INTEGRAL CALCULUS (MATH 106)

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Chapter 1: Integrals

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Antiderivative and indefinite integral

Definition

Let $f : I \to \mathbb{R}$ be a function defined on an interval I . A function $F : I \to \mathbb{R}$ is called an **antiderivative** of f on I if F is differentiable on I and $F'(x) = f(x)$, for all $x \in I$.

EXAMPLE

There are many antiderivatives of the function $f(x) = 2x$ on $\mathbb{R}$ such as:
$F_1(x) = x^2 + 1,\ F_2(x) = x^2,\ F_3(x) = x^2 + \frac{3}{5},\ F_4 = x^2 - 5.$
Thus, all function $F(x) = x^2 + c$, with c is a constant, is an antiderivative of $f(x) = 2x$.

Antiderivative and indefinite integral

Proposition

Let F and G be two antiderivatives of a function f on an interval I , then there is a constant $c \in \mathbb{R}$ such that

$$F(x) = G(x) + c; \;\; \forall x \in I$$

Definition

Let F be an anti-derivative of a function f on an interval I , we denote $\int f(x)dx$ any antiderivative i.e.

$$\int f(x)dx = F(x) + c;\; ;\; \forall x \in I \quad (1)$$

$\int f(x)dx$ is called the indefinite integral of f on I . In the equation (1),

- the constant c is called the constant of integration,
- x is called the variable of integration,
- $f(x)$ is called the integrand.

Basic rules of integrations

1. $\int 1 dx = x + C.$
2. $\int x^n dx = \dfrac{x^{n+1}}{n+1} + C$, where $n \neq 1, \quad n \in \mathbb{Q}$.
3. $\int \cos x \, dx = \sin x + C$
4. $\int \sin x \, dx = \cos x + C$
5. $\int \sec^2 x \, dx = \tan x + C$
6. $\int \csc^2 x \, dx = \cot x + C$
7. $\int \sec x \tan x \, dx = \sec x + C$
8. $\int \csc x \cot x \, dx = \csc x + C$

Basic rules of integrations

1. $$\int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} + C, \text{ where } n \neq 1, \quad n \in \mathbb{Q}.$$
2. $$\int \sin(f(x))f'(x)dx = \ \cos(f(x)) + C$$
3. $$\int \cos(f(x))f'(x)dx = \sin(f(x)) + C$$
4. $$\int \sec^2(f(x))f'(x)dx = \tan(f(x)) + C$$
5. $$\int \csc^2(f(x))f'(x)dx = \ \cot(f(x)) + C$$
6. $$\int \sec(f(x))\tan(f(x))f'(x)dx = \ sec(f(x)) + C$$
7. $$\int \csc(f(x))\cot(f(x))f'(x)dx = \ \csc(f(x)) + C$$

Example

- $\int \cos(3x+4)dx = \frac{1}{3}\int \cos(3x+4)3dx = \frac{1}{3}sin(3x+4) + C$
- $\int \tan^2 x \sec^2 x dx = \int (tanx)^2 sec^2 xdx = \frac{(tanx)^3}{3} + C$

Proposition (Some important formulas)

- If f is differntiable on an interval I , then

$$\int \frac{d}{dx} f(x)dx = f(x) + c$$

- If f has an antiderivative on an interval I , then

$$\frac{d}{dx} \int f(x)dx = f(x).$$

$$\int \alpha f(x)dx = \alpha \int f(x)dx.$$

$$\int f(x) + g(x)dx = \int f(x)dx + \int g(x)dx.$$

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Substitution Method (changing variable method)

Theorem (Substitution)

If F is an antiderivative of f , then $f(g(x))g'(x)$ has antiderivative $F(g(x))$. Or,

$$\int f(g(x))g'(x)dx = F(g(x)) + c.$$

This is obvious. It is called "substitution" since it can be obtained by substituting $u = g(x)$ and $du = g'(x)dx$ into $\int f(u)du = F(u) + c$

Substitution Method (changing variable method)

Example

Solve $\int (4x+1)^2 dx$.

