Integral Calculus (Math 228)

BEN AMIRA Aymen

Department of Mathematics

College of Sciences

King Saud University

Chapter 1: Integrals

Table of contents

Table of contents

Definition 1.1

Let $f : I \to \mathbb{R}$ be a function defined on an interval I. A function $F : I \to \mathbb{R}$ is called an **antiderivative** *of f on I if F is differentiable on I and $F'(x) = f(x)$, for all $x \in I$.*

Example 1.1

There are many antiderivatives of the function $f(x) = 2x$ on $\mathbb{R}$ such as:
$F_1(x) = x^2 + 1$, $F_2(x) = x^2$, $F_3(x) = x^2 + \dfrac{3}{5}$, $F_4 = x^2 - 5$.
Thus, all function $F(x) = x^2 + c$, with c is a constant, is an antiderivative of $f(x) = 2x$.

Proposition 1.1

Let F and G be two antiderivatives of a function f on an interval I , then there is a constant $c \in \mathbb{R}$ such that

$$F(x) = G(x) + c; \quad \forall x \in I$$

Antiderivative and indefinite integral

Definition 1.2

Let F be an anti-derivative of a function f on an interval I , we denote $\int f(x)dx$ any antiderivative i.e.

$$\int f(x)dx = F(x) + c; ; \ \forall x \in I \quad (1)$$

$\int f(x)dx$ is called the indefinite integral of f on I . In the equation (1),

- *the constant c is called the constant of integration,*
- *x is called the variable of integration,*
- *$f(x)$ is called the integrand.*

Basic rules of integrations

1. $\int 1dx = x + C.$
2. $\int x^n dx = \dfrac{x^{n+1}}{n+1} + C$, where $n \neq -1, \quad n \in \mathbb{Q}$.
3. $\int \cos x \; dx = \sin x + C$
4. $\int \sin x \; dx = -\cos x + C$
5. $\int \sec^2 x \; dx = \tan x + C$
6. $\int \csc^2 x \; dx = -\cot x + C$
7. $\int \sec x \tan x \; dx = \sec x + C$
8. $\int \csc x \cot x \; dx = -\csc x + C$

Basic rules of integrations

1. $\displaystyle\int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} + C$, where $n \neq -1, \quad n \in \mathbb{Q}$.
2. $\displaystyle\int \sin(f(x))f'(x)dx = -\cos(f(x)) + C$
3. $\displaystyle\int \cos(f(x))f'(x)dx = \sin(f(x)) + C$
4. $\displaystyle\int \sec^2(f(x))f'(x)dx = \tan(f(x)) + C$
5. $\displaystyle\int \csc^2(f(x))f'(x)dx = -\cot(f(x)) + C$
6. $\displaystyle\int \sec(f(x))\tan(f(x))f'(x)dx = \ sec(f(x)) + C$
7. $\displaystyle\int \csc(f(x))\cot(f(x))f'(x)dx = -\csc(f(x)) + C$

Example 1.2

- $\int \cos(3x+4)dx = \frac{1}{3}\int \cos(3x+4)3dx = \frac{1}{3}sin(3x+4)+C$
- $\int \tan^2 x \sec^2 x dx = \int (tanx)^2 sec^2 x dx = \frac{(tanx)^3}{3} + C$

Proposition 1.2 (Some important formulas)

- *If f is differntiable on an interval I , then*

$$\int \frac{d}{dx} f(x)dx = f(x) + c$$

- *If f has an antiderivative on an interval I , then*

$$\frac{d}{dx} \int f(x)dx = f(x).$$

$$\int \alpha f(x)dx = \alpha \int f(x)dx.$$

$$\int f(x) + g(x)dx = \int f(x)dx + \int g(x)dx.$$

Table of contents

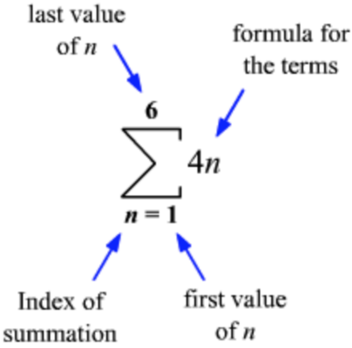
Summation Notation

- A series can be represented in a compact form, called summation or sigma notation.
- The Greek capital letter, $\sum$, is used to represent the sum.
- The series $4+8+12+16+20+24$ can be expressed as

$$\sum_{n=1}^{6} 4n$$

- The expression is read as the sum of $4n$ as n goes from 1 to 6.
- The variable n is called the index of summation.

