The Riemann Integral

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1 Anti-Derivatives, Indefinite Integrals

• In the classical analysis course, we defined the derivative f' of functions if it exists.
• In this section we are interested to the inverse problem. If f is a function, find if it is possible a function F such that $F' = f$.

Definition 1.1
A function F is called an anti-derivative of f on an interval I, if F is differentiable on I and

$$F'(x) = f(x), \quad \forall x \in I.$$

Example 1 :
Let $F(x) = x^2$, $x \in \mathbb{R}$; then F is differentiable on $\mathbb{R}$ and $F'(x) = 2x$. Therefore F is an anti-derivative of $2x$ on $\mathbb{R}$.

Note: There are many anti-derivatives of $f(x) = 2x$ on $\mathbb{R}$ such as

$$F_1(x) = x^2 + 1, \quad F_2(x) = x^2 - 7, \quad F_3(x) = x^2 + \frac{11}{2}, \quad F_4(x) = x^2 - \sqrt{2}.$$

In general the function $G(x) = x^2 + c$, where c is an arbitrary constant is an anti-derivative of $f(x) = 2x$.

Theorem 1.2
Let F and G be two anti-derivatives of f on an interval I, then there is a constant $c \in \mathbb{R}$ such that

$$F(x) = G(x) + c, \quad \forall x \in I.$$

Proof .
$(F - G)'(x) = F'(x) - G'(x) = 0$, then $F - G = c$ on the interval I. □

Example 2 :
Given $F(x) = \sin(x)$, its derivative is $F'(x) = \cos(x)$. Then an anti-derivative of $\cos(x)$ is $\sin(x)$. Also $G(x) = \sin(x) + c$ is an anti-derivative of $\cos(x)$.

Definition 1.3
Let F be an anti-derivative of f on an interval I, we denote $\int f(x)dx$ any anti-derivative i.e.

$$\int f(x)dx = F(x) + c, \quad \forall x \in I \tag{1.1}$$

$\int f(x)dx$ is called the indefinite integral of f on I.

In the equation (1.1),
• the constant c is called the constant of integration,
• x is called the variable of integration,
• $f(x)$ is called the integrand.

The mapping $f \longmapsto \int f(x)dx$ is called an indefinite integral or an integrating of f.

Basic table of indefinite integrals

$f(x)$	$\int f(x)dx$
1	$x+c$
$x^r;$	$\frac{x^{r+1}}{r+1}+c, r\in\mathbb{Q}\setminus\{-1\}$
$\cos(x)$	$\sin(x)+c$
$\sin(x)$	$-\cos(x)+c$
$\sec^2(x)$	$\tan(x)+c$
$\csc^2(x)$	$-\cot(x)+c$
$\sec(x)\tan(x)$	$\sec(x)+c$
$\csc(x)\cot(x)$	$-\csc(x)+c$

Theorem 1.4 *(Some important formulas)*
• *If f is differentiable on an interval I, then*

$$\int \frac{d}{dx}f(x)dx = f(x)+c.$$

• *If f has an anti-derivative on an interval I, then*

$$\frac{d}{dx}\int f(x)dx = f(x).$$

• *If f has an anti-derivative on an interval I, then for all $\alpha\in\mathbb{R}$ the function αf has an anti-derivative on the interval I and*

$$\int \alpha f(x)dx = \alpha\int f(x)dx.$$

• *If f and g have anti-derivatives on an interval I, then the function $f+g$ has an anti-derivative on the interval I and*

$$\int f(x)+g(x)dx = \int f(x)dx + \int g(x)dx.$$

2 Change of Variables; Substitution Method

Theorem 2.1 *(***Substitution***)*
If F is an anti-derivative of f, then $F(g(x))$ is an anti-derivative of $f(g(x))g'(x)$. Or,

$$\int f(g(x))g'(x)dx = F(g(x)) + c.$$

This is obvious. It is called "substitution" since it can be obtained by substituting $u = g(x)$ and $du = g'(x)dx$ into

$$\int f(u)du = F(u) + c.$$

Remark 2.2

Substitution method is also called changing variable method.

Example 1 :

$$\int (x^2+1)^n 2x dx \overset{u=x^2+1}{=} \int u^n du = \frac{u^{n+1}}{n+1} = \frac{(x^2+1)^{n+1}}{n+1} + c.$$

Example 2 :

$$\int \sin(2x+3)dx \overset{u=2x+3}{=} \frac{1}{2}\int \sin u du = -\frac{1}{2}\cos u + c = -\frac{1}{2}\cos(2x+3) + c.$$

Example 3 :

$$\int \frac{1}{\cos^2(\pi x)}dx \overset{u=\pi x}{=} \frac{1}{\pi}\int \frac{1}{\cos^2(u)}du = \frac{1}{\pi}\tan(\pi x) + c.$$

Example 4 :

$$\int \cos(x)dx = \sin(x) + c,$$

but

$$\int \cos(2x)dx \neq \sin(2x) + c,$$

because $\frac{d}{dx}(\sin(2x) + c) = 2\cos(2x) \neq \cos(2x)$.

To solve this problem, substituting $u = 2x$, then $du = 2dx$. Thus

$$\int \cos(2x)dx = \int \cos(u)\frac{1}{2}du = \frac{1}{2}\int \cos(u)du = \frac{1}{2}\sin(2x) + c.$$

Example 5 :

Let $n \neq -1$.

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c,$$

but

$$\int (3x+1)^n dx \neq \frac{(3x+1)^{n+1}}{n+1} + c,$$

because $\frac{d}{dx}(\frac{(3x+1)^{n+1}}{n+1}) = 3(3x+1)^n \neq (3x+1)^n$.

