

# *INTEGRAL CALCULUS (MATH 106)*

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# Chapter 1: Integrals

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# Antiderivative and indefinite integral

## Definition

Let  $f : I \rightarrow \mathbb{R}$  be a function defined on an interval  $I$ . A function  $F : I \rightarrow \mathbb{R}$  is called an **antiderivative** of  $f$  on  $I$  if  $F$  is differentiable on  $I$  and  $F'(x) = f(x)$ , for all  $x \in I$ .

## EXAMPLE

There are many antiderivatives of the function  $f(x) = 2x$  on  $\mathbb{R}$  such as:

$$F_1(x) = x^2 + 1, F_2(x) = x^2, F_3(x) = x^2 + \frac{3}{5}, F_4(x) = x^2 - 5.$$

Thus, all function  $F(x) = x^2 + c$ , with  $c$  is a constant, is an antiderivative of  $f(x) = 2x$ .

## Proposition

Let  $F$  and  $G$  be two antiderivatives of a function  $f$  on an interval  $I$ , then there is a constant  $c \in \mathbb{R}$  such that

$$F(x) = G(x) + c; \quad \forall x \in I$$

## Definition

Let  $F$  be an anti-derivative of a function  $f$  on an interval  $I$ , we denote  $\int f(x)dx$  any antiderivative i.e.

$$\int f(x)dx = F(x) + c; ; \forall x \in I \quad (1)$$

$\int f(x)dx$  is called the indefinite integral of  $f$  on  $I$ . In the equation (1),

- the constant  $c$  is called the constant of integration,
- $x$  is called the variable of integration,
- $f(x)$  is called the integrand.

# Basic rules of integrations

$$\textcircled{1} \int 1 dx = x + C.$$

$$\textcircled{2} \int x^n dx = \frac{x^{n+1}}{n+1} + C, \text{ where } n \neq -1, n \in \mathbb{Q}.$$

$$\textcircled{3} \int \cos x dx = \sin x + C$$

$$\textcircled{4} \int \sin x dx = -\cos x + C$$

$$\textcircled{5} \int \sec^2 x dx = \tan x + C$$

$$\textcircled{6} \int \csc^2 x dx = -\cot x + C$$

$$\textcircled{7} \int \sec x \tan x dx = \sec x + C$$

$$\textcircled{8} \int \csc x \cot x dx = -\csc x + C$$



# Basic rules of integrations

$$\textcircled{1} \int [f(x)]^n f'(x) dx = \frac{[f(x)]^{n+1}}{n+1} + C, \text{ where } n \neq -1, n \in \mathbb{Q}.$$

$$\textcircled{2} \int \sin(f(x)) f'(x) dx = -\cos(f(x)) + C$$

$$\textcircled{3} \int \cos(f(x)) f'(x) dx = \sin(f(x)) + C$$

$$\textcircled{4} \int \sec^2(f(x)) f'(x) dx = \tan(f(x)) + C$$

$$\textcircled{5} \int \csc^2(f(x)) f'(x) dx = -\cot(f(x)) + C$$

$$\textcircled{6} \int \sec(f(x)) \tan(f(x)) f'(x) dx = \sec(f(x)) + C$$

$$\textcircled{7} \int \csc(f(x)) \cot(f(x)) f'(x) dx = -\csc(f(x)) + C$$

## Example

- $\int \cos(3x + 4) dx = \frac{1}{3} \int \cos(3x + 4) 3 dx = \frac{1}{3} \sin(3x + 4) + C$
- $\int \tan^2 x \sec^2 x dx = \int (\tan x)^2 \sec^2 x dx = \frac{(\tan x)^3}{3} + C$

## Proposition (Some important formulas)

- If  $f$  is differentiable on an interval  $I$ , then

$$\int \frac{d}{dx} f(x) dx = f(x) + c$$

- If  $f$  has an antiderivative on an interval  $I$ , then

$$\frac{d}{dx} \int f(x) dx = f(x).$$

$$\int \alpha f(x) dx = \alpha \int f(x) dx.$$

$$\int f(x) + g(x) dx = \int f(x) dx + \int g(x) dx.$$

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# Substitution Method (changing variable method)

## Theorem (Substitution)

If  $F$  is an antiderivative of  $f$ , then  $f(g(x))g'(x)$  has antiderivative  $F(g(x))$ . Or,

$$\int f(g(x))g'(x)dx = F(g(x)) + c.$$

This is obvious. It is called "substitution" since it can be obtained by substituting  $u = g(x)$  and  $du = g'(x)dx$  into  $\int f(u)du = F(u) + c$

# Substitution Method (changing variable method)

## Example

Solve  $\int (4x + 1)^2 dx$ .