Summation Notation



Summation Notation

Definition 2.1

Given a set of numbers $\{a_1, a_2, \ldots, a_n\}$, *the symbol* $\sum\limits_{k=1}^{n} a_k$ *represents their sum as follows*

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$

Theorem 2.1

For every $c \in \mathbb{R}$ *(constant), and* $n \in \mathbb{N}$, *we have*

$$\sum_{k=1}^{n} c = cn$$

Theorem 2.2

Let $\alpha, \beta \in \mathbb{R}$*, and* $n \in \mathbb{N}$*. For every* $a_1, a_2, \ldots, b_1, b_2, \cdots_n \in \mathbb{R}$ *we have*

$$\sum_{k=1}^{n} (\alpha a_k + \beta b_k) = \alpha \sum_{k=1}^{n} a_k + \beta \sum_{k=1}^{n} b_k$$

Theorem 2.3

For all $n \in \mathbb{N}$, *we have*

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Example 2.1

$$\begin{aligned}\sum_{k=1}^{4}(k^3-k+2) &= \sum_{k=1}^{4}k^3-\sum_{k=1}^{4}k+\sum_{k=1}^{4}2\\ &= \left(\frac{4(4+1)}{2}\right)^2-\frac{4(4+1)}{2}+(2\times 4)\\ &= 98\end{aligned}$$

Example 2.2

$$\begin{aligned}\sum_{k=1}^{n}(3k^2-2k+1) &= \sum_{k=1}^{n}3k^2-\sum_{k=1}^{n}2k+\sum_{k=1}^{n}1\\ &= 3\frac{n(n+1)(2n+1)}{6}-2\frac{n(n+1)}{2}+n.1\\ &= \frac{n}{2}\left[(n+1)(2n+1)-2(n+1)+2\right]\\ &= \frac{n}{2}(2n^2+3n+1-2n-2+2)\\ &= \frac{n}{2}(2n^2+n+1).\end{aligned}$$

Exercise 2.1

Using the formulas and properties from above determine the value of the following summations.

1. $\sum_{i=1}^{100}(3-2i)^2$ — 1293700
2. $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n^3}(i-1)^2$ — $\frac{1}{3}$
3. $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{5k}{n^2}$ — $\frac{5}{2}$

The approach of the integral of function by areas gives the geometrical sense of integration. The second approach consists in introducing a priori the antiderivative of function. The idea of the first approach is to cut the interval $[a; b]$ by a subdivision in sub-intervals $[a_j; a_{j+1}]$, then to add the areas of rectangles based on the intervals $[a_j; a_{j+1}]$.

In this section we assume that the function $f(x) \geq 0$ on the interval $[a, b]$.

Definition 2.2

The set $\{a = x_0, x_1, \ldots, x_n = b\}$ *is called a* **regular partition** *of the interval* $[a, b]$
if $x_i = x_0 + i\triangle x$ *for every* $i = 1, 2, \ldots, n$, *and* $\triangle x = \frac{b-a}{n}$.
This regular partition divides the interval $[a, b]$ *into n subintervals of the form* $[x_{i-1}, x_i]$ *where* $i = 1, 2, \ldots, n$.

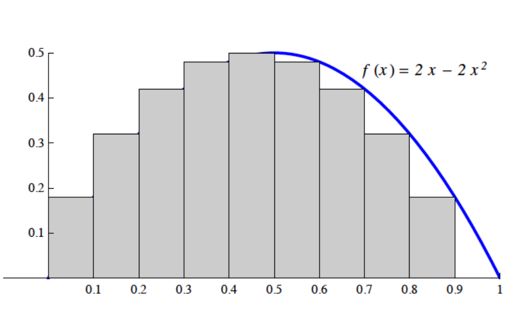
Area under the graph of a function :

If $f(x) \geq 0$ on the interval $[a, b]$ and $\{x_0 = a, x_1, \ldots, x_n = b\}$ is a regular partition of $[a, b]$, then the area under the graph of $f(x)$ can be approximated by n rectangles using the formula:

$$A_n = \sum_{k=1}^{n} f(x_i) \triangle x$$

Example 2.3

Approximate the area under the graph of $f(x) = 2x - 2x^2$ on the interval $[0, 1]$ using 10 rectangles .



