at any point on the surface
  - **E** is the *total electric field* and may have contributions from charges both inside and outside of the surface
- Although Gauss's law can, in theory, be solved to find **E** for any charge configuration, in practice it is limited to symmetric situations

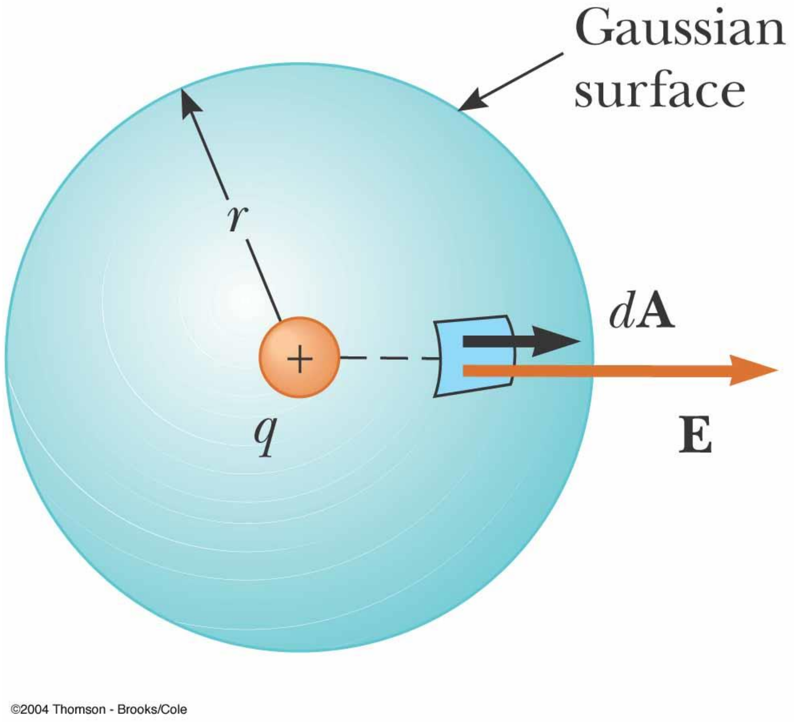
Field Due to a Point Charge

- Choose a sphere as the gaussian surface
  - **E** is parallel to *d***A** at each point on the surface

$$\Phi_E = \oint \mathbf{E} \cdot d\mathbf{A} = \oint E dA = \frac{q}{\varepsilon_o}$$

$$= E \oint dA = E(4\pi r^2)$$

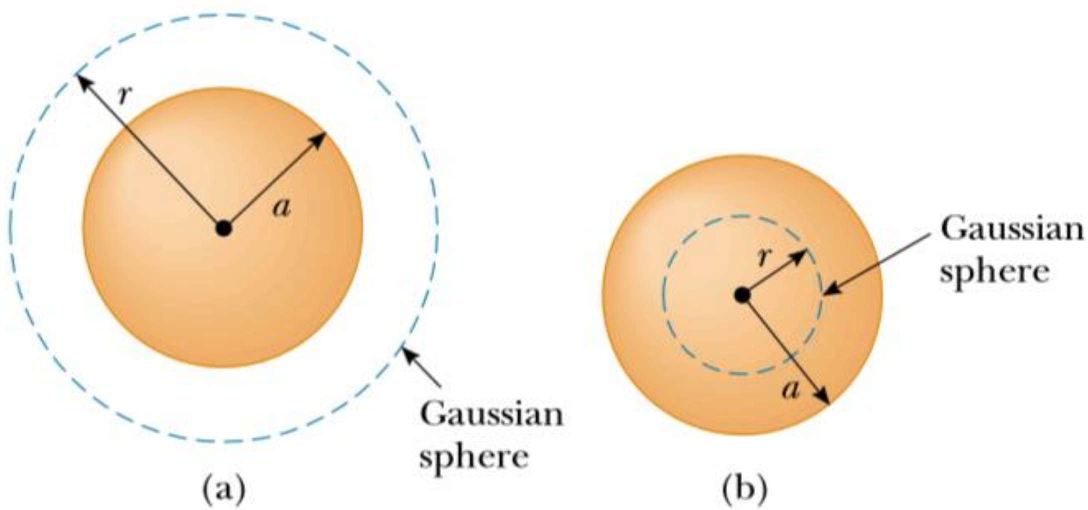
$$E = \frac{q}{4\pi\varepsilon_o r^2} = k_e \frac{q}{r^2}$$



EXAMPLE 24.5

An insulating solid sphere of radius a has a uniform volume charge density ρ and carries a total positive charge Q (Fig. 24.11). (a) Calculate the magnitude of the electric field at a point outside the sphere.