Put  $u = 4x + 1$  then  $du = 4dx$  hence  $\frac{1}{4} du = dx$

$$\begin{aligned}\int (4x + 1)^2 dx &= \int u^2 \frac{1}{4} du = \frac{1}{4} \int u^2 du = \frac{1}{4} \frac{u^3}{3} + C \\ &= \frac{1}{4} \frac{(4x + 1)^3}{3} + C\end{aligned}$$

## Example

$$\textcircled{1} \int (x^2 + 1)^n 2x dx \stackrel{u=x^2+1}{=} \int u^n du = \frac{u^{n+1}}{n+1} = \frac{(x^2 + 1)^{n+1}}{n+1} + c.$$

$$\textcircled{2} \int \sin(2x + 3) dx \stackrel{u=2x+3}{=} \frac{1}{2} \int \sin u du = -\frac{1}{2} \cos u + c =$$
$$-\frac{1}{2} \cos(2x + 3) + c.$$

$$\textcircled{3} \int \frac{1}{\cos^2(\pi x)} dx \stackrel{u=\pi x}{=} \frac{1}{\pi} \int \frac{1}{\cos^2(u)} du = \frac{1}{\pi} \tan(\pi x) + c.$$

## Example

$$\begin{aligned}\int (20x + 30)(x^2 + 3x - 5)^9 dx &= \int 10(2x + 3)(x^2 + 3x - 5)^9 dx \\ &= \int 10 \underbrace{(x^2 + 3x - 5)^9}_u \underbrace{(2x + 3) dx}_{du} \\ &= \int 10u^9 du \\ &= u^{10} + C \quad (\text{replace } u \text{ with } x^2 + 3x - 5) \\ &= (x^2 + 3x - 5)^{10} + C\end{aligned}$$



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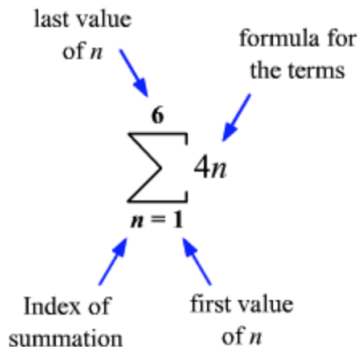
# Summation Notation

- A series can be represented in a compact form, called summation or sigma notation.
- The Greek capital letter,  $\sum$ , is used to represent the sum.
- The series  $4 + 8 + 12 + 16 + 20 + 24$  can be expressed as

$$\sum_{n=1}^6 4n$$

- The expression is read as the sum of  $4n$  as  $n$  goes from 1 to 6.
- The variable  $n$  is called the index of summation.

# Summation Notation



The diagram illustrates the components of the summation notation  $\sum_{n=1}^6 4n$ . The summation symbol  $\Sigma$  is in the center. Above it is the number 6, and below it is the expression  $n = 1$ . To the right of the symbol is the formula  $4n$ . Four blue arrows point from text labels to these parts: 'last value of  $n$ ' points to the 6; 'formula for the terms' points to  $4n$ ; 'Index of summation' points to  $n = 1$ ; and 'first value of  $n$ ' points to  $n = 1$ .

last value of  $n$

6

formula for the terms

$4n$

$n = 1$

Index of summation

first value of  $n$

# Summation Notation

## Definition

Given a set of numbers  $\{a_1, a_2, \dots, a_n\}$ , the symbol  $\sum_{k=1}^n a_k$  represents their sum as follows

$$\sum_{k=1}^n a_k = a_1 + a_2 + \dots + a_n$$

## Theorem

For every  $c \in \mathbb{R}$  (constant), and  $n \in \mathbb{N}$ , we have

$$\sum_{k=1}^n c = cn$$

## Theorem

Let  $\alpha, \beta \in \mathbb{R}$ , and  $n \in \mathbb{N}$ . For every  $a_1, a_2, \dots, b_1, b_2, \dots, b_n \in \mathbb{R}$  we have

$$\sum_{k=1}^n (\alpha a_k + \beta b_k) = \alpha \sum_{k=1}^n a_k + \beta \sum_{k=1}^n b_k$$

