Introduction to Systems of Linear Equations

Linear Equations

define a ***linear equation*** in the n variables $x_1, x_2, \ldots, x_n$ to be one that can be expressed in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \tag{1}$$

where $a_1, a_2, \ldots, a_n$ and b are constants, and the a's are not all zero.

In the special case where $b = 0$, Equation (1) has the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0 \tag{4}$$

which is called a ***homogeneous linear equation*** in the variables $x_1, x_2, \ldots, x_n$.

The following are linear equations:

$$x + 3y = 7 \qquad x_1 - 2x_2 - 3x_3 + x_4 = 0$$
$$\tfrac{1}{2}x - y + 3z = -1 \qquad x_1 + x_2 + \cdots + x_n = 1$$

The following are not linear equations:

$$x + 3y^2 = 4 \qquad 3x + 2y - xy = 5$$
$$\sin x + y = 0 \qquad \sqrt{x_1} + 2x_2 + x_3 = 1$$ ◄

A finite set of linear equations is called a ***system of linear equations***

For example:

$$\begin{aligned} 5x + y &= 3 \\ 2x - y &= 4 \end{aligned} \qquad \begin{aligned} 4x_1 - x_2 + 3x_3 &= -1 \\ 3x_1 + x_2 + 9x_3 &= -4 \end{aligned} \tag{5–6}$$

A general linear system of m equations in the n unknowns $x_1, x_2, \ldots, x_n$ can be written as

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array} \tag{7}$$

A ***solution*** of a linear system in n unknowns $x_1, x_2, \ldots, x_n$ is a sequence of n numbers $s_1, s_2, \ldots, s_n$ for which the substitution

$$x_1 = s_1, \quad x_2 = s_2, \ldots, \quad x_n = s_n$$

the system in (6) has the solution

$$x_1 = 1, \quad x_2 = 2, \quad x_3 = -1$$

solutions can be written more succinctly as

$(1, 2, -1)$

Remark:

$$(s_1, s_2, \ldots, s_n)$$

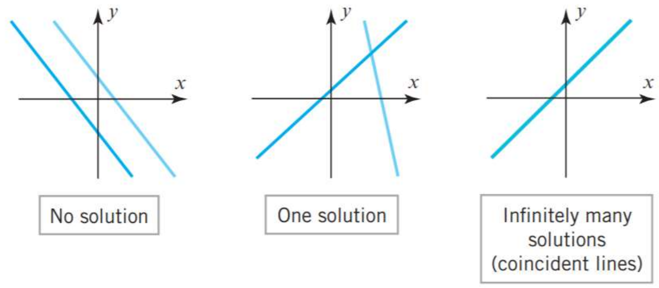
is called an ***ordered n-tuple***.

Remark:

Linear systems in two unknowns arise in connection with intersections of lines. For example, consider the linear system

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$$

in which the graphs of the equations are lines in the xy-plane.



Result:

Every system of linear equations has zero, one, or infinitely many solutions. There are no other possibilities.

▶ EXAMPLE 2 A Linear System with One Solution

Solve the linear system

$$\begin{aligned} x - y &= 1 \\ 2x + y &= 6 \end{aligned}$$

Solution We can eliminate x from the second equation by adding -2 times the first equation to the second. This yields the simplified system

$$\begin{aligned} x - y &= 1 \\ 3y &= 4 \end{aligned}$$

From the second equation we obtain $y = \frac{4}{3}$, and on substituting this value in the first equation we obtain $x = 1 + y = \frac{7}{3}$. Thus, the system has the unique solution

$$x = \tfrac{7}{3}, \quad y = \tfrac{4}{3}$$

▶ EXAMPLE 3 A Linear System with No Solutions

Solve the linear system

$$\begin{aligned} x + y &= 4 \\ 3x + 3y &= 6 \end{aligned}$$

Solution We can eliminate x from the second equation by adding -3 times the first equation to the second equation. This yields the simplified system

$$\begin{aligned} x + y &= 4 \\ 0 &= -6 \end{aligned}$$

▶ EXAMPLE 4 A Linear System with Infinitely Many Solutions

Solve the linear system

$$\begin{aligned} 4x - 2y &= 1 \\ 16x - 8y &= 4 \end{aligned}$$

Solution We can eliminate x from the second equation by adding -4 times the first equation to the second. This yields the simplified system

$$\begin{aligned} 4x - 2y &= 1 \\ 0 &= 0 \end{aligned}$$

The second equation does not impose any restrictions on x and y and hence can be omitted. Thus, the solutions of the system are those values of x and y that satisfy the single equation

$$4x - 2y = 1 \tag{8}$$

Geometrically, this means the lines corresponding to the two equations in the original system coincide. One way to describe the solution set is to solve this equation for x in terms of y to obtain $x = \frac{1}{4} + \frac{1}{2}y$ and then assign an arbitrary value t (called a ***parameter***)