Put $u = 4x + 1$ then $du = 4dx$ hence $\frac{1}{4}du = dx$

$\int (4x+1)^2 dx = \int u^2 \frac{1}{4} du = \frac{1}{4} \int u^2 du = \frac{1}{4} \frac{u^3}{3} + C$

$= \frac{1}{4} \frac{(4x+1)^3}{3} + C$

Example

1. $\displaystyle\int (x^2+1)^n 2x dx \overset{u=x^2+1}{=} \int u^n du = \frac{u^{n+1}}{n+1} = \frac{(x^2+1)^{n+1}}{n+1} + c.$
2. $\displaystyle\int \sin(2x+3) dx \overset{u=2x+3}{=} \frac{1}{2}\int \sin u du = -\frac{1}{2}\cos u + c = -\frac{1}{2}\cos(2x+3) + c.$
3. $\displaystyle\int \frac{1}{\cos^2(\pi x)} dx \overset{u=\pi x}{=} \frac{1}{\pi}\int \frac{1}{cos^2(u)} du = \frac{1}{\pi}\tan(\pi x) + c.$

Example

$$\int (20x+30)(x^2+3x-5)^9\,dx = \int 10(2x+3)(x^2+3x-5)^9\,dx$$

$$= \int 10(\underbrace{x^2+3x-5}_{u})^9\,\underbrace{(2x+3)\,dx}_{du}$$

$$= \int 10u^9\,du$$

$$= u^{10} + C \quad (\text{replace } u \text{ with } x^2+3x-5)$$

$$= (x^2+3x-5)^{10} + C$$

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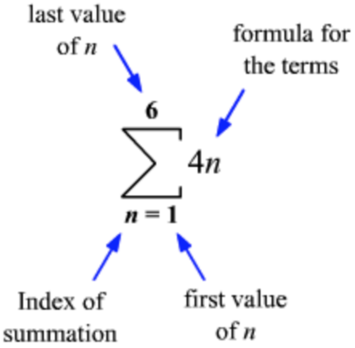
Summation Notation

- A series can be represented in a compact form, called summation or sigma notation.
- The Greek capital letter, $\sum$, is used to represent the sum.
- The series $4 + 8 + 12 + 16 + 20 + 24$ can be expressed as

$$\sum_{n=1}^{6} 4n$$

- The expression is read as the sum of $4n$ as n goes from 1 to 6.
- The variable n is called the index of summation.

Summation Notation



Summation Notation

Definition

Given a set of numbers $\{a_1, a_2, \ldots, a_n\}$, the symbol $\sum_{k=1}^{n} a_k$ represents their sum as follows

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$

Theorem

For every $c \in \mathbb{R}$ (constant), and $n \in \mathbb{N}$, we have

$$\sum_{k=1}^{n} c = cn$$

Theorem

Let $\alpha, \beta \in \mathbb{R}$, and $n \in \mathbb{N}$. For every $a_1, a_2, \ldots, b_1, b_2, \cdots_n \in \mathbb{R}$ we have

$$\sum_{k=1}^{n} (\alpha a_k + \beta b_k) = \alpha \sum_{k=1}^{n} a_k + \beta \sum_{k=1}^{n} b_k$$

Summation Notation

Theorem

For all $n \in \mathbb{N}$, we have

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Example

$$\sum_{k=1}^{4}(k^3 - k + 2) = \sum_{k=1}^{4} k^3 - \sum_{k=1}^{4} k + \sum_{k=1}^{4} 2$$
$$= \left(\frac{4(4+1)}{2}\right)^2 - \frac{4(4+1)}{2} + (2 \times 4)$$
$$= 98$$

Summation Notation

Example

$$\begin{aligned}\sum_{k=1}^{n}(3k^2-2k+1) &= \sum_{k=1}^{4}3k^2-\sum_{k=1}^{4}2k+\sum_{k=1}^{4}1\\ &= 3\frac{n(n+1)(2n+1)}{6}-2\frac{n(n+1)}{2}+n.1\\ &= \frac{n}{2}\left[(n+1)(2n+1)-2(n+1)+2\right]\\ &= \frac{n}{2}(2n^2+3n+1-2n-2+2)\\ &= \frac{n}{2}(2n^2+n+1).\end{aligned}$$

Exercises

Using the formulas and properties from above determine the value of the following summations.

1. $\sum_{i=1}^{100}(3-2i)^2$ 1293700
2. $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n^3}(i-1)^2$ $\frac{1}{3}$
3. $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{5k}{n^2}$ $\frac{5}{2}$

Riemann Sums and Area

The approach of the integral of function by areas gives the geometrical sense of integration. The second approach consists in introducing a priori the antiderivative of function. The idea of the first approach is to cut the interval $[a; b]$ by a subdivision in sub-intervals $[a_j; a_{j+1}]$, then to add the areas of rectangles based on the intervals $[a_j; a_{j+1}]$.