## Theorem

For all  $n \in \mathbb{N}$ , we have

$$\sum_{k=1}^n k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^n k^3 = \left( \frac{n(n+1)}{2} \right)^2$$

# Summation Notation

## Example

$$\begin{aligned}\sum_{k=1}^4 (k^3 - k + 2) &= \sum_{k=1}^4 k^3 - \sum_{k=1}^4 k + \sum_{k=1}^4 2 \\ &= \left(\frac{4(4+1)}{2}\right)^2 - \frac{4(4+1)}{2} + (2 \times 4) \\ &= 98\end{aligned}$$

# Summation Notation

## Example

$$\begin{aligned}\sum_{k=1}^n (3k^2 - 2k + 1) &= \sum_{k=1}^n 3k^2 - \sum_{k=1}^n 2k + \sum_{k=1}^n 1 \\ &= 3 \frac{n(n+1)(2n+1)}{6} - 2 \frac{n(n+1)}{2} + n \cdot 1 \\ &= \frac{n}{2} [(n+1)(2n+1) - 2(n+1) + 2] \\ &= \frac{n}{2} (2n^2 + 3n + 1 - 2n - 2 + 2) \\ &= \frac{n}{2} (2n^2 + n + 1).\end{aligned}$$



# Summation Notation

## Exercises

Using the formulas and properties from above determine the value of the following summations.

$$\textcircled{1} \sum_{i=1}^{100} (3 - 2i)^2 \qquad 1293700$$

$$\textcircled{2} \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{n^3} (i - 1)^2 \qquad \frac{1}{3}$$

$$\textcircled{3} \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{5k}{n^2} \qquad \frac{5}{2}$$

The approach of the integral of function by areas gives the geometrical sense of integration. The second approach consists in introducing a priori the antiderivative of function. The idea of the first approach is to cut the interval  $[a; b]$  by a subdivision in sub-intervals  $[a_j; a_{j+1}]$ , then to add the areas of rectangles based on the intervals  $[a_j; a_{j+1}]$ .

In this section we assume that the function  $f(x) \geq 0$  on the interval  $[a, b]$ .

## Definition

The set  $\{a = x_0, x_1, \dots, x_n = b\}$  is called a **regular partition** of the interval  $[a, b]$

if  $x_i = x_0 + i\Delta x$  for every  $i = 1, 2, \dots, n$ , and  $\Delta x = \frac{b-a}{n}$ .

This regular partition divides the interval  $[a, b]$  into  $n$  subintervals of the form  $[x_{i-1}, x_i]$  where  $i = 1, 2, \dots, n$ .

## Area under the graph of a function :

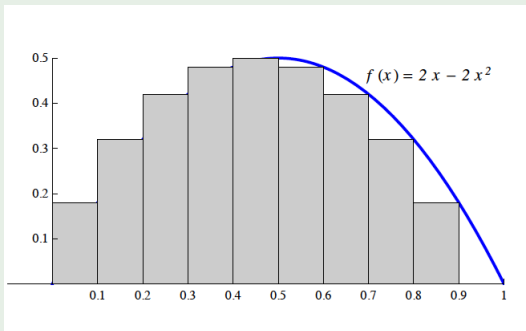
If  $f(x) \geq 0$  on the interval  $[a, b]$  and  $\{x_0 = a, x_1, \dots, x_n = b\}$  is a regular partition of  $[a, b]$ , then the area under the graph of  $f(x)$  can be approximated by  $n$  rectangles using the formula:

$$A_n = \sum_{k=1}^n f(x_k) \Delta x$$

# Riemann Sums and Area

## Example

Approximate the area under the graph of  $f(x) = 2x - 2x^2$  on the interval  $[0, 1]$  using 10 rectangles .