Solution

1. $\triangle x = \frac{1-0}{10} = 0.1$
2. $x_0 = 0, x_1 = 0.1, x_2 = 0.2, \ldots, x_9 = 0.9, x_{10} = 1$
3. $A_{10} = \sum_{i=1}^{10} f(x_i)\triangle x = \sum_{i=1}^{10} (2x_i - 2x_i^2)0.1$
4. $A_{10} = 0.1[0.18+0.32+0.42+0.48+0.5+0.48+0.42+0.32+0.18+0]$
5. $A_{10} = 0.1(3.3) = 0.33$

Definition 2.3

Let $\{x_0 = a, x_1, \ldots, x_n = b\}$ *be a* **regular partition** *of the interval* $[a, b]$ *with* $\triangle x = \frac{b-a}{n}$. *Pick points* $c_1, c_2, \ldots, c_n$ *where* c_i *is any point in the subintrval* $[x_{i-1}, x_i], i = 1, 2, \ldots, n$.

The Riemann sum is:

$$R_n = \sum_{i=1}^{n} f(c_i)\triangle x$$

The area under the curve of $f(x)$ *is the limit of the Riemann sum.*

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\triangle x$$

Riemann Sums and Area

Example 2.4

Find the area under the curve of the function $f(x) = 3x + 1$ on the interval $[1, 3]$ using Riemann sum and c_i is the middle point of the subinterval.

Solution

1. $\triangle x = \frac{b-a}{n} = \frac{2}{n}$
2. $x_0 = 1, x_i = x_0 + i\triangle x = 1 + \frac{2i}{n}$ for every $i = 1, 2, \dots, n$
3. For every $i = 1, 2, \dots, n, c_i \in [x_{i-1}, x_i], c_i = \frac{x_i + x_{i-1}}{2} = 1 + \frac{2i-1}{n}$.
4. $R_n = \sum_{i=1}^{n} f(c_i)\triangle x = \sum_{i=1}^{n} [3(1 + \frac{2i-1}{n}) + 1]\frac{2}{n} = 8 + 6\frac{n(n+1)}{n^2} - \frac{6}{n}$.
5. The desired area A is: $A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} 8 + 6\frac{n(n+1)}{n^2} - \frac{6}{n} = 14$

Exercise 2.2

Do the last example where c_i is the end point of the subinterval.

Table of contents

The definite Integral

Definition 3.1

For any continuous function f defined on the interval $[a, b]$ the definite integral of f from a to b is:

$$\int_a^b f(x)dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\triangle x,$$

whenever the limit exists.
(where c_i is any point in the subintrval $[x_{i-1}, x_i], i = 1, 2, \ldots, n$).

Remark 3.1

1. Rieman Sum is the same for any choice of the points $c_1, c_2, \ldots, c_n$
2. When the limit exists we say that the function f is integrable.

The definite Integral

Remark 3.2

If the function f is continuous on $[a,b]$ and $f(x) \geq 0$ for every $x \in [a,b]$, then

1. $\displaystyle\int\limits_a^b f(x)dx \geq 0$
2. $\displaystyle\int\limits_a^b f(x)dx =$ The area under the curve of f

Example 3.1

$\displaystyle\int\limits_1^3 (3x+1)dx =$ Area under the curve of $f = \lim_{n\to\infty} R_n = 14.$

Exercise 3.1

Estimate the area of the region between the function and the x-axis on the given interval using $n = 6$ and using, the midpoints of the subintervals for the height of the rectangles.

1. $f(x) = x^3 - 2x^2 + 4$ *on* $[1, 4]$ $\quad A = 33.40625$
2. $g(x) = 4 - \sqrt{x^2 + 2}$ *on* $[-1, 3]$ $\quad A = 8.031494$
3. $h(x) = -x \cos\left(\dfrac{x}{3}\right)$ *on* $[0, 3]$ $\quad A = -3.449532$

Observation

In the last exercise, do not get excited about the negative area here. As we discussed in this section this just means that the graph, in this case, is below the x-axis.

The definite Integral

Theorem 3.1

If the function f is continuous on the interval $[a, b]$ then f is integrable on $[a, b]$.

