1 Indefinite Integrals & Anti-derivative Function

Solution of the Exercise 1:

1. $\int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1} + c, \quad \alpha \in \mathbb{Q} \setminus \{-1\}$,
2. $\int \sec^2 x dx = \tan x + c$,
3. $\int \csc^2 x dx = -\cot x + c$,
4. $\int \tan^2 x dx = \tan x - x + c$,
5. $\int \cot^2 x dx = -\cot x - x + c$,
6. $\int \sec x \tan x dx = \sec x + c$,
7. $\int \csc x \cot x dx = -\csc x + c$,
8. $\int (x - \frac{1}{x^{\frac{2}{3}}} + \frac{1}{x^2}) dx = \frac{x^2}{2} - 3x^{\frac{1}{3}} - \frac{1}{x} + c$,
9. $\int x + 2 + \frac{4}{(x+1)^2} dx = \frac{x^2}{2} + 2x - \frac{4}{x+1} + c$,
10. $\int (\frac{1}{\sec x} - \frac{1}{\csc x}) dx = \int (\cos x - \sin x) dx = \sin x + \cos x + c$.

2 Change of variables, Substitution Method

Solution of the Exercise 2:

1. $\int x\sqrt{x+1}dx \overset{t=\sqrt{x+1}}{=} 2\int (t^2-1)t^2 dt = \frac{2}{5}t^5 - \frac{2}{3}t^3 + c = \frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} + c$,
2. $\int \frac{x}{\sqrt{3-4x^2}} dx \overset{t=\sqrt{3-4x^2}}{=} -\frac{1}{4}\int dt = -\frac{1}{4}u + c = -\frac{1}{4}\sqrt{3-4x^2} + c$,
3. $\int \frac{1}{\sqrt{x}\cos^2(\sqrt{x})} dx \overset{t=\sqrt{x}}{=} 2\int \sec^2 t dt = 2\tan t + c = 2\tan(\sqrt{x}) + c$,
4.
$$\begin{aligned}\int \frac{x^2+3x+6}{\sqrt{x+1}} dx &\overset{t=\sqrt{x+1}}{=} 2\int (t^2-1)^2 + 3(t^2-1) + 6dt \\ &= \frac{2}{5}(x+1)^{\frac{5}{2}} + \frac{2}{3}(x+1)^{\frac{3}{2}} + 4(x+1)^{\frac{1}{2}} + c,\end{aligned}$$
5. $\int (x^2+1)^n 2x dx \overset{t=x^2+1}{=} \int t^n dt = \frac{t^{n+1}}{n+1} = \frac{(x^2+1)^{n+1}}{n+1} + c$,
6. $\int \sin(2x+3) dx \overset{t=2x+3}{=} \frac{1}{2}\int \sin t dt = -\frac{1}{2}\cos t + c = -\frac{1}{2}\cos(2x+3) + c$,

7. $\displaystyle\int \frac{1}{\cos^2(\pi x)}dx \overset{t=\pi x}{=} \frac{1}{\pi}\int \frac{1}{\cos^2 t}dt = \frac{1}{\pi}\tan(\pi x) + c,$

8. $\displaystyle\int \frac{1}{\sqrt{x}\cos^2(\sqrt{x})}dx \overset{t=\sqrt{x}}{=} 2\int \sec^2 t dt = 2\tan(\sqrt{x}) + c,$

9. $\displaystyle\int xe^{-x^2}dx \overset{t=x^2}{=} \frac{1}{2}\int e^{-t}dt = -\frac{1}{2}e^{-x^2} + c,$

10. $\displaystyle\int \frac{\sin(\ln x)}{x}dx \overset{t=\ln x}{=} \int \sin t dt = -\cos(\ln x) + c.$

3 Riemann Sums, Area and the Definite Integral

Solution of the Exercise 3:

1. $\displaystyle\sum_{k=1}^{n}(2k^2 - 5k + 1) = \frac{n}{6}\left(4n^2 - 9n - 7\right).$

   The value of n such that $\displaystyle\sum_{k=1}^{n}(2k^2 - 5k + 1) = 147$ is 7.