## Solution

①  $\Delta x = \frac{1-0}{10} = 0.1$

②  $x_0 = 0, x_1 = 0.1, x_2 = 0.2, \dots, x_9 = 0.9, x_{10} = 1$

③  $A_{10} = \sum_{i=1}^{10} f(x_i)\Delta x = \sum_{i=1}^{10} (2x_i - 2x_i^2)0.1$

④  $A_{10} = 0.1[0.18 + 0.32 + 0.42 + 0.48 + 0.5 + 0.48 + 0.42 + 0.32 + 0.18 + 0]$

⑤  $A_{10} = 0.1(3.3) = 0.33$

## Definition

Let  $\{x_0 = a, x_1, \dots, x_n = b\}$  be a **regular partition** of the interval  $[a, b]$  with  $\Delta x = \frac{b-a}{n}$ . Pick points  $c_1, c_2, \dots, c_n$  where  $c_i$  is any point in the subinterval  $[x_{i-1}, x_i]$ ,  $i = 1, 2, \dots, n$ .

The Riemann sum is:

$$R_n = \sum_{i=1}^n f(c_i) \Delta x$$

The area under the curve of  $f(x)$  is the limit of the Riemann sum.

$$A = \lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(c_i) \Delta x$$

## Example

Find the area under the curve of the function  $f(x) = 3x + 1$  on the interval  $[1, 3]$  using Riemann sum and  $c_i$  is the **middle** point of the subinterval.

## Solution

- ①  $\Delta x = \frac{b-a}{n} = \frac{2}{n}$
- ②  $x_0 = 1, x_i = x_0 + i\Delta x = 1 + \frac{2i}{n}$  for every  $i = 1, 2, \dots, n$
- ③ For every  $i = 1, 2, \dots, n, c_i \in [x_{i-1}, x_i], c_i = \frac{x_i + x_{i-1}}{2} = 1 + \frac{2i-1}{n}$ .
- ④  $R_n = \sum_{i=1}^n f(c_i)\Delta x = \sum_{i=1}^n [3(1 + \frac{2i-1}{n}) + 1]\frac{2}{n} = 8 + 6\frac{n(n+1)}{n^2} - \frac{6}{n}$ .
- ⑤ The desired area A is:  $A = \lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} 8 + 6\frac{n(n+1)}{n^2} - \frac{6}{n} = 14$



## Exercises

Do the last example where  $c_i$  is the **end point** of the subinterval.

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# The definite Integral

## Definition

For any continuous function  $f$  defined on the interval  $[a, b]$  the definite integral of  $f$  from  $a$  to  $b$  is:

$$\int_a^b f(x)dx = \lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(c_i)\Delta x,$$

whenever the limit exists.

(where  $c_i$  is any point in the subinterval  $[x_{i-1}, x_i], i = 1, 2, \dots, n$ ).

## Remark

- 1 Riemann Sum is the same for any choice of the points  $c_1, c_2, \dots, c_n$
- 2 When the limit exists we say that the function  $f$  is integrable.

# The definite Integral

## Remark

If the function  $f$  is continuous on  $[a, b]$  and  $f(x) \geq 0$  for every  $x \in [a, b]$ , then

$$\textcircled{1} \int_a^b f(x) dx \geq 0$$

$$\textcircled{2} \int_a^b f(x) dx = \text{The area under the curve of } f$$

## Example

$$\int_1^3 (3x + 1) dx = \text{Area under the curve of } f = \lim_{n \rightarrow \infty} R_n = 14.$$

# The definite Integral

## Exercises

Estimate the area of the region between the function and the  $x$ -axis on the given interval using  $n = 6$  and using, the midpoints of the subintervals for the height of the rectangles.

①  $f(x) = x^3 - 2x^2 + 4$  on  $[1, 4]$   $A = 33.40625$

②  $g(x) = 4 - \sqrt{x^2 + 2}$  on  $[-1, 3]$   $A = 8.031494$

③  $h(x) = -x \cos\left(\frac{x}{3}\right)$  on  $[0, 3]$   $A = -3.449532$

## Observation

In the last exercise, do not get excited about the **negative area** here. As we discussed in this section this just means that the graph, in this case, is below the  $x$ -axis.

# The definite Integral

## Theorem

If the function  $f$  is continuous on the interval  $[a, b]$  then  $f$  is integrable on  $[a, b]$  .