In this section we assume that the function $f(x) \geq 0$ on the interval $[a, b]$.

Definition

The set $\{a = x_0, x_1, \ldots, x_n = b\}$ is called a **regular partition** of the interval $[a, b]$
if $x_i = x_0 + i\triangle x$ for every $i = 1, 2, \ldots, n$, and $\triangle x = \frac{b-a}{n}$.
This regular partition divides the interval $[a, b]$ into n subintervals of the form $[x_{i-1}, x_i]$ where $i = 1, 2, \ldots, n$.

Area under the graph of a function :

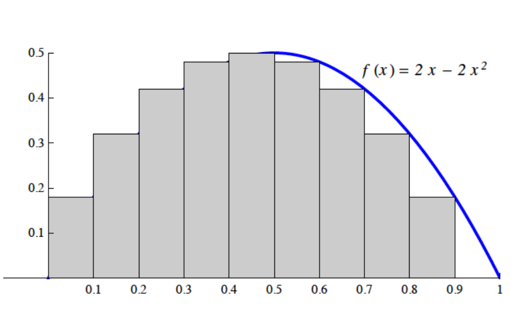
If $f(x) \geq 0$ on the interval $[a, b]$ and $\{x_0 = a, x_1, \ldots, x_n = b\}$ is a regular partition of $[a, b]$, then the area under the graph of $f(x)$ can be approximated by n rectangles using the formula:

$$A_n = \sum_{k=1}^{n} f(x_i) \triangle x$$

Riemann Sums and Area

Example

Approximate the area under the graph of $f(x) = 2x - 2x^2$ on the interval $[0, 1]$ using 10 rectangles .



Solution

1. $\triangle x = \frac{1-0}{10} = 0.1$
2. $x_0 = 0, x_1 = 0.1, x_2 = 0.2, \ldots, x_9 = 0.9, x_{10} = 1$
3. $A_{10} = \sum_{i=1}^{10} f(x_i)\triangle x = \sum_{i=1}^{10} (2x_i - 2x_i^2)0.1$
4. $A_{10} = 0.1[0.18+0.32+0.42+0.48+0.5+0.48+0.42+0.32+0.18+0]$
5. $A_{10} = 0.1(3.3) = 0.33$

Definition

Let $\{x_0 = a, x_1, \ldots, x_n = b\}$ be a **regular partition** of the interval $[a, b]$ with $\triangle x = \frac{b-a}{n}$. Pick points $c_1, c_2, \ldots, c_n$ where c_i is any point in the subintrval $[x_{i-1}, x_i], i = 1, 2, \ldots, n$.

The Riemann sum is:

$$R_n = \sum_{i=1}^{n} f(c_i)\triangle x$$

The area under the curve of $f(x)$ is the limit of the Riemann sum.

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\triangle x$$

Riemann Sums and Area

Example

Find the area under the curve of the function $f(x) = 3x + 1$ on the interval $[1, 3]$ using Riemann sum and c_i is the middle point of the subinterval.

Solution

1. $\triangle x = \frac{b-a}{n} = \frac{2}{n}$
2. $x_0 = 1, x_i = x_0 + i\triangle x = 1 + \frac{2i}{n}$ for every $i = 1, 2, \ldots, n$
3. For every $i = 1, 2, \ldots, n, c_i \in [x_{i-1}, x_i], c_i = \frac{x_i + x_{i-1}}{2} = 1 + \frac{2i-1}{n}$.
4. $R_n = \sum_{i=1}^{n} f(c_i)\triangle x = \sum_{i=1}^{n} [3(1 + \frac{2i-1}{n}) + 1]\frac{2}{n} = 8 + 6\frac{n(n+1)}{n^2} - \frac{6}{n}$.
5. The desired area A is: $A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} 8 + 6\frac{n(n+1)}{n^2} - \frac{6}{n} = 14$

Exercises

Do the last example where c_i is the end point of the subinterval.

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The definite Integral

Definition

For any continuous function f defined on the interval $[a, b]$ the definite integral of f from a to b is:

$$\int_a^b f(x)dx = \lim_{n \to \infty} R_n = \lim_{n \to \infty} \sum_{i=1}^{n} f(c_i) \triangle x,$$

whenever the limit exists.
(where c_i is any point in the subintrval $[x_{i-1}, x_i], i = 1, 2, \ldots, n$).