2. $\displaystyle\sum_{k=1}^{6}\left(k^2 + 3k + 2\alpha\right) = \frac{6(7)(13)}{6} + 3\frac{6(7)}{2} + 12\alpha = 154 + 12\alpha.$ Then the value of α such that $\displaystyle\sum_{k=1}^{6}\left(k^2 + 3k + 2\alpha\right) = 130$ is -2.

Solution of the Exercise 4:

$$\sum_{k=1}^{n}k(k+1) = \sum_{k=1}^{n}k^2 + k = \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{3}.$$

Solution of the Exercise 5:

1. $\displaystyle\frac{1}{n^2}\sum_{k=1}^{n}(3k-2) = \frac{1}{n^2}\left(\frac{3n(n+1)}{2} - 2n\right).$ Then $\displaystyle\lim_{n\to+\infty}\frac{1}{n^2}\sum_{k=1}^{n}(3k-2) = \frac{3}{2}.$

2. $\displaystyle\sum_{k=1}^{n}(2\frac{k}{n^2} - \frac{3}{n}) = \frac{n+1}{n} - 3.$ Then $\displaystyle\lim_{n\to+\infty}\sum_{k=1}^{n}(2\frac{k}{n^2} - \frac{3}{n}) = -2$

3. $\displaystyle\sum_{k=1}^{n}(3k^2 - 2k + 1) = \frac{n(n+1)(2n+1)}{2} - n(n+1) + n = \frac{n(n+1)(2n+1)}{2} - n^2.$

   Then $\displaystyle\lim_{n\to+\infty}\frac{1}{n^3}\sum_{k=1}^{n}(3k^2 - 2k + 1) = 1.$

Solution of the Exercise 6:

1. $R_P = ((-8)(2) - 2 + (0.5) + (2.5)(0.5)) = -16.25$

2. $R_P = ((2)(-2) + 1 + (0.5)(2.5) + (4)(0.5)) = 0.25$

3. $R_P = ((-5)(2) + (0.5)(1.25) + (0.5)(1.75)) = -8$

Solution of the Exercise 7:

1. $\int_0^1 (3x+7)dx = \lim_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n} 3\frac{k}{n} + 7 = \lim_{n\to+\infty} \frac{3}{n^2}\frac{n(n+1)}{2} + 7 = \frac{3}{2} + 7 = \frac{17}{2}.$

2.
$$\begin{aligned}
\int_1^4 (x^2+x+2)dx &= \lim_{n\to+\infty} \frac{3}{n}\sum_{k=1}^{n} \left(1+3\frac{k}{n}\right)^2 + (1+3\frac{k}{n}) + 2 \\
&= \lim_{n\to+\infty} \frac{3}{n}\sum_{k=1}^{n} \left(1+6\frac{k}{n}+9\frac{k^2}{n^2}+1+3\frac{k}{n}+2\right) \\
&= \lim_{n\to+\infty} \frac{3}{n}\left(n + \frac{6}{n}\frac{n(n+1)}{2} + \frac{9}{n^2}\frac{n(n+1)(2n+1)}{6} + n + \frac{3}{n}\frac{n(n+1)}{2} + 2n\right) \\
&= \frac{69}{2}.
\end{aligned}$$

3.
$$\begin{aligned}
\int_0^2 (6x^3+1)dx &= \lim_{n\to+\infty} \frac{2}{n}\sum_{k=1}^{n} 6\left(2\frac{k}{n}\right)^3 + 1 \\
&= \lim_{n\to+\infty} \frac{2}{n}\left(12\frac{(n+1)^2}{n} + n\right) = 26.
\end{aligned}$$

Solution of the Exercise 8:

1. Let $f(x) = \frac{1}{1+x}$ on the interval $[0,1]$. The Riemann sum of f is

$$\frac{1}{n}\sum_{k=1}^{n} f(\frac{k}{n}) = \sum_{k=1}^{n}\frac{1}{n+k},$$

then

$$\lim_{n\to+\infty}\sum_{k=1}^{n}\frac{1}{n+k} = \ln 2.$$

2. Let $f(x) = \frac{1}{2+x}$ on the interval $[0,1]$. The Riemann sum of f is $\frac{1}{n}\sum_{k=1}^{n} f(\frac{k}{n}) = \sum_{k=1}^{n}\frac{1}{2n+k}$, then

$$\lim_{n\to+\infty}\sum_{k=1}^{n}\frac{1}{2n+k} = \int_0^1 \frac{dx}{2+x} = \ln 3 - \ln 2.$$

3. Let $f(x) = x^2$ on the interval $[0, 1]$. The Riemann sum of f is $\frac{1}{n}\sum_{k=1}^{n}\frac{k^2}{n^2}$, then

$$\lim_{n \longrightarrow +\infty} \frac{1}{n}\sum_{k=1}^{n}\frac{k^2}{n^2} = \frac{1}{3}.$$

4. Let $f(t) = \sin(xt)$ on the interval $[0, 1]$. The Riemann sum of f is $\frac{1}{n}\sum_{k=1}^{n}\sin(\frac{kx}{n})$, then

$$\lim_{n \longrightarrow +\infty} \frac{1}{n}\sum_{k=1}^{n}\sin(\frac{kx}{n}) = \frac{1-\cos x}{x}.$$

5. $\lim_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n}\frac{\frac{k}{n}}{1+\frac{k^2}{n^2}} = \int_0^1 \frac{x}{1+x^2}dx = \frac{1}{2}\ln(2).$

6. Let $f(t) = \frac{1}{1+x^2}$ on the interval $[0, 1]$. The Riemann sum of f is $\frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+\frac{k^2}{n^2}} = \frac{n}{n^2+k^2}$, then

$$\lim_{n \longrightarrow +\infty} \sum_{k=1}^{n}\frac{n}{n^2+k^2} = \frac{\pi}{4}.$$

7. Let $f(t) = \frac{1}{\sqrt{1+x^2}}$ on the interval $[0, 1]$. The Riemann sum of f is $\sum_{k=1}^{n}\frac{1}{\sqrt{n^2+k^2}}$, then

$$\lim_{n \longrightarrow +\infty} \sum_{k=1}^{n}\frac{1}{\sqrt{n^2+k^2}} = \sinh^{-1}(1) = \ln(1+\sqrt{2}).$$

8. Let $f(t) = x^2\sin(\pi x)$ on the interval $[0, 1]$. The Riemann sum of f is $\frac{1}{n^3}\sum_{k=1}^{n}k^2\sin(\frac{k\pi}{n})$, then

$$\lim_{n \longrightarrow +\infty} \frac{1}{n^3}\sum_{k=1}^{n}k^2\sin(\frac{k\pi}{n}) = \int_0^1 x^2\sin(\pi x)dx = \frac{1}{\pi} - \frac{4}{\pi^3}.$$

9. $\lim_{n\to+\infty} \sum_{k=1}^{n}\frac{1}{n}\cos\left(\frac{k\pi}{n}\right) = \int_0^1 \cos(\pi x)dx = 0.$

10. $\lim_{n\to+\infty} \sum_{k=1}^{n}\frac{n}{n^2+k^2} = \int_0^1 \frac{1}{1+x^2}dx = \frac{\pi}{4}.$

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11. $\lim_{n\to+\infty}\sum_{k=1}^{2^n}\frac{k^3}{2^{4n}} = \lim_{n\to+\infty}\frac{1}{2^n}\sum_{k=1}^{2^n}\frac{k^3}{2^{3n}} = \int_0^1 x^3 dx = \frac{1}{4}.$

4 The Fundamental Theorem of Calculus

Solution of the Exercise 9:

1. $\lim\limits_{n\to+\infty}\frac{1}{n}\sum_{k=1}^{n}\sec^2\left(\frac{k}{n}\right) = \int_0^1 \sec^2 x dx = \tan 1.$