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## Properties of the definite integral

( $P_1$ ) If  $c$  is a real number, then

$$\int_a^b c dx = c(b - a)$$

( $P_2$ ) If  $k$  is a real number and  $f : [a; b] \rightarrow \mathbb{R}$  is an integrable function, then  $kf$  is integrable on  $[a; b]$  and

$$\int_a^b kf(x) dx = k \int_a^b f(x) dx$$



# The definite Integral

## Properties of the definite integral

**(P<sub>3</sub>)** If  $f$  and  $g$  are two integrable functions on  $[a; b]$ , then  $f + g$  is integrable on  $[a; b]$  and

$$\int_a^b [f(x) + g(x)] dx = \int_a^b f(x) dx + \int_a^b g(x) dx$$

**(P<sub>4</sub>)** If  $f$  and  $g$  are two integrable functions on  $[a; b]$ , then  $f - g$  is integrable on  $[a; b]$  and

$$\int_a^b [f(x) - g(x)] dx = \int_a^b f(x) dx - \int_a^b g(x) dx$$

# The definite Integral

## Properties of the definite integral

( $P_5$ ) If  $a < c < b$  and if  $f$  is an integrable function on  $[a; b]$ , then  $f$  is integrable on  $[a; c]$  and on  $[c; b]$ , moreover

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

( $P_6$ ) If  $f$  is integrable on  $[a; b]$  and  $\forall x \in [a, b], f(x) \geq 0$  then

$$\int_a^b f(x)dx \geq 0$$

# The definite Integral

## Properties of the definite integral

( $P_7$ ) If  $f$  and  $g$  are integrable on  $[a; b]$  and  $\forall x \in [a, b], f(x) \geq g(x)$  then

$$\int_a^b f(x)dx \geq \int_a^b g(x)dx$$

( $P_8$ ) If  $f$  is integrable on  $[a; b]$  then

$$\int_a^b f(x)dx = - \int_b^a f(x)dx$$

# The definite Integral

## Examples

$$\textcircled{1} \int_7^2 3(x^2 - 3)dx = 3 \int_7^2 (x^2 - 3)dx = -3 \int_2^7 (x^2 - 3)dx = -290$$

$$\textcircled{2} \int_7^2 3(x^2 - 3)dx = -3 \int_2^7 (x^2 - 3)dx =$$
$$-3 \int_2^5 (x^2 - 3)dx - 3 \int_5^7 (x^2 - 3)dx = -290$$

$$\textcircled{3} x^2 \geq \frac{x^2}{x^2+4} \text{ then, } \int_{-1}^1 x^2 dx \geq \int_{-1}^1 \frac{x^2}{x^2+4} dx$$

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# The fundamental Theorem of Calculus

## The fundamental Theorem of Calculus (Part I)

If  $f$  is a continuous function on the interval  $[a, b]$ , and  $G(x)$  is the antiderivative of  $f(x)$  on  $[a, b]$  then:

$$\int_a^b f(x)dx = [G(x)]_a^b = G(b) - G(a)$$

### Remark

$$\int_a^b \frac{d}{dx} G(x)dx = G(b) - G(a)$$

# The fundamental Theorem of Calculus

## Examples

① 
$$\int_0^2 (x^2 - 2x) dx = \left[ \frac{x^3}{3} - x^2 \right]_0^2 = \left( \frac{8}{3} - 4 \right) - \left( \frac{0}{3} - 0 \right) = -\frac{4}{3}$$

- ② Find the area under the graph of  $f(x) = \sin x$ , on  $[0, \pi]$ .  
The area:

$$A = \int_0^{\pi} \sin x \, dx = [-\cos x]_0^{\pi} = (-\cos \pi) - (-\cos 0) = 2$$

# The fundamental Theorem of Calculus

## The fundamental Theorem of Calculus (Part II)

If  $f$  is a continuous function on the interval  $[a, b]$  and  $G(x) = \int_a^x f(t)dt$  for every  $x \in [a, b]$  then  $G'(x) = f(x)$  for every  $x \in [a, b]$ .