Table of contents

Properties of the definite integral

(P_1) If c is a real number, then

$$\int_a^b c\,dx = c(b-a)$$

(P_2) If k is a real number and $f : [a;b] \to \mathbb{R}$ is an integrable function, then kf is integrable on $[a;b]$ and

$$\int_a^b kf(x)dx = k\int_a^b f(x)dx$$

Properties of the definite integral

(P_3) If f and g are two integrable functions on $[a;b]$, then $f+g$ is integrable on $[a;b]$ and

$$\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

(P_4) If f and g are two integrable functions on $[a;b]$, then $f-g$ is integrable on $[a;b]$ and

$$\int_a^b [f(x)-g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$$

The definite Integral

Properties of the definite integral

(P_5) If $a < c < b$ and if f is an integrable function on $[a;b]$, then f is integrable on $[a;c]$ and on $[c;b]$, moreover

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

(P_6) If f is integrable on $[a;b]$ and $\forall x \in [a,b], \quad f(x) \geq 0$ then

$$\int_a^b f(x)dx \geq 0$$

Properties of the definite integral

(P_7) If f and g are integrable on $[a;b]$ and $\forall x \in [a,b], \quad f(x) \geq g(x)$ then

$$\int_a^b f(x)dx \geq \int_a^b g(x)dx$$

(P_8) If f is integrable on $[a;b]$ then

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

The definite Integral

Example 4.1

1. $$\int_7^2 3(x^2-3)dx = 3\int_7^2 (x^2-3)dx = -3\int_2^7 (x^2-3)dx = -290$$
2. $$\int_7^2 3(x^2-3)dx = -3\int_2^7 (x^2-3)dx =$$
   $$-3\int_2^5 (x^2-3)dx - 3\int_5^7 (x^2-3)dx = -290$$
3. $x^2 \geq \dfrac{x^2}{x^2+4}$ then, $\displaystyle\int_{-1}^1 x^2 dx \geq \int_{-1}^1 \frac{x^2}{x^2+4}dx$

Table of contents

The fundamental Theorem of Calculus

Theorem 5.1 (The fundamental Theorem of Calculus (Part I))

If f is a continuous function on the interval $[a, b]$, and $G(x)$ is the antiderivative of $f(x)$ on $[a, b]$ then:

$$\int_a^b f(x)dx = [G(x)]_a^b = G(b) - G(a)$$

Remark 5.1

$$\int_a^b \frac{d}{dx}G(x)dx = G(b) - G(a)$$

Example 5.1

1. $$\int_0^2 (x^2 - 2x)dx = \left[\frac{x^3}{3} - x^2\right]_0^2 = \left(\frac{8}{3} - 4\right) - \left(\frac{0}{3} - 0\right) = -\frac{4}{3}$$
2. Find the area under the graph of $f(x) = \sin x$, on $[0, \pi]$.
   The area:
   $$A = \int_0^\pi \sin x \; dx = [-\cos x]_0^\pi = (-\cos \pi) - (-\cos 0) = 2$$

Theorem 5.2 (The fundamental Theorem of Calculus (Part II))

If f is a continuous function on the interval $[a, b]$ and $G(x) = \int_a^x f(t)dt$ for every $x \in [a, b]$ then $G'(x) = f(x)$ for every $x \in [a, b]$.

Example 5.2

1. $\frac{d}{dx} \int_0^x \sqrt{t^2 + 1}\, dt = \sqrt{x^2 + 1}$
2. $\frac{d}{dx} \int_1^x \frac{1}{t^2 + 1}\, dt = \frac{1}{x^2 + 1}$

Theorem 5.3

If f is a continuous function , g and h are deifferentiable functions then

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\ dt = f(h(x))h'(x) - f(g(x))g'(x)$$

Example 5.3

Find $G'(x)$, if $G(x) = \int_{1-x}^{x^2} \frac{1}{4+3t^2} dt$

Solution

$$G'(x) = \frac{d}{dx} \int_{1-x}^{x^2} \frac{1}{4+3t^2}\, dt =$$

$$= \frac{1}{4+3(x^2)^2}(2x) - \frac{1}{4+3(1-x)^2}(-1)$$

$$G'(x) = \frac{2x}{4+3(x^2)^2} + \frac{1}{4+3(1-x)^2}$$

Remark 5.2

1. If $g(x) = a$ and $h(x) = b$ then $\frac{d}{dx}\int\limits_a^b f(t)\ dt = f(b)(0) - f(a)(0) = 0$
2. If $g(x) = a$ and $h(x) = x$ then
$\frac{d}{dx}\int\limits_a^x f(t)\ dt = f(x)(1) - f(a)(0) = f(x)$

The fundamental Theorem of Calculus

Example 5.4

Find $F'(2)$, if $F(x) = \int\limits_{1}^{x^2} \frac{1}{t} dt$.