(b) Find the magnitude of the electric field at a point inside the sphere.



(a)

$$E = k_e \frac{Q}{r^2} \qquad \text{(for } r > a\text{)}$$

(b)

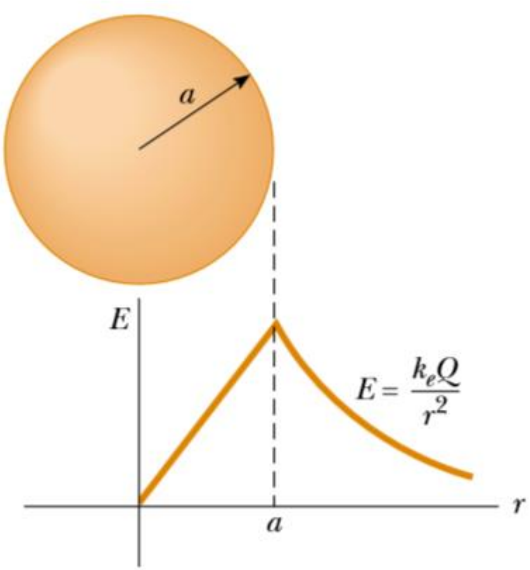
$$q_{\text{in}} = \rho V' = \rho\left(\tfrac{4}{3}\pi r^3\right)$$

$$\oint E\, dA = E \oint dA = E(4\pi r^2) = \frac{q_{\text{in}}}{\epsilon_0}$$

$$E = \frac{q_{\text{in}}}{4\pi\epsilon_0 r^2} = \frac{\rho \frac{4}{3}\pi r^3}{4\pi\epsilon_0 r^2} = \frac{\rho}{3\epsilon_0} r$$

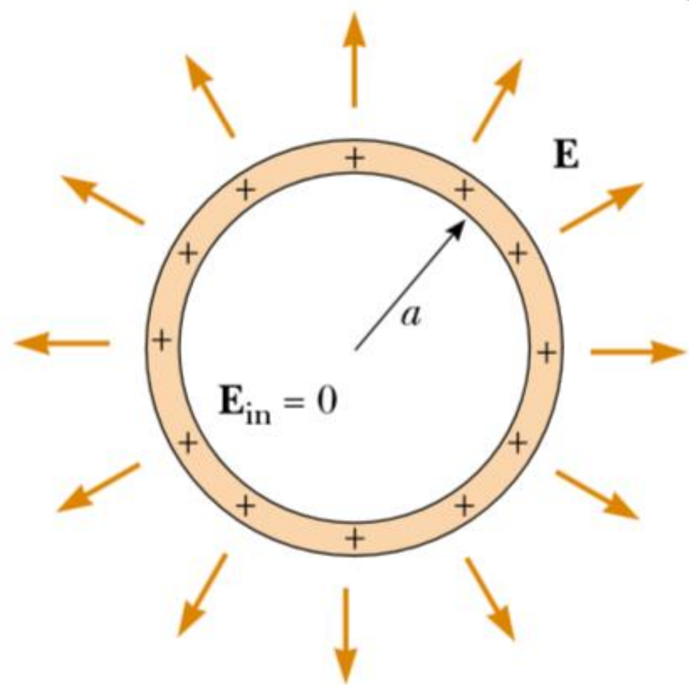
Because $\rho = Q/\frac{4}{3}\pi a^3$ by definition and since $k_e = 1/(4\pi\epsilon_0)$, this expression for E can be written as

$$E = \frac{Qr}{4\pi\epsilon_0 a^3} = \frac{k_e Q}{a^3}\, r \qquad (\text{for } r < a)$$