To solve this problem, substituting $u = 3x+1$, then $du = 3dx$. Thus

$$\int (3x+1)^n dx = \int u^n \frac{1}{3}du = \frac{1}{3}\int u^n du = \frac{u^{n+1}}{n+1} + c = \frac{1}{3}\frac{(3x+1)^{n+1}}{n+1} + c.$$

Theorem 2.3
Let I be an interval. Let $r \in \mathbb{Q} \setminus \{-1\}$ and $f : I \to \mathbb{R}$ be a differentiable function. Assume that $f^r(x)$ is defined for every $x \in I$. Then

$$\int f^r(x) f'(x) dx = \frac{f^{r+1}(x)}{r+1} + c.$$

Example 6 :
Find the values of the following integrals

$$\begin{aligned} K_1 &= \int (2x^3+1)^7 6x^2 dx \\ K_2 &= \int (7-6x^2)^{1/2} x dx \\ K_3 &= \int \frac{x^2-1}{(x^3-3x+1)^6} dx. \end{aligned}$$

Solution.
• To compute K_1, set $f(x) = 2x^3 + 1$, then $f'(x) = 6x^2$, therefore

$$\begin{aligned} K_1 &= \int f^7(x) f'(x) dx \\ &= \frac{f^8(x)}{8} + c \\ &= \frac{(2x^3+1)^8}{8} + c. \end{aligned}$$

• To compute K_2, set $f(x) = 7 - 6x^2$, then $f'(x) = -12x$, therefore

$$\begin{aligned} K_2 &= -\frac{1}{12} \int (7-6x^2)^{1/2}(-12x) dx \\ &= \int f^{1/2}(x) f'(x) dx \\ &= \frac{f^{3/2}(x)}{3/2} + c \\ &= \frac{2}{3}(7-6x^2)^{3/2} + c. \end{aligned}$$

• To compute K_3, set $f(x) = x^3 - 3x + 1$, then $f'(x) = 3x^2 - 3 = 3(x^2 - 1)$, therefore

$$\begin{aligned} K_3 &= \frac{1}{3} \int \frac{3(x^2-1)}{(x^3-3x+1)^6} dx \\ &= \frac{1}{3} \int (x^3-3x+1)^{-6}.3(x^2-1) dx \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{3}\int (f(x))^{-6} f'(x)dx \\
&= \frac{1}{3}\frac{(f(x))^{-5}}{-5} + c \\
&= -\frac{1}{15(x^3-3x+1)^5} + c.
\end{aligned}
$$

3 Summation Notation

Definition 3.1

Given a set of numbers $\{a_1, a_2, ..., a_n\}$*, the symbol* $\sum_{k=1}^{n} a_k$ *represents their sum as follows*

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + ... + a_n.$$

3.1 Summation Properties.

Let C be a constant and let $m, n \in \mathbb{N}$. We have the following summation properties

1. $\sum_{k=1}^{n} C = \underbrace{C + C + C + \ldots + C}_{n \text{ terms}} = nC.$
2. $\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k.$
3. $\sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k.$
4. $\sum_{k=1}^{n} C a_k = C \sum_{k=1}^{n} a_k.$
5. If $1 \le m \le n$ then $\sum_{k=1}^{m} a_k + \sum_{k=m+1}^{n} a_k = \sum_{k=1}^{n} a_k.$

Example 1 :

Evaluate the sum $\sum_{k=1}^{3} (k+1)^2 k^3$.

Solution. Here $n = 3$ and $a_k = (k+1)^2 k^3$. Then

$$a_1 = 4, \quad a_2 = 72, \quad a_3 = 432.$$

Therefore

$$\sum_{k=1}^{3} (k+1)^2 k^3 = 4 + 72 + 432 = 508.$$

Example 2 :

Evaluate the sum $\sum_{k=1}^{10}(1+(-1)^k)$.

Solution. Here $n=10$ and $a_k = 1+(-1)^k$. Then

$a_1 = 0, \quad a_2 = 2, \quad a_3 = 0, \quad a_4 = 2, \quad a_5 = 0, \quad a_6 = 2, \quad a_7 = 0, \quad a_8 = 2, \quad a_9 = 0, \quad a_{10} = 2.$

Therefore

$$\sum_{k=1}^{10}(1+(-1)^k) = 0+2+0+2+0+2+0+2+0+2 = 10.$$

Theorem 3.2

For every $c \in \mathbb{R}$ and $n \in \mathbb{N}$, we have

$$\sum_{k=1}^{n} c = nc.$$

Example 3 :

Evaluate the sum $\sum_{k=1}^{123} 4$.

Solution. We have

$$\sum_{k=1}^{123} 4 = 123 \times 4 = 492.$$

Theorem 3.3

Let $\alpha, \beta \in \mathbb{R}$ and $n \in \mathbb{N}$. For every $a_1, a_2, ..., a_n, b_1, b_2, ..., b_n \in \mathbb{R}$ we have

$$\sum_{k=1}^{n}(\alpha a_k + \beta b_k) = \alpha \sum_{k=1}^{n} a_k + \beta \sum_{k=1}^{n} b_k.$$

Example 4 :

Evaluate the sum $\sum_{k=1}^{3}\Big[5(1+(-1)^k) + 2(k+1)^2k^3\Big]$.

Solution. We have

$$\sum_{k=1}^{3}\Big[5(1+(-1)^k)+2(k+1)^2k^3\Big] = 5\sum_{k=1}^{3}(1+(-1)^k)+2\sum_{k=1}^{3}(k+1)^2k^3 = 5\times 2+2\times 508 = 1026.$$

Theorem 3.4

Let $n \in \mathbb{N}$. Then

$$\begin{aligned}
\sum_{k=1}^{n} k &= \frac{n(n+1)}{2} \\
\sum_{k=1}^{n} k^2 &= \frac{n(n+1)(2n+1)}{6} \\
\sum_{k=1}^{n} k^3 &= \Big[\frac{n(n+1)}{2}\Big]^2.
\end{aligned}$$

Example 5 :
Evaluate the following sums

$$\sum_{k=1}^{100} k,\ \sum_{k=1}^{20} k^2,\ \sum_{k=1}^{10} k^3.$$

Solution. We have

$$\begin{aligned}
\sum_{k=1}^{100} k &= \frac{100\times(100+1)}{2} \qquad = 50\times 101 = 5050,\\
\sum_{k=1}^{20} k^2 &= \frac{20\times(20+1)\times(2\times 20+1)}{6} = \frac{20\times 21\times 41}{6} = 2870,\\
\sum_{k=1}^{10} k^3 &= \Big[\frac{10\times(10+1)}{2}\Big]^2 = 55^2 = 3025.
\end{aligned}$$

Example 6 :
Express the sum in terms of n

$$\sum_{k=1}^{n}(3k^2-2k+1).$$

Solution. We have

$$\begin{aligned}
\sum_{k=1}^{n}(3k^2-2k+1) &= 3\sum_{k=1}^{n} k^2 - 2\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1,\\
&= 3.\frac{n(n+1)(2n+1)}{6} - 2.\frac{n(n+1)}{2} + 1.n,\\
&= \frac{n}{2}\Big[(n+1)(2n+1) - 2(n+1) + 2\Big],\\
&= \frac{n}{2}(2n^2+3n+1-2n-2+2),\\
&= \frac{n}{2}(2n^2+n+1).
\end{aligned}$$

4 Riemann Sums, Area and the Definite Integral

The approach of the integral of function by areas gives the geometrical sense of integration. The second approach consists in introducing a priori the anti-derivative of function. The idea of the first approach is to cut the interval $[a,b]$ by a subdivision in sub-intervals $[a_j, a_{j+1}]$, then to add the areas of rectangles based on the intervals $[a_j, a_{j+1}]$.