2. $\lim\limits_{n\to+\infty}\frac{1}{n^4}\sum_{k=1}^{n}\sum_{k=1}^{n}(k-1)(k+2) = \lim\limits_{n\to+\infty}\frac{1}{n^2}\sum_{k=1}^{n}\sum_{k=1}^{n}\left(\frac{k^2}{n^2}+\frac{k}{n}-2\right) = \int_0^1 x^2 dx = \frac{1}{3}.$

Solution of the Exercise 10:

1. $\int_{-\frac{\pi}{2}}^{\pi} f(t)dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos(t)dt + \int_{\frac{\pi}{2}}^{\pi}\sin(t)dt = 3.$

2. $\int_0^2 |x-1|\,dx = \int_0^1(1-x)dx + \int_1^2(x-1)dx = \left[x-\frac{x^2}{2}\right]_0^1 + \left[\frac{x^2}{2}-x\right]_1^2 = 1.$

Solution of the Exercise 11:

1. $I = \int_0^{\frac{\pi}{2}}\frac{\sin x}{\sin x+\cos x}dx \overset{t=\frac{\pi}{2}-x}{=} \int_0^{\frac{\pi}{2}}\frac{\cos t}{\sin t+\cos t}dt = J.$

2. $I+J = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2}.$

3. $2I = I+J = \frac{\pi}{2}$, then $I = \frac{\pi}{4} = J.$

Solution of the Exercise 12:

1. $\int_0^{\frac{\pi}{4}}\frac{\sin x}{\cos^3 x}dx \overset{t=\cos x}{=} -\int_1^{\frac{\sqrt{2}}{2}}\frac{dt}{t^3} = \left[\frac{1}{2t^2}\right]_1^{\frac{\sqrt{2}}{2}} = \frac{1}{2},$

2. $\int_1^2\frac{dx}{x(x^3+1)} \overset{t=x^3+1}{=} \frac{1}{3}\int_2^9\frac{dt}{t(t-1)} = \frac{1}{3}\left[\ln(\frac{t-1}{t})\right]_2^9 = \frac{1}{3}\ln(\frac{16}{9}) + \frac{1}{3}\ln 2,$

Solution of the Exercise 13:
$F'(x) = \pi\cos(\pi^2).$

Solution of the Exercise 14:

1. $f'(x) = \sin^3(x),$

2. $f'(x) = \ 2x\cos^5(x^2) - \cos^5(x)$,

3. $f'(x) = -\sin x(1-\cos^2 x)^{\frac{3}{2}} - \cos x(1-\sin^2 x)^{\frac{3}{2}} = -\sin x|\sin^3 x| - \cos x|\cos^3 x|$,

4. $f'(x) = \sec x \tan x(1+\sec^3 x)^{\frac{1}{2}} - \sec^2 x(1+\tan^3 x)^{\frac{1}{2}}$,

5. $f'(x) = \dfrac{1}{x^2}(4+\dfrac{1}{x^2})^{\frac{5}{2}}$

6. $f'(x) = 2e^{2x}(1+4e^{4x})^{\frac{1}{2}} - e^x(1+4e^{2x})^{\frac{1}{2}}$.

Solution of the Exercise 15:
The average value of f on $[a,b]$ is 0. Then by the Mean Value Theorem, f assumes this value at some point $c \in [a,b]$.

Solution of the Exercise 16:

1. We know that $\displaystyle\int_0^1 (x^2+x+2)dx = \frac{69}{2}$. Then the point c where f reached its average value verifies $c^2+c+2 = 696$, then $c = \dfrac{-1+\sqrt{39}}{2}$.

2. We know that $\displaystyle\int_0^1 (x^2+x+2)dx = \frac{2}{6}$. Then the point c where f reached its average value verifies $6c^3+1 = 26$, then $c = (\dfrac{25}{6})^{\frac{1}{3}}$.

3. $\dfrac{1}{\beta-\alpha}\displaystyle\int_\alpha^\beta f(x)dx = \frac{a}{2}(\beta+\alpha)+b$. Then $c = \dfrac{\beta+\alpha}{2}$.