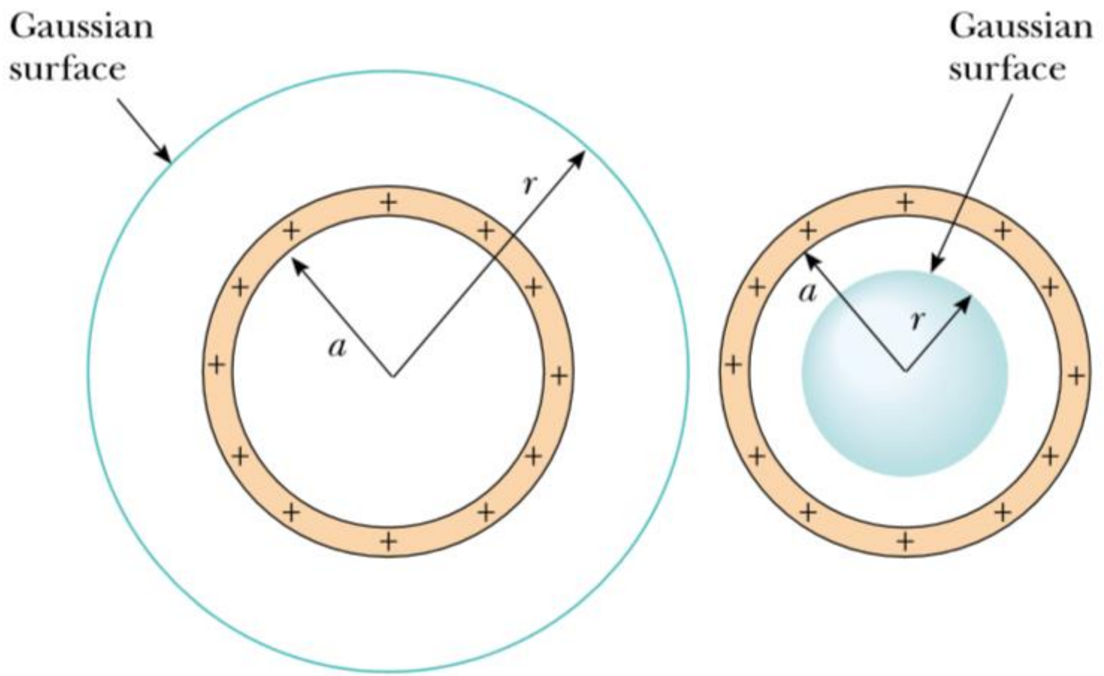


EXAMPLE 24.6 The Electric Field Due to a Thin Spherical Shell

A thin spherical shell of radius a has a total charge Q distributed uniformly over its surface (Fig. 24.13a). Find the electric field at points (a) outside and (b) inside the shell.