Augmented Matrices and Elementary Row Operations

$$\begin{aligned} a_{11}x_1 &+ a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 &+ a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ &\vdots \\ a_{m1}x_1 &+ a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{aligned}$$

we can abbreviate the system by writing only the rectangular array of numbers

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

This is called the ***augmented matrix*** for the system. For example, the augmented matrix for the system of equations

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 9 \\ 2x_1 + 4x_2 - 3x_3 &= 1 \\ 3x_1 + 6x_2 - 5x_3 &= 0 \end{aligned} \quad \text{is} \quad \begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

elementary row operations

The basic method for solving a linear system is to perform algebraic operations on the system that do not alter the solution set and that produce a succession of increasingly simpler systems, until a point is reached where it can be ascertained whether the system is consistent, and if so, what its solutions are. Typically, the algebraic operations are:

1. Multiply an equation through by a nonzero constant.
2. Interchange two equations.
3. Add a constant times one equation to another.

$$\begin{aligned} x + y + 2z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0 \end{aligned} \qquad \begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

Add -2 times the first equation to the second to obtain

$$\begin{aligned} x + y + 2z &= 9 \\ 2y - 7z &= -17 \\ 3x + 6y - 5z &= 0 \end{aligned}$$

Add -2 times the first row to the second to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

Matrices and Matrix Operations

DEFINITION 1 A ***matrix*** is a rectangular array of numbers. The numbers in the array are called the ***entries*** in the matrix.

EXAMPLE 1 Examples of Matrices

Some examples of matrices are

$$\begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -1 & 4 \end{bmatrix}, \quad [2 \quad 1 \quad 0 \quad -3], \quad \begin{bmatrix} e & \pi & -\sqrt{2} \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad [4]$$

The ***size*** of a matrix is described in terms of the number of rows (horizontal lines) and columns (vertical lines) it contains. For example, the first matrix in Example 1 has three rows and two columns, so its size is 3 by 2 (written 3×2). In a size description, the first number always denotes the number of rows, and the second denotes the number of columns. The remaining matrices in Example 1 have sizes 1×4, 3×3, 2×1, and 1×1, respectively.

A matrix with only one row, such as the second in Example 1, is called a ***row vector*** (or a ***row matrix***), and a matrix with only one column, such as the fourth in that example, is called a ***column vector*** (or a ***column matrix***). The fifth matrix in that example is both a row vector and a column vector.

The entry that occurs in row i and column j of a matrix A will be denoted by a_{ij}. Thus a general 3×4 matrix might be written as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

and a general $m \times n$ matrix as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \tag{1}$$

When a compact notation is desired, the preceding matrix can be written as

$$[a_{ij}]_{m\times n} \quad \text{or} \quad [a_{ij}]$$

A matrix A with n rows and n columns is called a ***square matrix of order n***, and the shaded entries $a_{11}, a_{22}, \ldots, a_{nn}$ in (2) are said to be on the ***main diagonal*** of A.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \tag{2}$$

Operations on Matrices

DEFINITION 2 Two matrices are defined to be ***equal*** if they have the same size and their corresponding entries are equal.

▶ EXAMPLE 2 Equality of Matrices

Consider the matrices

$$A = \begin{bmatrix} 2 & 1 \\ 3 & x \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \end{bmatrix}$$

If $x = 5$, then $A = B$, but for all other values of x the matrices A and B are not equal, since not all of their corresponding entries are equal. There is no value of x for which $A = C$ since A and C have different sizes. ◀

Example:1. Write down the system of equation, if matrices A and B are equal

$$A = \begin{bmatrix} x-2 & y-3 \\ x+y & z+3 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 3+z \\ z & y \end{bmatrix}$$

Solution: A and B are of the same size, hence

$$A = B \Rightarrow$$

$$\begin{aligned} x-2 &= 1 \\ y-3 &= 3+z \\ x+y &= z \\ z+3 &= y \end{aligned}$$

System of equations are

$$\begin{aligned} x \qquad &= 3 \\ y - z &= 6 \\ x + y - z &= 0 \\ -y + z &= -3 \end{aligned}$$

DEFINITION 3 If A and B are matrices of the same size, then the ***sum*** $A + B$ is the matrix obtained by adding the entries of B to the corresponding entries of A, and the ***difference*** $A - B$ is the matrix obtained by subtracting the entries of B from the corresponding entries of A. Matrices of different sizes cannot be added or subtracted.