Remark

1. Rieman Sum is the same for any choice of the points $c_1, c_2, \ldots, c_n$
2. When the limit exists we say that the function f is integrable.

The definite Integral

Remark

If the function f is continuous on $[a,b]$ and $f(x) \geq 0$ for every $x \in [a,b]$, then

1. $\displaystyle\int_a^b f(x)dx \geq 0$
2. $\displaystyle\int_a^b f(x)dx =$ The area under the curve of f

Example

$$\int_1^3 (3x+1)dx = \text{Area under the curve of } f = \lim_{n\to\infty} R_n = 14.$$

Exercises

Estimate the area of the region between the function and the x-axis on the given interval using $n = 6$ and using, the midpoints of the subintervals for the height of the rectangles.

1. $f(x) = x^3 - 2x^2 + 4$ on $[1, 4]$ $A = 33.40625$
2. $g(x) = 4 - \sqrt{x^2 + 2}$ on $[-1, 3]$ $A = 8.031494$
3. $h(x) = -x \cos\left(\frac{x}{3}\right)$ on $[0, 3]$ $A = -3.449532$

Observation

In the last exercise, do not get excited about the negative area here. As we discussed in this section this just means that the graph, in this case, is below the x-axis.

Theorem

If the function f is continuous on the interval $[a, b]$ then f is integrable on $[a, b]$.

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Properties of the definite integral

(P_1) If c is a real number, then

$$\int_a^b c\,dx = c(b-a)$$

(P_2) If k is a real number and $f : [a; b] \to \mathbb{R}$ is an integrable function, then kf is integrable on $[a; b]$ and

$$\int_a^b kf(x)dx = k\int_a^b f(x)dx$$

The definite Integral

Properties of the definite integral

(P_3) If f and g are two integrable functions on $[a;b]$, then $f+g$ is integrable on $[a;b]$ and

$$\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

(P_4) If f and g are two integrable functions on $[a;b]$, then $f-g$ is integrable on $[a;b]$ and

$$\int_a^b [f(x)-g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$$

Properties of the definite integral

(P_5) If $a < c < b$ and if f is an integrable function on $[a; b]$, then f is integrable on $[a; c]$ and on $[c; b]$, moreover

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

(P_6) If f is integrable on $[a; b]$ and $\forall x \in [a, b], \quad f(x) \geq 0$ then

$$\int_a^b f(x)dx \geq 0$$

Properties of the definite integral

(P_7) If f and g are integrable on $[a;b]$ and $\forall x \in [a,b], \quad f(x) \geq g(x)$ then

$$\int_a^b f(x)dx \geq \int_a^b g(x)dx$$

(P_8) If f is integrable on $[a;b]$ then

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

The definite Integral

Examples

1. $$\int_{7}^{2} 3(x^2-3)dx = 3\int_{7}^{2}(x^2-3)dx = -3\int_{2}^{7}(x^2-3)dx = -290$$
2. $$\int_{7}^{2} 3(x^2-3)dx = -3\int_{2}^{7}(x^2-3)dx =$$
   $$-3\int_{2}^{5}(x^2-3)dx - 3\int_{5}^{7}(x^2-3)dx = -290$$
3. $x^2 \geq \frac{x^2}{x^2+4}$ then, $\int_{-1}^{1} x^2 dx \geq \int_{-1}^{1}\frac{x^2}{x^2+4}dx$

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The fundamental Theorem of Calculus (Part I)

If f is a continuous function on the interval $[a, b]$, and $G(x)$ is the antiderivative of $f(x)$ on $[a, b]$ then:

$$\int_a^b f(x)dx = [G(x)]_a^b = G(b) - G(a)$$

Remark

$$\int_a^b \frac{d}{dx} G(x)dx = G(b) - G(a)$$

Examples

1. $$\int_0^2 (x^2 - 2x)dx = \left[\frac{x^3}{3} - x^2\right]_0^2 = \left(\frac{8}{3} - 4\right) - \left(\frac{0}{3} - 0\right) = -\frac{4}{3}$$
2. Find the area under the graph of $f(x) = \sin x$, on $[0, \pi]$.
   The area:
   $$A = \int_0^\pi \sin x \; dx = [-\cos x]_0^\pi = (-\cos \pi) - (-\cos 0) = 2$$

The fundamental Theorem of Calculus (Part II)

If f is a continuous function on the interval $[a, b]$ and $G(x) = \int_a^x f(t)dt$ for every $x \in [a, b]$ then $G'(x) = f(x)$ for every $x \in [a, b]$.