Example 24.6 The Electric Field Due to a Thin Spherical Shell



Example 24.6 The Electric Field Due to a Thin Spherical Shell

(a) Outside the ring

$$E = k_e \frac{Q}{r^2} \qquad (\text{for } r > a)$$

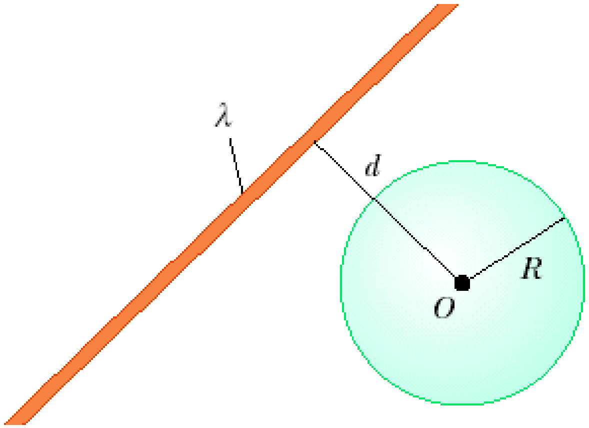
(b) E=0 inside the ring

Chapter 24

Gauss's Law Applications

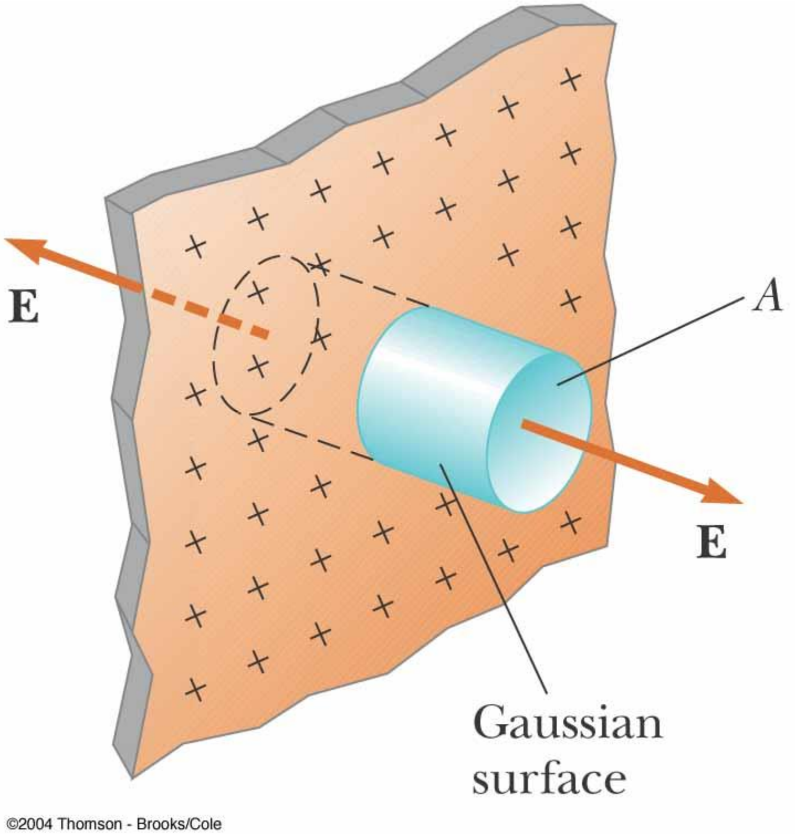
Quiz

- An infinitely long line charge having a uniform charge per unit length λ lies a distance d from point O as shown in Figure P24.19. Determine the total electric flux through the surface of a sphere of radius R centered at O resulting from this line charge. Consider both cases, where $R < d$ and $R > d$.



Field Due to a Plane of Charge

- **E** must be perpendicular to the plane and must have the same magnitude at all points equidistant from the plane
- Choose a small cylinder whose axis is perpendicular to the plane for the gaussian surface



Field Due to a Plane of Charge, cont

- **E** is parallel to the curved surface and there is no contribution to the surface area from this curved part of the cylinder
- The flux through each end of the cylinder is *EA* and so the total flux is 2*EA*

Field Due to a Plane of Charge, final

- The total charge in the surface is σA
- Applying Gauss's law

$$\Phi_E = 2EA = \frac{\sigma A}{\varepsilon_o} \; and \; E = \frac{\sigma}{2\varepsilon_o}$$

- Note, this does not depend on r
- Therefore, the field is uniform everywhere

Electrostatic Equilibrium

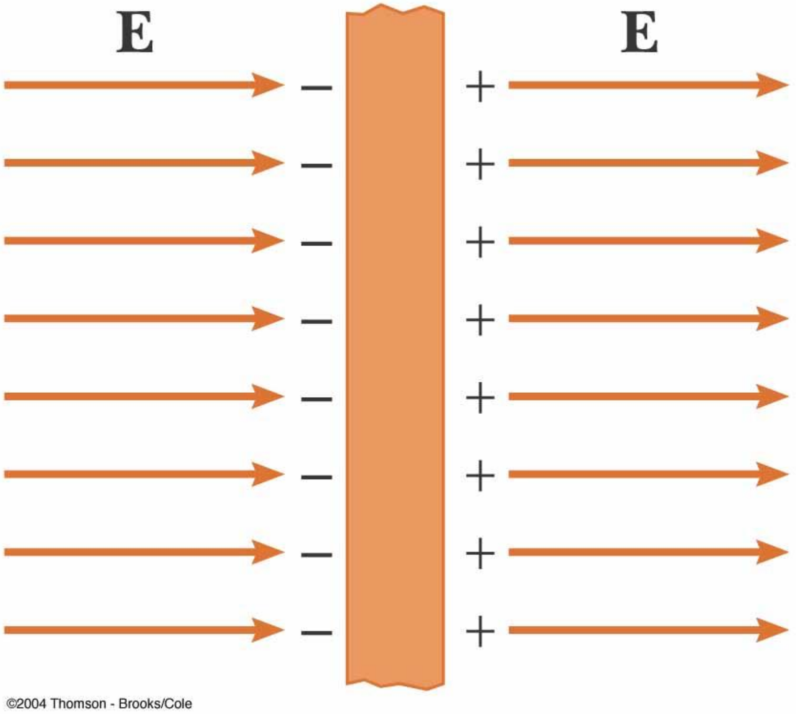
- When there is no net motion of charge within a conductor, the conductor is said to be in **electrostatic equilibrium**