Definitions 4.1

1. *A partition P of the closed interval $[a, b]$ is a finite set of points $P = \{a_0, a_1, a_2, \ldots, a_n\}$ such that $a = a_0 < a_1 < a_2 < \ldots < a_{n-1} < a_n = b$. Each $[a_{j-1}, a_j]$ is called a subinterval of the partition and the number $h_j = a_j - a_{j-1}$ is called the amplitude of this interval.*

2. *The norm of a partition is defined to be the length of the longest subinterval $[a_j, a_{j+1}]$, that is, it is $||P|| = \max\{a_j - a_{j-1}, j = 1, \ldots, n\}$.*

3. *A partition $P = \{a_0, a_1, a_2, \ldots, a_n\}$ of the closed interval $[a, b]$ is a called uniform if $a_{k+1} - a_k = \dfrac{b-a}{n}$. Then in this case*

$$a_k = a + k\frac{b-a}{n}.$$

4. *A mark on the partition $P = \{a_0, a_1, a_2, \ldots, a_n\}$ is a set of points $w = \{x_1, \ldots, x_n\}$ such that $x_j \in [a_{j-1}, a_j]$.*

5. *A pointed partition of an interval is a partition of an interval together with a finite sequence of numbers $x_1, x_2, \ldots, x_n$ such that*

$$\forall j = 1, \ldots, n, \qquad x_j \in [a_{j-1}, a_j].$$

This pointed partition will be denoted by:

$$P = \{([a_{j-1}, a_j], x_j)\}_{1 \le j \le n}.$$

Definition 4.2
Let $P = \{([a_{j-1}, a_j], x_j)\}_{1 \le j \le n}$ be a pointed partition of the interval $[a, b]$. The Riemann sum of f with respect to the pointed partition P is the number

$$R(f, P) = \sum_{j=1}^{n} f(x_j)(a_j - a_{j-1}) = \sum_{j=1}^{n} f(x_j)\Delta_j \qquad (4.2)$$

Each term in the sum is the product of the value of the function at a given point and the length of an interval. Consequently, each term represents the area of a rectangle with height $f(x_j)$ and length $a_j - a_{j-1}$. The Riemann sum is the signed area under all the rectangles.

The Riemann sum $R(f, P)$ is the algebraic area of the union of the rectangles of width Δ_j and height $f(x_j)$. This is an algebraic area since $f(x_j)\Delta_j$ is counted positively if $f(x_j) > 0$ and negatively if $f(x_j) < 0$.

Intuitively the algebraic area A under the graph of f is the limit of $R(f, P)$ when the Δ_j tend to 0. One possible choice is a uniform partition of the interval $[a, b]$

$$a_j = a + j\frac{b-a}{n}, \quad 0 \le j \le n \quad \text{where} \quad \Delta_j = h = \frac{b-a}{n},$$

which could be combined with any choice of x_j.

Example 1 :
As a first example, consider the identity function $f(x) = x$ on the interval $[0,1]$. For $n \geq 1$ let:

$$a_0 = 0,\ a_1 = \frac{1}{n}, \ldots,\ a_j = \frac{j}{n}, \ldots,\ a_n = 1.$$

This partition is uniform and of norm equal to $\frac{1}{n}$. We present three cases of Riemann sums, as we put the x_j at the beginning, middle or at the end of the intervals $[a_{j-1}, a_j]$.

$$x_j = a_{j-1}\colon \qquad R(f,P) = \sum_{j=1}^{n} \frac{j-1}{n}\frac{1}{n} = \frac{1}{n^2}\sum_{j=0}^{n-1} j = \frac{n-1}{2n},$$

$$x_j = \frac{a_{j-1}+a_j}{2}\colon \qquad R(f,P) = \sum_{j=1}^{n} \frac{2j-1}{2n}\frac{1}{n} = \frac{1}{2n^2}\sum_{j=0}^{n-1} 2j+1 = \frac{1}{2}$$

$$x_j = a_j\colon \qquad R(f,P) = \sum_{j=1}^{n} \frac{j}{n}\frac{1}{n} = \frac{n+1}{2n}.$$

The second sum is equal to $\frac{1}{2}$ for every n, the other tends to $\frac{1}{2}$ when n tends to infinity. The area of the triangle under the graph of the function is equal to $\frac{1}{2}$.

Definition 4.3
For any bounded function f defined on the closed interval $[a,b]$, the definite integral of f from a to b is

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(w_k)\Delta x_k,\ \ (\|P\| \to 0)$$

whenever the limit exists. (The limit is over all pointed partitions $P = \{([x_{j-1}, x_j], w_j)\}_{1\leq j\leq n}$). When the limit exists, we say that f is Riemann integrable (or integrable) on $[a,b]$.

Example 2 :
Let $f\colon [0,1] \longrightarrow \mathbb{R}$ defined by: $f(x) = 1$ if $x \in \mathbb{Q}\cap[0,1]$ and $f(x) = 0$ if $x \notin \mathbb{Q}\cap[0,1]$. If $P = \{x_0, x_1, x_2, \ldots, x_n\}$ a partition of the closed interval $[0,1]$, we take the marks $t = \{t_0, t_1, t_2, \ldots, t_n\}$, $t' = \{t'_0, t'_1, t'_2, \ldots, t'_n\}$ such that $t_k \in [x_k, x_{k+1}] \cap \mathbb{Q}$ and $t'_k \in [x_k, x_{k+1}] \cap (\mathbb{R}\setminus\mathbb{Q})$, for all $k = 0, \ldots, n-1$. Then $R(f,P,t) = 1$ and $R(f,P,t') = 0$. Then f is not Riemann integrable.