Examples

1. $\frac{d}{dx} \int_0^x \sqrt{t^2 + 1}\, dt = \sqrt{x^2 + 1}$
2. $\frac{d}{dx} \int_1^x \frac{1}{t^2 + 1}\, dt = \frac{1}{x^2 + 1}$

The fundamental Theorem of Calculus

Theorem

If f is a continuous function , g and h are deifferentiable functions then

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f(h(x))h'(x) - f(g(x))g'(x)$$

Example

Find $G'(x)$, if $G(x) = \int_{1-x}^{x^2} \frac{1}{4+3t^2} dt$

Solution

$$G'(x) = \frac{d}{dx} \int_{1-x}^{x^2} \frac{1}{4+3t^2}\, dt =$$

$$= \frac{1}{4+3(x^2)^2}(2x) - \frac{1}{4+3(1-x)^2}(-1)$$

$$G'(x) = \frac{2x}{4+3(x^2)^2} + \frac{1}{4+3(1-x)^2}$$

Remark

1. If $g(x) = a$ and $h(x) = b$ then $\frac{d}{dx}\int_a^b f(t)\ dt = f(b)(0) - f(a)(0) = 0$
2. If $g(x) = a$ and $h(x) = x$ then
$\frac{d}{dx}\int_a^x f(t)\ dt = f(x)(1) - f(a)(0) = f(x)$

Example

Find $F'(2)$, if $F(x) = \int_{1}^{x^2} \frac{1}{t} dt$.

Solution

$$F'(2) = \frac{d}{dx} \int_{1}^{x^2} \frac{1}{t} dt \,\Big|_{x=2} = \left(\frac{1}{x^2}(2x) - 0 \right)_{x=2} = \left(\frac{2x}{x^2} \right)_{x=2} = \frac{2}{2} = 1$$

Example

Find the derivative of $F(x) = \int_2^{x^2} \ln(t)\ dt$.

Solution

$$\frac{d}{dx} \int_{g(x)}^{h(x)} f(t)\ dt = f(h(x))h'(x) - f(g(x))g'(x)$$

$$F'(x) = \frac{d}{dx} \int_2^{x^2} \ln(t)\ dt = \ln(h(x))h'(x) - 0$$

where $h(x) = x^2$, so, we find $F'(x) = \ln(x^2)2x = 2x\ln(x^2)$

Example

Find the derivative of $F(x) = \int_{\cos(x)}^{5} t^3 \, dt$.

Solution

$$\frac{d}{dx} \int_{g(x)}^{h(x)} f(t)\, dt = f(h(x))h'(x) - f(g(x))g'(x)$$

$$F'(x) = \frac{d}{dx} \int_{\cos(x)}^{5} t^3 \, dt = 0 - f(g(x))g'(x)$$

Where $g(x) = \cos x$, so we find: $F'(x) = -\cos^3 x\,(-\sin x) = \cos^3 x \sin x$.

Average value of a function

Mean Value Theorem for the definite integrals

If f is continuous on $[a;b]$, then there is a number $c \in [a;b]$ such that

$$\int_a^b f(x)dx = (b-a)f(c)$$

Example

Find the value that satisfies the integral Mean value theorem for the function $f(x) = 4x^3 - 1$ on the interval $[1,2]$

$$f(c) = \frac{\int_1^2 (4x^3 - 1)}{2-1} \Rightarrow 4c^3 - 1 = [x^4 - x]_1^2 \Rightarrow 4c^3 - 1 = 14 \Rightarrow c = \sqrt[3]{\frac{15}{4}}$$

Note that $\sqrt[3]{\frac{15}{4}} \in [1,2]$

Average value of a function

Definition

Let f be a continuous on $[a; b]$. Then the average value f_av of f is given by

$$f_{av} = \frac{1}{b-a}\int_a^b f(x)dx$$

Example

Let $f(x) = 3x + 7$ on the interval $[0; 1]$. We know that $\int_0^1 (3x+7)dx = \left[\frac{3x^2}{2} + 7x\right]_0^1 = \frac{3}{2} + 7 = \frac{17}{2}$. Then the point c where f assumed its average value verify $3c + 7 = \frac{17}{2}$, then $c = \frac{1}{2}$