Properties of a Conductor in Electrostatic Equilibrium

- The electric field is zero everywhere inside the conductor
- If an isolated conductor carries a charge, the charge resides on its surface
- The electric field just outside a charged conductor is perpendicular to the surface and has a magnitude of σ/ε_o
- On an irregularly shaped conductor, the surface charge density is greatest at locations where the radius of curvature is the smallest

Property 1: $E_{inside} = 0$

- Consider a conducting slab in an external field **E**
- If the field inside the conductor were not zero, free electrons in the conductor would experience an electrical force
- These electrons would accelerate
- These electrons would not be in equilibrium
- Therefore, there cannot be a field inside the conductor

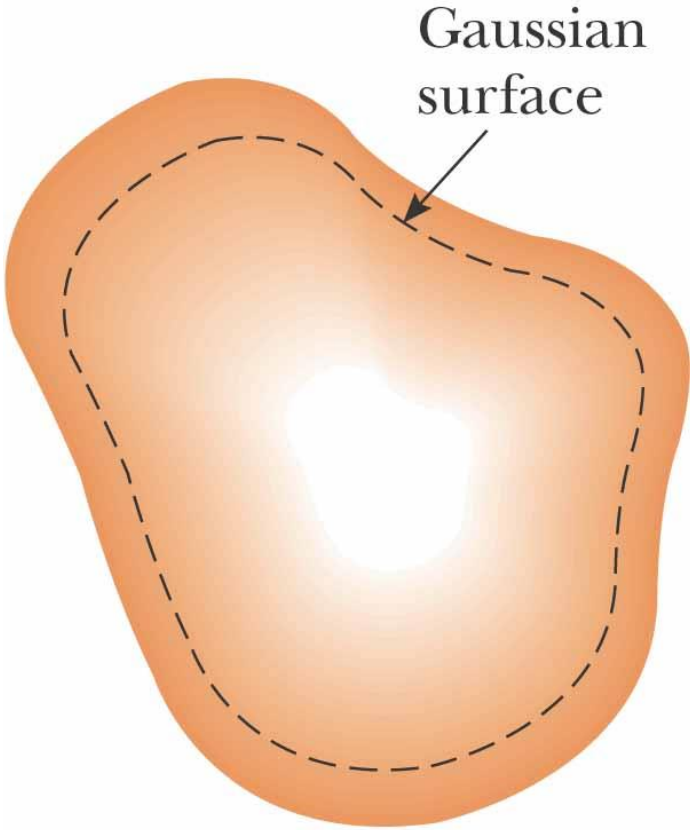


Property 1: $\mathbf{E}_{inside} = 0$, cont.

- Before the external field is applied, free electrons are distributed throughout the conductor
- When the external field is applied, the electrons redistribute until the magnitude of the internal field equals the magnitude of the external field
- There is a net field of zero inside the conductor
- This redistribution takes about 10^{-15}s and can be considered instantaneous

Property 2: Charge Resides on the Surface

- Choose a gaussian surface inside but close to the actual surface
- The electric field inside is zero (prop. 1)
- There is no net flux through the gaussian surface
- Because the gaussian surface can be as close to the actual surface as desired, there can be no charge inside the surface