## Examples

$$\textcircled{1} \quad \frac{d}{dx} \int_0^x \sqrt{t^2 + 1} dt = \sqrt{x^2 + 1}$$

$$\textcircled{2} \quad \frac{d}{dx} \int_1^x \frac{1}{t^2 + 1} dt = \frac{1}{x^2 + 1}$$



# The fundamental Theorem of Calculus

## Theorem

If  $f$  is a continuous function ,  $g$  and  $h$  are deifferentiable functions then

$$\frac{d}{dx} \int_{g(x)}^{h(x)} f(t) dt = f(h(x))h'(x) - f(g(x))g'(x)$$

# The fundamental Theorem of Calculus

## Example

Find  $G'(x)$ , if  $G(x) = \int_{1-x}^{x^2} \frac{1}{4+3t^2} dt$

## Solution

$$\begin{aligned} G'(x) &= \frac{d}{dx} \int_{1-x}^{x^2} \frac{1}{4+3t^2} dt = \\ &= \frac{1}{4+3(x^2)^2} (2x) - \frac{1}{4+3(1-x)^2} (-1) \\ G'(x) &= \frac{2x}{4+3(x^2)^2} + \frac{1}{4+3(1-x)^2} \end{aligned}$$

# The fundamental Theorem of Calculus

## Remark

1 If  $g(x) = a$  and  $h(x) = b$  then  $\frac{d}{dx} \int_a^b f(t) dt = f(b)(0) - f(a)(0) = 0$

2 If  $g(x) = a$  and  $h(x) = x$  then

$$\frac{d}{dx} \int_a^x f(t) dt = f(x)(1) - f(a)(0) = f(x)$$

# The fundamental Theorem of Calculus

## Example

Find  $F'(2)$ , if  $F(x) = \int_1^{x^2} \frac{1}{t} dt$ .

## Solution

$$F'(2) = \frac{d}{dx} \int_1^{x^2} \frac{1}{t} dt \Big|_{x=2} = \left( \frac{1}{x^2} (2x) - 0 \right)_{x=2} = \left( \frac{2x}{x^2} \right)_{x=2} = \frac{2}{2} = 1$$

# The fundamental Theorem of Calculus

## Example

Find the derivative of  $F(x) = \int_2^{x^2} \ln(t) dt$ .

## Solution

$$\frac{d}{dx} \int_{g(x)}^{h(x)} f(t) dt = f(h(x))h'(x) - f(g(x))g'(x)$$

$$F'(x) = \frac{d}{dx} \int_2^{x^2} \ln(t) dt = \ln(h(x))h'(x) - 0$$

where  $h(x) = x^2$ , so, we find  $F'(x) = \ln(x^2)2x = 2x \ln(x^2)$

# The fundamental Theorem of Calculus

## Example

Find the derivative of  $F(x) = \int_{\cos(x)}^5 t^3 dt$ .

## Solution

$$\frac{d}{dx} \int_{g(x)}^{h(x)} f(t) dt = f(h(x))h'(x) - f(g(x))g'(x)$$

$$F'(x) = \frac{d}{dx} \int_{\cos(x)}^5 t^3 dt = 0 - f(g(x))g'(x)$$

Where  $g(x) = \cos x$ , so we find:  $F'(x) = -\cos^3 x (-\sin x) = \cos^3 x \sin x$ .

# Average value of a function

## Mean Value Theorem for the definite integrals

If  $f$  is continuous on  $[a; b]$ , then there is a number  $c \in [a; b]$  such that

$$\int_a^b f(x) dx = (b - a)f(c)$$

### Example

Find the value that satisfies the integral Mean value theorem for the function  $f(x) = 4x^3 - 1$  on the interval  $[1, 2]$

$$f(c) = \frac{\int_1^2 (4x^3 - 1) dx}{2 - 1} \Rightarrow 4c^3 - 1 = [x^4 - x]_1^2 \Rightarrow 4c^3 - 1 = 14 \Rightarrow c = \sqrt[3]{\frac{15}{4}}$$

Note that  $\sqrt[3]{\frac{15}{4}} \in [1, 2]$

# Average value of a function

## Definition

Let  $f$  be a continuous on  $[a; b]$ . Then the average value  $f_{av}$  of  $f$  is given by

$$f_{av} = \frac{1}{b-a} \int_a^b f(x) dx$$

## Example

Let  $f(x) = 3x + 7$  on the interval  $[0; 1]$ . We know that

$$\int_0^1 (3x + 7) dx = \left[ \frac{3x^2}{2} + 7x \right]_0^1 = \frac{3}{2} + 7 = \frac{17}{2}. \text{ Then the point } c \text{ where } f$$

assumed its average value verify  $3c + 7 = \frac{17}{2}$ , then  $c = \frac{1}{2}$