Average value of a function

Example

Find f_{av} of the following function: $f(x) = x^2 - 2x$ on the interval $[1, 4]$

$$\int_1^4 (x^2 - 2x)dx = \left[\frac{x^3}{3} - x^2\right]_1^4 = 6$$

Hence $f_{av} = \dfrac{\int_1^4 (x^2 - 2x)dx}{4-1} = \frac{6}{3} = 2$

Exercise

1. Find f_{av} of the function $f(x) = (2x+1)^2$ on the interval $[0,1]$
2. Find f_{av} of the function $f(x) = \sin^2 x \cos x$ on the interval $[0, \frac{\pi}{2}]$

Definition

1. A function $f : [-a; a] \to \mathbb{R}$ is odd if $f(-x) = -f(x)$ for all $x \in [-a; a]$.
2. A function $f : [-a; a] \to \mathbb{R}$ is even if $f(-x) = f(x)$ for all $x \in [-a; a]$.
3. A function $f : \mathbb{R} \to \mathbb{R}$ is T-periodic if $f(x + T) = f(x)$ for all $x \in \mathbb{R}$.

The fundamental Theorem of Calculus

Theorem

1. If f is an odd function on $[-a;a]$, then

$$\int_{-a}^{a} f(x)dx = 0$$

2. If f is an even function on $[-a;a]$, then

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

3. If f is T-periodic, then, for all $a \in \mathbb{R}$

$$\int_{a}^{a+T} f(x)dx = \int_{0}^{T} f(x)dx$$

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Trapezoidal Rule

It is used to approximate $\int_a^b f(x)dx$ with a regular partition of the interval $[a, b]$, where $\triangle x = \frac{b-a}{n}$, by using the formula

$$\int_a^b f(x)dx \approx \frac{b-a}{2n}[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

Numerical Intergration

Example

Approximate the integral $\int_0^1 \sqrt{x+x^2}\, dx$ using Trapezoidal rule with $n = 4$.

Solution

$[a,b] = [0,1], f(x) = \sqrt{x+x^2}, \ \triangle x = \frac{1-0}{4} = 0.25$

n	x_n	$f(x_n)$	m	$mf(x_n)$
0	0	0	1	0
1	0.25	0.559017	2	1.11803
2	0.5	0.86625	2	1.73205
3	0.75	1.14564	2	2.29129
4	1	1.41421	1	1.41421
				6.55559

$$\int_0^1 \sqrt{x+x^2}\, dx \approx \frac{0-1}{2(4)}[f(0) + 2f(0.25) + 2f(0.5) + 2f(0.75) + f(1)]$$

$$\int_0^1 \sqrt{x+x^2}\, dx \approx 0.819448$$

Trapezoidal Rule

It is used to approximate $\int_a^b f(x)dx$ with a regular partition of the interval $[a, b]$, where $\triangle x = \frac{b-a}{n}$, and n even, by using the formula

$$\int_a^b f(x)dx \approx \frac{b-a}{3n}[f(x_0) + 4f(x_1) + 2f(x_2) +$$

$$+ \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

Example

Approximate the integral $\int_0^{10} \sqrt{10x - x^2}\, dx$ using Simpson's rule with $n = 4$.

Solution

$[a, b] = [0, 10], f(x) = \sqrt{10x - x^2},\ \triangle x = \frac{10-0}{4} = 2.5$

n	x_n	$f(x_n)$	m	$mf(x_n)$
0	0	0	1	0
1	2.5	4.33013	4	17.3204
2	5	5	2	10
3	7.5	4.33013	4	17.3204
4	10	0	1	0
				44.6408

$$\int_0^{10} \sqrt{10x - x^2}\, dx \approx \frac{10-0}{3(4)}[f(0) + 4f(2.5) + 2f(5) + 4f(7.5) + f(10)]$$

$$\int_0^{10} \sqrt{10x - x^2}\, dx \approx 37.2007$$

Exercise

1. Approximate the integral $\int_2^4 \frac{1}{x-1}dx$ using Trapezoidal rule with $n = 4$.
2. Approximate the integral $\int_0^2 \frac{x}{x+1}dx$ using Simpson's rule with $n = 4$.