Theorem 4.4
If $f\colon [a,b] \to \mathbb{R}$ is continuous, then f is integrable on $[a,b]$.

Definition 4.5
A function $f\colon [a,b] \longrightarrow \mathbb{R}$ *is called piecewise continuous on a closed interval* $[a,b]$ *if there exists a partition* $P = \{x_0, x_1, x_2, \dots, x_n\}$ *of the closed interval* $[a,b]$ *such that* f *is continuous on every interval* $]x_k, x_{k+1}[$, $\lim_{x\to x_k^+} f(x)$ *and* $\lim_{x\to x_{k+1}^-} f(x)$ *exist in* $\mathbb{R}$*, for all* $k = 0, \dots, n-1$.

Theorem 4.6
Any piecewise continuous function $f\colon [a,b] \longrightarrow \mathbb{R}$ *is Riemann integrable and then*

$$\int_a^b f(x)dx = \lim_{n\to+\infty} \frac{b-a}{n} \sum_{k=1}^{n} f(x_k).$$

Example 3 :

$$\int_0^1 (3x+7)dx = \lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} 3\frac{k}{n} + 7 = \lim_{n\to+\infty} \frac{3}{n^2}\frac{n(n+1)}{2} + 7 = \frac{3}{2} + 7 = \frac{17}{2}.$$

Example 4 :

$$\begin{aligned}
\int_1^4 (x^2+x+2)dx &= \lim_{n\to+\infty} \frac{3}{n} \sum_{k=1}^{n} \left(1+3\frac{k}{n}\right)^2 + (1+3\frac{k}{n}) + 2 \\
&= \lim_{n\to+\infty} \frac{3}{n} \sum_{k=1}^{n} \left(1 + 6\frac{k}{n} + 9\frac{k^2}{n^2} + 1 + 3\frac{k}{n} + 2\right) \\
&= \lim_{n\to+\infty} \frac{3}{n} \left(n + \frac{6}{n}\frac{n(n+1)}{2} + \frac{9}{n^2}\frac{n(n+1)(2n+1)}{6} + n + \frac{3}{n}\frac{n(n+1)}{2} + 2n\right) \\
&= \frac{69}{2}.
\end{aligned}$$

Example 5 :

$$\begin{aligned}
\int_0^2 (6x^3+1)dx &= \lim_{n\to+\infty} \frac{2}{n} \sum_{k=1}^{n} 6\left(2\frac{k}{n}\right)^3 + 1 \\
&= \lim_{n\to+\infty} \frac{2}{n} \left(12\frac{(n+1)^2}{n} + n\right) = 26.
\end{aligned}$$

4.1 Fundamental Properties.

1. Linearity. If $f, g\colon [a,b] \to \mathbb{R}$ are two functions and α,β two reals numbers, then

$$S(\alpha f + \beta g, P) = \alpha S(f,P) + \beta S(g,P).$$

2. Monotony. If $f, g\colon [a,b] \to \mathbb{R}$ are two functions, then

$$f \le g \quad \Rightarrow \quad S(f,P) \le S(g,P).$$

In particular, if $f \ge 0$, then $S(f,P) \ge 0$.

3. Chasles's Formula. If $a < c < b$ are three reals numbers and f be a function defined on $[a, b]$. If P_1 is a pointed partition of $[a, c]$ and P_2 be a pointed partition of $[b, c]$, then $P_1 \cup P_2$ is a pointed partition of $[a, b]$ and

$$S(f, P_1 \cup P_2) = S(f, P_1) + S(f, P_2).$$

Example 6 :
Find the Riemann sum for the following function

$$f(x) = 4x + 1$$

on the partition $P = \{-1,\ 0,\ 2,\ 4,\ 6\}$ of the interval $[-1, 6]$ by choosing in each subinterval of P

a) the left hand end point,
b) the right hand end point,
c) the middle point.

Solution.
Start by given Δx_k for every k:

$$\Delta x_1 = 0 - (-1) = 1, \quad \Delta x_2 = 2 - 0 = 2, \quad \Delta x_3 = 4 - 2 = 2, \quad \Delta x_4 = 6 - 4 = 2.$$

a) The left hand endpoints are

$$w_1 = -1,\ w_2 = 0,\ w_3 = 2,\ w_4 = 4,$$

and

$$f(w_1) = -3, \quad f(w_2) = 1, \quad f(w_3) = 9, \quad f(w_4) = 17.44$$

Therefore

$$\begin{aligned} S(f, P) &= \sum_{k=1}^{4} f(w_k)\Delta x_k \\ &= (-3) \times 1 + 1 \times 2 + 9 \times 2 + 17 \times 2 \\ &= 1 + 2 + 18 + 34 \\ &= 56. \end{aligned}$$

b) The right hand endpoint are

$$w_1 = 0,\ w_2 = 2,\ w_3 = 4,\ w_4 = 6.$$

Then

$$f(w_1) = 1, \quad f(w_2) = 9, \quad f(w_3) = 17, \quad f(w_4) = 25.$$

Therefore

$$\begin{aligned} S(f,P) &= \sum_{k=1}^{4} f(w_k)\Delta x_k \\ &= 1\times 1 + 9\times 2 + 17\times 2 + 25\times 2 \\ &= 1+18+34+50 \\ &= 103. \end{aligned}$$

c) The middle points are

$$w_1 = \frac{-1+0}{2} = -\frac{1}{2},\ w_2 = \frac{0+2}{2} = 1,\ w_3 = \frac{2+4}{2} = 3,\ w_4 = \frac{4+6}{2} = 5.$$

Then

$$f(w_1) = -1, \quad f(w_2) = 5, \quad f(w_3) = 13, \quad f(w_4) = 21.$$

Therefore

$$\begin{aligned} S(f,P) &= \sum_{k=1}^{4} f(w_k)\Delta x_k \\ &= (-1)\times 1 + 5\times 2 + 13\times 2 + 21\times 2 \\ &= -1+10+26+42 \\ &= 77. \end{aligned}$$

Example 7 :
Use the above definition to express each the following limits as a definite integral

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \pi(w_k^2 - 4), \quad [a,b] = [2,3], w_k = 2 + \frac{k}{n},$$
$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} (w_k^{1/3} + 4w_k), \quad [a,b] = [-4,-3], w_k = -4 + \frac{k}{n}.$$

Solution. Using the definition and the above theorem, we obtain

$$\begin{aligned} \lim_{n\to\infty} \sum_{k=1}^{n} \pi(w_k^2 - 4)\Delta x_k &= \int_2^3 (x^2-4)dx \\ \lim_{n\to\infty} \sum_{k=1}^{n} (w_k^{1/3} + 4w_k)\Delta x_k &= \int_{-4}^{-3} (x^{1/3}+4x)dx. \end{aligned}$$

Conventions and Notations:
If $c > d$, then $\int_c^d f(x)dx = -\int_d^c f(x)dx$.
If $f(a)$ exists, then $\int_a^a f(x)dx = 0$.