5 Numerical Integration

Solution of the Exercise 17:

1.
$$\begin{aligned}
\int_0^\pi \sqrt{1+\sin x}dx &= 2\int_0^{\frac{\pi}{2}} \sqrt{1+\sin x}dx \\
&\overset{t=\tan(\frac{x}{2})}{=} 4\int_0^1 \frac{1+t}{(1+t^2)^{\frac{3}{2}}}dt \\
&\overset{t=\tan(\theta)}{=} 4\int_0^{\frac{\pi}{4}} (\cos\theta+\sin\theta)d\theta \\
&= 4.
\end{aligned}$$

k	x_k	$f(x_k)$	m	$mf(x_k)$
0	0	1	1	1
1	$\frac{\pi}{4}$	$\sqrt{1+\frac{\sqrt{2}}{2}}$	2	$2\sqrt{1+\frac{\sqrt{2}}{2}}$
2	$\frac{\pi}{2}$	$\sqrt{2}$	2	$2\sqrt{2}$
3	$\frac{3\pi}{4}$	$\sqrt{1+\frac{\sqrt{2}}{2}}$	2	$2\sqrt{1+\frac{\sqrt{2}}{2}}$
4	π	1	1	1
				10.0546

$\displaystyle\int_0^{\pi} \sqrt{1+\sin x}dx \approx \frac{\pi}{8}(10.0546) \approx 3.9484632.$

If $f(x) = \sqrt{1+\sin x}$, $f''(x) = -\dfrac{\sqrt{1+\sin x}}{4}$. The reminder R_2 verifies $|R_2| \leq 610^{-2}$.

2.

$$\begin{aligned}\int_0^2 \frac{x}{\sqrt{x+1}}dx &\overset{t=\sqrt{x+1}}{=} 2\int_1^{\sqrt{3}}(t^2-1)dt \\ &= \frac{4}{3}\end{aligned}$$

k	x_k	$f(x_k)$	m	$mf(x_k)$
0	0	0	1	0
1	$\frac{1}{2}$	$\dfrac{\sqrt{2}}{2\sqrt{3}}$	4	$\dfrac{2\sqrt{2}}{\sqrt{3}}$
2	1	$\dfrac{1}{\sqrt{2}}$	2	$\sqrt{2}$
3	$\frac{3}{2}$	$\dfrac{3}{\sqrt{10}}$	4	$\dfrac{12}{\sqrt{10}}$
4	2	$\dfrac{2}{\sqrt{3}}$	1	$\dfrac{2}{\sqrt{3}}$
				7.9966404

$\displaystyle\int_0^2 \frac{x}{\sqrt{x+1}}dx \approx 1.332773.$

$f''(x) = -4(4+x)(x+1)^{-\frac{5}{2}}$, We can prove easily that $f^{(5)} \geq 0$, then $\displaystyle\operatorname*{Sup}_{x\in[0,2]} |f''(x)| = 8.3^{-\frac{3}{2}}$.

The reminder R_4 verifies $|R_4| \leq 3^{-\frac{5}{2}} \leq 7.10^{-2}$.

For $n = 8$

k	x_k	$f(x_k)$	m	$mf(x_k)$
0	0	0	1	0
1	$\frac{1}{4}$	$\frac{1}{2\sqrt{5}}$	4	$\frac{2}{\sqrt{5}} \approx 0.894427$
2	$\frac{1}{2}$	$\frac{1}{\sqrt{6}}$	2	$\frac{2}{\sqrt{6}} \approx 0.8164965$
3	$\frac{3}{4}$	$\frac{3}{2\sqrt{7}}$	4	$\frac{6}{\sqrt{7}} \approx 2.2677868$
4	1	$\frac{1}{\sqrt{2}}$	2	$\sqrt{2} \approx 1.41421356$
5	$\frac{5}{4}$	$\frac{5}{6}$	4	$\frac{10}{3} \approx 3.33333333$
6	$\frac{3}{2}$	$\frac{3}{\sqrt{10}}$	2	$\frac{6}{\sqrt{10}} \approx 1.89736659$
7	$\frac{7}{4}$	$\frac{7}{2\sqrt{11}}$	4	$\frac{14}{\sqrt{11}} \approx 4.2211588$
8	2	$\frac{2}{\sqrt{3}}$	1	$\frac{2}{\sqrt{3}} \approx 1.154701$
				15.99948