Solution

$$F'(2) = \frac{d}{dx} \int\limits_{1}^{x^2} \frac{1}{t} dt \, |_{x=2} = \left(\frac{1}{x^2}(2x) - 0 \right)_{x=2} = \left(\frac{2x}{x^2} \right)_{x=2} = \frac{2}{2} = 1$$

Example 5.5

Find the derivative of $F(x) = \int_2^{x^2} \ln(t)\ dt$.

Solution

$$\frac{d}{dx} \int\limits_{g(x)}^{h(x)} f(t)\ dt = f(h(x))h'(x) - f(g(x))g'(x)$$

$$F'(x) = \frac{d}{dx} \int_2^{x^2} \ln(t)\ dt = \ln(h(x))h'(x) - 0$$

where $h(x) = x^2$, so, we find $F'(x) = \ln(x^2)2x = 2x\ln(x^2)$

Example 5.6

Find the derivative of $F(x) = \int_{\cos(x)}^{5} t^3 \; dt$.

Solution

$$\frac{d}{dx} \int\limits_{g(x)}^{h(x)} f(t)\; dt = f(h(x))h'(x) - f(g(x))g'(x)$$

$$F'(x) = \frac{d}{dx} \int_{\cos(x)}^{5} t^3 \; dt = 0 - f(g(x))g'(x)$$

Where $g(x) = \cos x$, so we find: $F'(x) = -\cos^3 x\,(-\sin x) = \cos^3 x \sin x$.

Average value of a function

Theorem 5.4 (Mean Value Theorem for the definite integrals)

If f is continuous on $[a;b]$, then there is a number $c \in [a;b]$ such that

$$\int_a^b f(x)dx = (b-a)f(c)$$

Example 5.7

Find the value that satisfies the integral Mean value theorem for the function $f(x) = 4x^3 - 1$ on the interval $[1,2]$

$$f(c) = \frac{\int_1^2 (4x^3 - 1)}{2-1} \Rightarrow 4c^3 - 1 = [x^4 - x]_1^2 \Rightarrow 4c^3 - 1 = 14 \Rightarrow c = \sqrt[3]{\frac{15}{4}}$$

Note that $\sqrt[3]{\frac{15}{4}} \in [1,2]$

Average value of a function

Definition 5.1

Let f be a continuous on $[a;b]$. Then the average value $f_a v$ of f is given by

$$f_{av} = \frac{1}{b-a}\int_a^b f(x)dx$$

Example 5.8

Let $f(x) = 3x + 7$ on the interval $[0;1]$. We know that $\int_0^1 (3x+7)dx = \left[\frac{3x^2}{2} + 7x\right]_0^1 = \frac{3}{2} + 7 = \frac{17}{2}$. Then the point c where f assumed its average value verify $3c + 7 = \frac{17}{2}$, then $c = \frac{1}{2}$

Average value of a function

Example

Find f_{av} of the following function: $f(x) = x^2 - 2x$ on the interval $[1, 4]$

$$\int_1^4 (x^2 - 2x)dx = \left[\frac{x^3}{3} - x^2\right]_1^4 = 6$$

Hence $f_{av} = \frac{\int_1^4 (x^2 - 2x)dx}{4-1} = \frac{6}{3} = 2$

Exercise 5.1

1. *Find* f_{av} *of the function* $f(x) = (2x+1)^2$ *on the interval* $[0, 1]$
2. *Find* f_{av} *of the function* $f(x) = \sin^2 x \cos x$ *on the interval* $[0, \frac{\pi}{2}]$

Definition 5.2

1. *A function* $f : [-a; a] \to \mathbb{R}$ *is odd if* $f(-x) = -f(x)$ *for all* $x \in [-a; a]$.
2. *A function* $f : [-a; a] \to \mathbb{R}$ *is even if* $f(-x) = f(x)$ *for all* $x \in [-a; a]$.
3. *A function* $f : \mathbb{R} \to \mathbb{R}$ *is T-periodic if* $f(x + T) = f(x)$ *for all* $x \in \mathbb{R}$.

The fundamental Theorem of Calculus

Theorem 5.5

1. *If f is an odd function on $[-a;a]$, then*

$$\int_{-a}^{a} f(x)dx = 0$$

2. *If f is an even function on $[-a;a]$, then*

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

3. *If f is T-periodic, then, for all $a \in \mathbb{R}$*

$$\int_{a}^{a+T} f(x)dx = \int_{0}^{T} f(x)dx$$