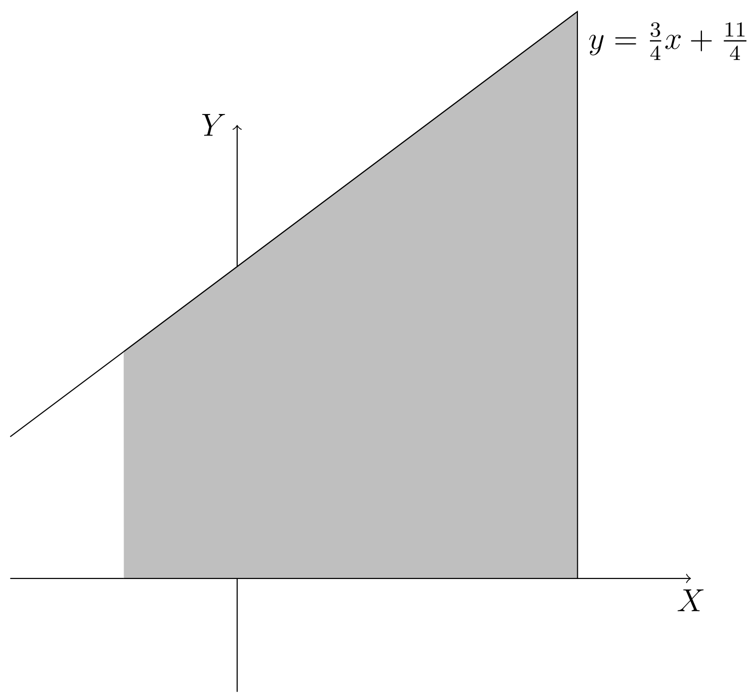
Theorem 4.7 *If* $f : [a, b] \to \mathbb{R}$ *is integrable and*

$$f(x) \geq 0, \quad \forall x \in [a, b],$$

then the area A *of the region under the graph of* f *from* a *to* b *is*

$$A = \int_a^b f(x)dx.$$

Example 8 :
Express the area A of the shaded region as a definite integral



Solution. We have

$$3x - 4y = 11 \iff y = \frac{3}{4}x + \frac{11}{4} = f(x).$$

Clearly f is continuous and positive on the interval $[-1, 3]$, by the above theorems we obtain

$$A = \int_{-1}^{3} f(x)dx = \int_{-1}^{3} \frac{3}{4}x + \frac{11}{4}dx.$$

Example 9 :
Evaluate the definite integrals by regarding it as the area under the graph of function

$$I_1 = \int_{-1}^{3} 4dx, \quad I_2 = \int_0^4 xdx, \quad I_3 = \int_{-1}^{4} |x|dx.$$

Solution.

The functions wich we integrate are continuous.
$I_1 = \int_{-1}^{3} 4dx = \lim_{n\to+\infty} \frac{4}{n}\sum_{k=1}^{n} 4 = \lim_{n\to+\infty} \frac{16n}{n} = 16.$
$I_2 = \int_{0}^{4} xdx = \lim_{n\to+\infty} \frac{4}{n}\sum_{k=1}^{n} \frac{4k}{n} = \lim_{n\to+\infty} \frac{8(n+1)}{n} = 8.$
$I_3 = \int_{-1}^{4} |x|dx = \int_{-1}^{0}(-x)dx + \int_{0}^{4} xdx,$
$\int_{-1}^{0}(-x)dx = \lim_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n}(1-\frac{k}{n}) = \frac{1}{n}(n-\frac{n+1}{2}) = \frac{1}{2}, \int_{0}^{4} xdx = 8$, then $I_3 = \frac{1}{2}+8.$

Theorem 4.8
We have the following properties of the definite integrals:
(P_1) *If α is a real number, then*

$$\int_a^b \alpha dx = \alpha(b-a).$$

(P_2) *If α is a real number and $f : [a,b] \to \mathbb{R}$ is an integrable function, then αf is integrable on $[a,b]$ and*

$$\int_a^b \alpha f(x)dx = \alpha \int_a^b f(x)dx.$$

(P_3)*If f and g are two integrable functions on $[a,b]$, then $f+g$ is integrable on $[a,b]$ and*

$$\int_a^b f(x)+g(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$$

(P_4)*If f and g are two integrable functions on $[a,b]$, then $f-g$ is integrable on $[a,b]$ and*

$$\int_a^b f(x)-g(x)dx = \int_a^b f(x)dx - \int_a^b g(x)dx.$$

(P_5) *If $a<c<b$ and if f is an integrable function on $[a,b]$, then f is integrable on $[a,c]$ and on $[c,b]$, moreover*

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

(P_6)*If f is integrable on a closed interval I and if a, b and c three numbers in I, then*

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

(P_7) *If f is integrable on $[a,b]$ and*

$$\forall x \in [a,b], \quad f(x) \geq 0.$$

Then

$$\int_a^b f(x)dx \geq 0.$$

(P_8) *If f and g are integrable on $[a, b]$ and*

$$\forall x \in [a, b], \quad f(x) \geq g(x).$$

Then

$$\int_a^b f(x)dx \geq \int_a^b f(x)dx.$$

Example 10 :
Verify the inequality

$$\int_1^4 (2x + 2)dx \leq \int_1^4 (3x + 1)dx.$$

Solution. Here $[a, b] = [1, 4]$ and

$$f(x) = 3x + 1, \quad g(x) = 2x + 2.$$

We have

$$f(x) - g(x) = (3x + 1) - (2x + 2) = x - 1 \geq 0, \quad \forall x \in [1, 4],$$

then

$$f(x) \geq g(x), \quad \forall x \in [1, 4].$$

Using the property **(P_8)**, we obtain

$$\int_1^4 g(x)dx \leq \int_1^4 f(x)dx$$

or

$$\int_1^4 (2x + 2)dx \leq \int_1^4 (3x + 1)dx.$$

5 The Fundamental Theorem of Calculus

Theorem 5.1 (Mean Value Theorem for the definite integrals)
If f is continuous on $[a, b]$, then there is a number $c \in [a, b]$ such that

$$\int_a^b f(x)dx = (b - a)f(c).$$

Proof .
Let $m = \inf_{x \in [a,b]} f(x)$ and $M = \text{Sup}_{x \in [a,b]} f(x)$.