$\int_0^2 \frac{x}{\sqrt{x+1}} dx \approx 1.33329.$

$f^{(4)}(x) = -\frac{15}{16}(8+x)(x+1)^{-\frac{9}{2}}$. We can prove easily that $f^{(5)} \geq 0$, then

$\operatorname{Sup}_{x\in[0,2]} |f^{(4)}(x)| = \frac{15}{16}(12)(3)^{-\frac{9}{2}} = \frac{5}{36\sqrt{3}} \leq 0.081.$

The reminder R_8 verifies $|R_8| \leq \frac{5}{2880.9.8^3} \leq 3.10^{-4}$.

Solution of the Exercise 18:

$x_k = 1 + \frac{2k}{5}$, $f(x_k) = 1 + \frac{4k}{5}$ and $g(x_k) = 3 + 2k + \frac{4k^2}{25}$, $k = 0, \ldots, 5$.

$$\int_1^3 (2x-1)dx \approx \frac{1}{5}\left(1 + 5 + 2\sum_{k=1}^{4}(1 + \frac{4k}{5})\right) = 6.$$

The exact value of the integral $\int_1^3 (2x-1)dx$ is 6.

$$\begin{aligned}\int_1^3 (x^2+3x-1)dx &\approx \frac{1}{5}\left(3 + 17 + 2\sum_{k=1}^{4}(1+\frac{4k}{5})^2 + 3(1+\frac{4k}{5}) - 1\right)\\ &= \frac{1}{5}\left(20 + 2\sum_{k=1}^{4}(\frac{4k^2}{25} + 2k + 3)\right) = \frac{1}{5}(93 + \frac{3}{5}) = 18.72.\end{aligned}$$

The exact value of the integral $\int_1^3 (x^2+3x-1)dx$ is $19 - \frac{1}{3}$.

Solution of the Exercise 19:

$x_k = 1 + \frac{k}{4}$, $g(x_k) = 3 + \frac{5k}{4} + \frac{k^2}{16}$ and $h(x_k) = 1 + \frac{3k}{4} + \frac{3k^2}{16} + \frac{k^3}{64}$, $k = 0, \ldots, 8$.

k	x_k	$g(x_k)$	m	$mg(x_k)$
0	1	3	1	3
1	$\frac{5}{4}$	$\frac{69}{16}$	4	$\frac{69}{4} = 17.25$
2	$\frac{3}{2}$	$\frac{23}{4}$	2	$\frac{23}{2} = 11.5$
3	$\frac{7}{4}$	7.3125	4	29.25
4	2	9	2	18
5	$\frac{9}{4}$	10.8125	4	43.25
6	$\frac{5}{2}$	12.75	2	25.5
7	$\frac{11}{4}$	14.8125	4	59.25
8	3	17	1	17
				233

$$\int_1^3 (x^2 + 3x - 1)dx \approx \frac{224}{12} = 18.66.$$

k	x_k	$h(x_k)$	m	$mh(x_k)$
0	1	1	1	1
1	$\frac{5}{4}$	1.953125	4	7.8125
2	$\frac{3}{2}$	$\frac{27}{8}$	2	$\frac{27}{4} = 6.75$
3	$\frac{7}{4}$	5.359375	4	21.4375
4	2	8	2	16
5	$\frac{9}{4}$	11.390625	4	45.5625
6	$\frac{5}{2}$	15.625	2	31.25
7	$\frac{11}{4}$	20.796875	4	83.1875
8	3	27	1	27
				240

$$\int_1^3 x^3 dx \approx \frac{240}{12} = 20$$