# Average value of a function

## Example

Find  $f_{av}$  of the following function:  $f(x) = x^2 - 2x$  on the interval  $[1, 4]$

$$\int_1^4 (x^2 - 2x) dx = \left[ \frac{x^3}{3} - x^2 \right]_1^4 = 6$$

$$\text{Hence } f_{av} = \frac{\int_1^4 (x^2 - 2x) dx}{4-1} = \frac{6}{3} = 2$$

## Exercise

- 1 Find  $f_{av}$  of the function  $f(x) = (2x + 1)^2$  on the interval  $[0, 1]$
- 2 Find  $f_{av}$  of the function  $f(x) = \sin^2 x \cos x$  on the interval  $[0, \frac{\pi}{2}]$

## Definition

- 1 A function  $f : [-a; a] \rightarrow \mathbb{R}$  is odd if  $f(-x) = -f(x)$  for all  $x \in [-a; a]$ .
- 2 A function  $f : [-a; a] \rightarrow \mathbb{R}$  is even if  $f(-x) = f(x)$  for all  $x \in [-a; a]$ .
- 3 A function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is T-periodic if  $f(x + T) = f(x)$  for all  $x \in \mathbb{R}$ .

# The fundamental Theorem of Calculus

## Theorem

- ① If  $f$  is an odd function on  $[-a; a]$ , then

$$\int_{-a}^a f(x) dx = 0$$

- ② If  $f$  is an even function on  $[-a; a]$ , then

$$\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$$

- ③ If  $f$  is  $T$ -periodic, then, for all  $a \in \mathbb{R}$

$$\int_a^{a+T} f(x) dx = \int_0^T f(x) dx$$

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## Trapezoidal Rule

It is used to approximate  $\int_a^b f(x)dx$  with a regular partition of the interval  $[a, b]$ , where  $\Delta x = \frac{b-a}{n}$ , by using the formula

$$\int_a^b f(x)dx \approx \frac{b-a}{2n} [f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

# Numerical Intergration

## Example

Approximate the integral  $\int_0^1 \sqrt{x+x^2} dx$  using **Trapezoidal rule** with  $n = 4$ .

## Solution

$[a, b] = [0, 1]$ ,  $f(x) = \sqrt{x+x^2}$ ,  $\Delta x = \frac{1-0}{4} = 0.25$

$n$	$x_n$	$f(x_n)$	$m$	$mf(x_n)$
0	0	0	1	0
1	0.25	0.559017	2	1.11803
2	0.5	0.86625	2	1.73205
3	0.75	1.14564	2	2.29129
4	1	1.41421	1	1.41421
				6.55559

$$\int_0^1 \sqrt{x+x^2} dx \approx \frac{0-1}{2(4)} [f(0) + 2f(0.25) + 2f(0.5) + 2f(0.75) + f(1)]$$

$$\int_0^1 \sqrt{x+x^2} dx \approx 0.819448$$

## Trapezoidal Rule

It is used to approximate  $\int_a^b f(x)dx$  with a regular partition of the interval  $[a, b]$ , where  $\Delta x = \frac{b-a}{n}$ , and  $n$  **even**, by using the formula

$$\int_a^b f(x)dx \approx \frac{b-a}{3n} [f(x_0) + 4f(x_1) + 2f(x_2) + \dots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$



# Numerical Intergration

## Example

Approximate the integral  $\int_0^{10} \sqrt{10x - x^2} dx$  using **Simpson's rule** with  $n = 4$ .

## Solution

$$[a, b] = [0, 10], f(x) = \sqrt{10x - x^2}, \Delta x = \frac{10-0}{4} = 2.5$$

$n$	$x_n$	$f(x_n)$	$m$	$mf(x_n)$
0	0	0	1	0
1	2.5	4.33013	4	17.3204
2	5	5	2	10
3	7.5	4.33013	4	17.3204
4	10	0	1	0
				44.6408

$$\int_0^{10} \sqrt{10x - x^2} dx \approx \frac{10-0}{3(4)} [f(0) + 4f(2.5) + 2f(5) + 4f(7.5) + f(10)]$$

$$\int_0^{10} \sqrt{10x - x^2} dx \approx 37.2007$$

## Exercise

- 1 Approximate the integral  $\int_2^4 \frac{1}{x-1} dx$  using Trapezoidal rule with  $n = 4$ .
- 2 Approximate the integral  $\int_0^2 \frac{x}{x+1} dx$  using Simpson's rule with  $n = 4$ .