Since $m \leq f \leq M$, $m \leq \dfrac{1}{b-a} \displaystyle\int_a^b f(x)dx \leq M$. By the Intermediate Value Theorem, there exists $c \in [a, b]$ such that $\dfrac{1}{b-a} \displaystyle\int_a^b f(x)dx = (b - a)f(c)$. □

Definition 5.2
Let f be a continuous on $[a,b]$. Then the average value f_{av} of f is given by

$$f_{av} = \frac{1}{b-a}\int_a^b f(x)dx.$$

Example 11 :
We give $\int_0^3 3x^2 dx = 27$. Let $f(x) = 3x^2,\ x \in [0,3]$.

a) Find the average value of f on $[0,3]$.

b) Find a number c that satisfies the conclusion of the Mean Value Theorem.

Solution.

a) $\frac{1}{3}\int_0^3 3x^2 dx = 9$.

b) $\frac{1}{3}\int_0^3 3x^2 dx = 9 = 3c^2$. Then $c^2 = 3$ and $c = \sqrt{3}$.

Remark 5.3
The continuity of f is important here. It is possible that a discontinuous function never equals its average value. We can take $f(x) = 0$ on the interval $[0,1]$ and $f(x) = 1$ on the interval $[1,2]$. The average of f on the interval $[0,2]$ is equal to $\frac{1}{2}\int_0^2 f(x)dx = \frac{1}{2}\int_1^2 dx = \frac{1}{2}$. But $f(x) = ne\frac{1}{2}$, for all $x \in [0,2]$.

Example 12 :
Let $f(x) = 3x+7$ on the interval $[0,1]$. We know that $\int_0^1 (3x+7)dx = \frac{17}{2}$. Then the point c where f reaches its average value verifies $3c + 7 = 172$, then $c = \frac{1}{2}$.

Example 13 :
Let $f(x) = x^2 + x + 2$ on the interval $[1,4]$. We know that $\int_0^1 (x^2 + x + 2)dx = \frac{69}{2}$. Then the point c where f reaches its average value verifies $c^2 + c + 2 = 696$, then $c = \frac{-1+\sqrt{39}}{2}$.

Example 14 :
Let $f(x) = 6x^3 + 1$ on the interval $[0,2]$. We know that $\int_0^1 (x^2 + x + 2)dx = \frac{2}{6}$. Then the point c where f reaches its average value verifies $6c^3 + 1 = 26$, then $c = (\frac{25}{6})^{\frac{1}{3}}$.

Example 15 :
Let f be a continuous function on $[a,b]$, $b \neq a$ and if $\int_a^b f(x)dx = 0$, then then $f(x) = 0$ at least once in $[a,b]$.

The average value of f on $[a,b]$ is 0. Then by the Mean Value Theorem, f reaches this value at some point $c \in [a,b]$.

Theorem 5.4 *(***The Fundamental Theorem of Calculus) part I**
If f is a continuous function on $[a, b]$, then $F(x) = \int_a^x f(t)dt$ is continuous on $[a, b]$ and differentiable on $[a, b]$ and its derivative is $F'(x) = f(x)$.

Proof .
Let $x \in [a, b]$ and $h \neq 0$ such that $x + h \in [a, b]$. Then it results from the The Mean Value Theorem for Definite Integrals that there exists $c \in [x, x+h]$ or $c \in [x+h, x]$ such that $f(c) = \dfrac{F(x+h) - F(x)}{h} = \dfrac{1}{h}\int_x^{x+h} f(t)dt$. As f is continuous $\lim_{h\to 0} f(c) = f(x) = F'(x)$. □

Theorem 5.5 *(***The Fundamental Theorem of Calculus) part II**
If f is a continuous function on $[a, b]$ and F is an anti-derivative of f on $[a, b]$, then $\int_a^b f(t)dt = F(b) - F(a)$

Proof .
Let $G(x) = \int_a^x f(t)dt$. We know that $G'(x) = f(x)$, then there exists $c \in \mathbb{R}$ such that $F(x) = G(x) + C$ for some constant C for $a \leq x \leq b$. Since $G(a) = 0$, then $C = F(a)$, and $G(x) = F(x) - F(a)$, for all $x \in [a, b]$. □

Notations:
$\Big[F(x)\Big]_a^b = F(b) - F(a)$.

Theorem 5.6
Let f be continuous on the closed interval $[a, b]$. Let $c \in [a, b]$ and

$$G(x) = \int_c^x f(t)dt; \quad x \in [a, b].$$

Then

$$G'(x) = f(x); \quad \forall x \in [a, b].$$

Proof .
$G(x) = \int_a^x f(t)dt - \int_a^c f(t)dt$, then $G'(x) = f(x)$. □

Example 1 :
If $G(x) = \int_1^x \frac{1}{t}dt$ and $x > 0$, find $G'(x)$.
Solution.
$G'(x) = \frac{1}{x}$.

Example 2 :
Find $\dfrac{d}{dx}\int_0^1 t\sqrt{t^2+4}dt$.
Solution.
$\int_0^1 t\sqrt{t^2+4}dt$ is a constant, $\dfrac{d}{dx}\int_0^1 t\sqrt{t^2+4}dt = 0$.