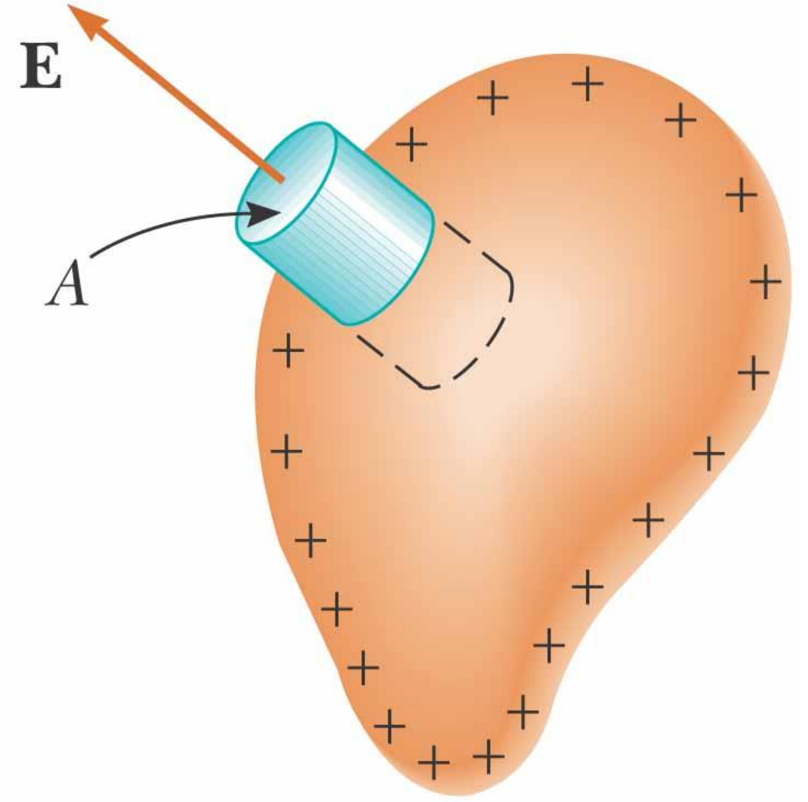


Property 2: Charge Resides on the Surface, cont

- Since no net charge can be inside the surface, any net charge must reside ***on*** the surface
- Gauss's law does not indicate the distribution of these charges, only that it must be on the surface of the conductor

Property 3: Field's Magnitude and Direction

- Choose a cylinder as the gaussian surface
- The field must be perpendicular to the surface



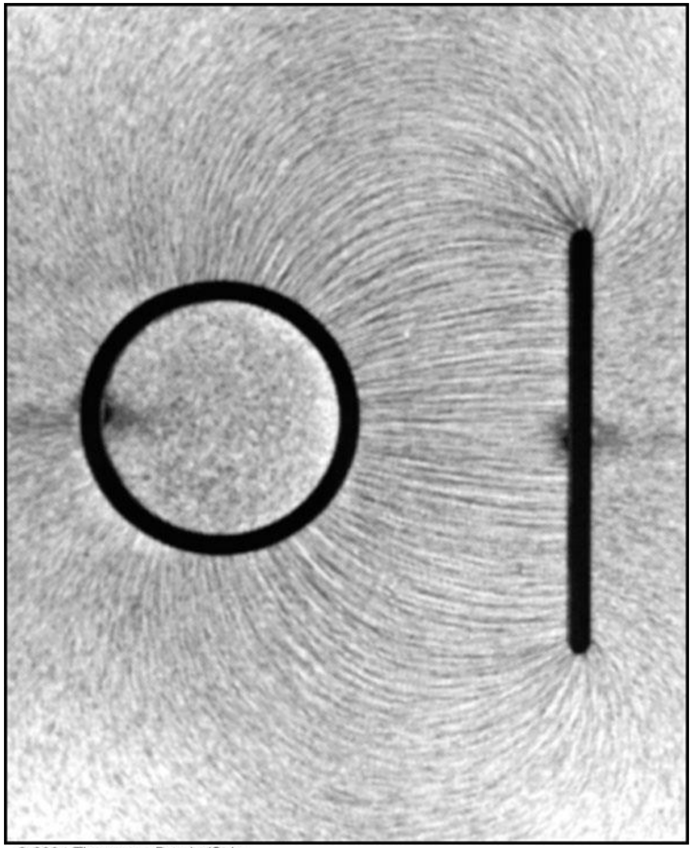
Property 3: Field's Magnitude and Direction, cont.

- The net flux through the gaussian surface is through only the flat face outside the conductor
  - The field here is perpendicular to the surface
- Applying Gauss's law

$$\Phi_E = EA = \frac{\sigma A}{\varepsilon_o} \text{ and } E = \frac{\sigma}{\varepsilon_o}$$

Conductors in Equilibrium, example

- The field lines are perpendicular to both conductors
- There are no field lines inside the cylinder



EXAMPLE 24.10 A Sphere Inside a Spherical Shell

A solid conducting sphere of radius a carries a net positive charge $2Q$. A conducting spherical shell of inner radius b and outer radius c is concentric with the solid sphere and carries a net charge $-Q$. Using Gauss's law, find the electric field in the regions labeled ①, ②, ③, and ④ in Figure 24.19 and the charge distribution on the shell when the entire system is in electrostatic equilibrium.