Theorem 5.7
Let f be continuous on an interval I. If v and u be two differentiable functions on an interval J such that $v(J) \subset I$ and $u(J) \subset I$, then the function

$$x \longmapsto \int_{u(x)}^{v(x)} f(t)dt$$

is defined and differentiable on the interval J. Moreover

$$\frac{d}{dx}\Big(\int_{u(x)}^{v(x)} f(t)dt\Big) = v'(x)f(v(x)) - u'(x)f(u(x)); \quad \forall x \in J.$$

Proof .
Let $F(x) = \displaystyle\int_a^x f(t)dt$, where $a \in I$. $\displaystyle\int_{u(x)}^{v(x)} f(t)dt = F(u(x)) - F(v(x))$. Since $F'(x) = f(x)$, the Chain Rule Formula yields

$$\frac{d}{dx}\Big(\int_{u(x)}^{v(x)} f(t)dt\Big) = v'(x)f(v(x)) - u'(x)f(u(x)); \quad \forall x \in J.$$

Example 3 :
Find $\displaystyle\frac{d}{dx}\Big(\int_{3x}^{x^2} (t^3+1)^7 dt\Big)$.
Solution.
$\displaystyle\frac{d}{dx}\Big(\int_{3x}^{x^2} (t^3+1)^7 dt\Big) = 2x(x^6+1)^7 - 3(27x^3+1)^7$.

6 Numerical Integration

Very often definite integrations cannot be done in closed form. When this happens we need some simple and useful techniques for approximating definite integrals. In this section we discuss three such simple and useful methods.

6.1 Trapezoidal Rule

Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be a non negative continuous function. In this method, to approximate the area under the graph of f, we join the point $(x_j, f(x_j))$ with the point $(x_{j+1}, f(x_{j+1}))$ for each sub-interval $[x_j, x_{j+1}]$, by a straight line and find the area under this line. which means that we replace f on $[x_j, x_{j+1}]$ by the polynomial P of degree 1 such that $P(x_j) = f(x_j)$ and $P(x_{j+1}) = f(x_{j+1})$. We say that P interpolates f on the points x_j and x_{j+1}. Then

$$P(x) = f(x_j)\frac{x_{j+1} - x}{x_{j+1} - x_j} + f(x_{j+1})\frac{x - x_j}{x_{j+1} - x_j}.$$

The area under the graph of P on the interval $[x_j, x_{j+1}]$ is the area of a trapezoid with value equal to

$$\frac{1}{2}(x_{j+1}-x_j)(f(x_{j+1})+f(x_{j+1})).$$

The area under the graph of f is approximated by:

$$\sum_{j=1}^{n}\frac{1}{2}(x_{j+1}-x_j)(f(x_{j+1})+f(x_j)). \tag{6.3}$$

In the case where $x_{j+1}-x_j=\dfrac{b-a}{n}$, this area is approximated by

$$\int_a^b f(x)dx \approx \frac{b-a}{2n}\left(f(a)+2\sum_{j=1}^{n-1}f(x_j) \quad +f(b)\right). \tag{6.4}$$

This formula is called the **Trapezoidal Rule**.
This formula is exact for the polynomial of degree at most 1.

Theorem 6.1
Let $f\colon[a,b]\longrightarrow\mathbb{R}$ be a twice continuously differentiable function. The reminder for this method is approximated as follows

$$|R_n|\leq\frac{(b-a)^3M_2}{12n^2}, \qquad M_2=\operatorname*{Sup}_{x\in[a,b]}|f^{(2)}(x)|.$$

Example 1 :
Let $f(x)=2x-1$ and $g(x)=x^2+3x-1$ defined on the interva] $[1,3]$. Use the trapezoidal Mathod for $n=5$ to give an approximation of the integrals $\displaystyle\int_1^3 f(x)dx$ and $\displaystyle\int_1^3 g(x)dx$.

Solution $x_k=1+\dfrac{2k}{5}$, $f(x_k)=1+\dfrac{4k}{5}$ and $g(x_k)=3+2k+\dfrac{4k^2}{25}$.

$$\int_1^3(2x-1)dx\approx\frac{1}{5}\left(1+5+2\sum_{k=1}^{4}(1+\frac{4k}{5})\right)=6.$$

$$\begin{aligned}\int_1^3(x^2+3x-1)dx &\approx \frac{1}{5}\left(3+17+2\sum_{k=1}^{4}(1+\frac{4k}{5})^2+3(1+\frac{4k}{5})-1\right)\\ &= \frac{1}{5}\left(20+2\sum_{k=1}^{4}(\frac{4k^2}{25}+2k+3)\right)=\frac{1}{5}(93+\frac{3}{5})=18.72.\end{aligned}$$

6.2 Simpson Method

In this method, we replace f on $[x_j, x_{j+1}]$ by the polynomial P of degree 2 which interpolates f to the points x_j, x_{j+1} and the middle point $m_j = \frac{x_j+x_{j+1}}{2}$.

$$\int_{x_j}^{x_{j+1}} f(x)dx \approx \int_{x_j}^{x_{j+1}} P_j(x)dx = \frac{x_{j+1}-x_j}{6}(f(x_j)+f(x_{j+1})+4f(m_j)).$$

$$\begin{aligned} P_j(x) &= f(x_j)\frac{(x_{j+1}-x)(x-m_j)}{(x_{j+1}-x_j)(x_j-m_j)} + f(m_j)\frac{(x_{j+1}-x)(x-x_j)}{(x_{j+1}-m_j)(m_j-x_j)} \\ &+ f(x_{j+1})\frac{(x-x_j)(x-m_j)}{(x_{j+1}-x_j)(x_{j+1}-m_j)}. \end{aligned}$$

$$\int_{x_j}^{x_{j+1}} f(x)dx \approx \int_{x_j}^{x_{j+1}} P_2(x)dx = \frac{x_{j+1}-x_j}{6}(f(x_j)+f(x_{j+1})+4f(m_j)).$$

If the the partition is uniform, $x_{j+1} - x_j = \dfrac{b-a}{n}$, then

$$S_n(f) = \frac{b-a}{6n}\sum_{j=0}^{n-1}(f(x_j)+f(x_{j+1})+4f(m_j)) = \frac{b-a}{6}\Big(f(a)+f(b)+2\sum_{j=1}^{n-1}f(x_j)+4\sum_{j=0}^{n-1}f(m_j)\Big).$$

This formula is called **The Simpson Formula** and it is exact for polynomials of degree at most 3.

An other expression of the Simpson formula If $n = 2m$ is an even integer and $P = \{x_0, x_1, \ldots, x_{2m-1}\}$ is a partition of the interval $[a, b]$, **The Simpson Formula** takes the following form

$$S_n(f) = \frac{b-a}{3n}\Big(f(a)+f(b)+4\sum_{j=0}^{m-1}f(x_{2j+1})+2\sum_{j=1}^{m-1}f(x_{2j})\Big).$$

Example 2 :

Let $g(x) = x^2 + 3x - 1$ and $h(x) = x^3$ defined on the interval $[1, 3]$. Use the Simpson Method for $n = 8$ to give an approximation of the integrals $\int_1^3 x^2 + 3x - 1dx$ and $\int_1^3 x^3 dx$.

Solution

$x_k = 1 + \dfrac{k}{4}$, $x_{2k} = 1 + \dfrac{k}{2}$ and $x_{2k+1} = 1 + \dfrac{2k+1}{4}$, $g(x_k) = 3 + \dfrac{5k}{4} + \dfrac{k^2}{16}$ and $h(x_k) = 1 + \dfrac{3k}{4} + \dfrac{3k^2}{16} + \dfrac{k^3}{64}$.

$$\begin{aligned} \int_1^3 (x^2+3x-1)dx &\approx \frac{1}{12}\left(3+17+4\sum_{k=0}^{3}g(x_{2k+1})+2\sum_{k=1}^{3}g(x_{2k})\right) \\ &= 18.5. \end{aligned}$$

$$\int_1^3 x^3 dx \approx \frac{1}{12}\left(1+27+4\sum_{k=0}^{3} h(x_{2k+1}) + 2\sum_{k=1}^{3} h(x_{2k})\right)$$
$$= .$$

Example 3 :
Let $f(x) = \sqrt{1+x^3}$ defined on the interval $[0,3]$. Use the Simpson Method for $n=6$ to give an approximation of the integral $\int_0^3 \sqrt{1+x^3}dx$.

Solution
$x_k = \frac{k}{2}$,

k	x_k	m	$mf(x_k)$
0	1	1	1
1	$\frac{1}{2}$	4	4.24264
2	1	2	2.82842
3	$\frac{3}{2}$	4	8.3666
4	2	2	3.4641
5	$\frac{5}{2}$	4	16.3095
6	3	1	5.2915
			41.50276

$\int_0^3 \sqrt{1+x^3}dx \approx 6.9171.$

Theorem 6.2
Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be a function. If the function f is C^4 on the interval $[a,b]$, the reminder for this method is approximated as follows

$$|R_n| \leq \frac{(b-a)^5 M_4}{2880n^4}, \qquad M_4 = \operatorname*{Sup}_{x\in[a,b]} |f^{(4)}(x)|.$$

7 Symmetry and Definite Integrals

Definition 7.1

1. *A function $f\colon [-a,a] \longrightarrow \mathbb{R}$ is odd if $f(-x) = -f(x)$ for all $x \in [-a,a]$.*
2. *A function $f\colon (-a,a) \longrightarrow \mathbb{R}$ is even if $f(-x) = f(x)$ for all $x \in [-a,a]$.*
3. *A function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ is $T-$periodic if $f(x+T) = f(x)$ for all $x \in \mathbb{R}$*

Theorem 7.2

1. *If f is an odd function on $[-a,a]$, then*

$$\int_{-a}^{a} f(x)\,dx = 0.$$

2. *If f is an even function on $[-a,a]$, then*

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$

3. *If f is $T-$periodic, then* $\int_{a}^{a+T} f(x)\,dx = \int_{0}^{T} f(x)\,dx$, *for all* $a \in \mathbb{R}$.

Proof .

1. If f is an odd function on $[-a,a]$, then $\int_{-a}^{0} f(x)\,dx \overset{t=-x}{=} -\int_{a}^{0} f(-t)\,dt = -\int_{0}^{a} f(t)\,dt$, then

$$\int_{-a}^{a} f(x)\,dx = 0.$$

2. If f is an even function on $[-a,a]$,then $\int_{-a}^{0} f(x)\,dx \overset{t=-x}{=} -\int_{a}^{0} f(-t)\,dt = \int_{0}^{a} f(t)\,dt$, then

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$

3. If f is $T-$periodic, then $\int_{T}^{a+T} f(x)\,dx \overset{t=x-T}{=} \int_{0}^{a} f(t+T)\,dt = \int_{0}^{a} f(t)\,dt$, then

$$\begin{aligned}\int_{a}^{a+T} f(x)\,dx &= \int_{a}^{0} f(x)\,dx + \int_{0}^{T} f(x)\,dx + \int_{T}^{a+T} f(x)\,dx\\ &= -\int_{0}^{a} f(x)\,dx + \int_{0}^{T} f(x)\,dx + \int_{0}^{a} f(x)\,dx\\ &= \int_{0}^{T} f(x)\,dx.\end{aligned}$$

□

Example 4 :
Compute $\int_{-1}^{1} x^2\,dx$, $\int_{-1}^{1} x^3\,dx$ and $\int_{5-\pi}^{5+\pi} \sin(x)\,dx$
Solution

$$\int_{-1}^{1} x^2\, dx = 2\int_{0}^{1} x^2\, dx = 2\left[\frac{1}{3}x^3\right]_0^1 = \frac{2}{3}.$$
$$\int_{-1}^{1} x^3\, dx = \left[\frac{1}{4}x^4\right]_{-1}^1 = 0.$$
$$\int_{5-\pi}^{5+\pi} \sin(x)\, dx = \int_{-\pi}^{\pi} \sin(x)\, dx = 0.$$

8 Exercises

Exercise 1 :
Evaluate the given integrals.

1) $\displaystyle\int \frac{e^{\tan^{-1} x}}{1+x^2}dx,$ 2) $\displaystyle\int \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}}dx,$ 3) $\displaystyle\int e^{\sin 2x} \cos 2x dx,$

4) $\displaystyle\int x^2 e^{x^3} dx,$ 5) $\displaystyle\int \frac{e^{2x}}{1+e^{2x}}dx,$ 6) $\displaystyle\int e^x \cos(1+2e^x)dx,$

7) $\displaystyle\int e^{3x} \sec^2(2+e^{3x})dx,$ 8) $\displaystyle\int 10^{\cos x} \sin x dx,$ 9) $\displaystyle\int \frac{4^{\sec^{-1} x}}{x\sqrt{x^2-1}}dx,$

10) $\displaystyle\int x 10^{x^2+3} dx.$