In matrix notation, if $A = [a_{ij}]$ and $B = [b_{ij}]$ have the same size, then

$$(A + B)_{ij} = (A)_{ij} + (B)_{ij} = a_{ij} + b_{ij} \quad \text{and} \quad (A - B)_{ij} = (A)_{ij} - (B)_{ij} = a_{ij} - b_{ij}$$

EXAMPLE 3 Addition and Subtraction

Consider the matrices

$$A = \begin{bmatrix} 2 & 1 & 0 & 3 \\ -1 & 0 & 2 & 4 \\ 4 & -2 & 7 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -4 & 3 & 5 & 1 \\ 2 & 2 & 0 & -1 \\ 3 & 2 & -4 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

Then

$$A + B = \begin{bmatrix} -2 & 4 & 5 & 4 \\ 1 & 2 & 2 & 3 \\ 7 & 0 & 3 & 5 \end{bmatrix} \quad \text{and} \quad A - B = \begin{bmatrix} 6 & -2 & -5 & 2 \\ -3 & -2 & 2 & 5 \\ 1 & -4 & 11 & -5 \end{bmatrix}$$

The expressions $A + C$, $B + C$, $A - C$, and $B - C$ are undefined. ◀

DEFINITION 4 If A is any matrix and c is any scalar, then the ***product*** cA is the matrix obtained by multiplying each entry of the matrix A by c. The matrix cA is said to be a ***scalar multiple*** of A.

Example:3. Find the value of x and y in the following matrix equation

$$\begin{bmatrix} 5 & x \\ 3y & 2 \end{bmatrix} + \begin{bmatrix} -3 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 5 & 7 \end{bmatrix}$$

Solution. Using concept of addition of matrices, we simplify left hand side

$$\begin{bmatrix} 5-3 & x+2 \\ 3y-1 & 2+5 \end{bmatrix} = \begin{bmatrix} 2 & x+2 \\ 3y-1 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 5 & 7 \end{bmatrix}$$

Two matrices are equal when their correspoding entries are equal

$x + 2 = 4$

$2y - 1 = 5$

Solving these equations

$x = 4 - 2 = 2$

▶ **EXAMPLE 4 Scalar Multiples**

For the matrices

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 2 & 7 \\ -1 & 3 & -5 \end{bmatrix}, \quad C = \begin{bmatrix} 9 & -6 & 3 \\ 3 & 0 & 12 \end{bmatrix}$$

we have

$$2A = \begin{bmatrix} 4 & 6 & 8 \\ 2 & 6 & 2 \end{bmatrix}, \quad (-1)B = \begin{bmatrix} 0 & -2 & -7 \\ 1 & -3 & 5 \end{bmatrix}, \quad \tfrac{1}{3}C = \begin{bmatrix} 3 & -2 & 1 \\ 1 & 0 & 4 \end{bmatrix}$$

It is common practice to denote $(-1)B$ by $-B$. ◀

DEFINITION 5 If A is an $m \times r$ matrix and B is an $r \times n$ matrix, then the ***product*** AB is the $m \times n$ matrix whose entries are determined as follows: To find the entry in row i and column j of AB, single out row i from the matrix A and column j from the matrix B. Multiply the corresponding entries from the row and column together, and then add up the resulting products.

▶ **EXAMPLE 5 Multiplying Matrices**

Consider the matrices

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}$$

Since A is a 2×3 matrix and B is a 3×4 matrix, the product AB is a 2×4 matrix. To determine, for example, the entry in row 2 and column 3 of AB, we single out row 2 from A and column 3 from B. Then, as illustrated below, we multiply corresponding entries together and add up these products.

▶ **EXAMPLE 5 Multiplying Matrices**

Consider the matrices

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}$$

Since A is a 2×3 matrix and B is a 3×4 matrix, the product AB is a 2×4 matrix. To determine, for example, the entry in row 2 and column 3 of AB, we single out row 2 from A and column 3 from B. Then, as illustrated below, we multiply corresponding entries together and add up these products.

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} \square & \square & \square & \square \\ \square & \square & \boxed{26} & \square \end{bmatrix}$$

$$(2 \cdot 4) + (6 \cdot 3) + (0 \cdot 5) = 26$$

The entry in row 1 and column 4 of AB is computed as follows:

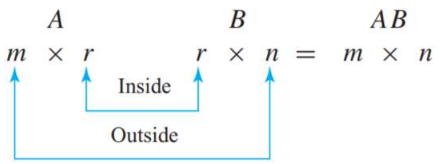
$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} \square & \square & \square & \boxed{13} \\ \square & \square & \square & \square \end{bmatrix}$$

$$(1 \cdot 3) + (2 \cdot 1) + (4 \cdot 2) = 13$$

The computations for the remaining entries are

$$\begin{aligned} (1 \cdot 4) + (2 \cdot 0) + (4 \cdot 2) &= 12 \\ (1 \cdot 1) - (2 \cdot 1) + (4 \cdot 7) &= 27 \\ (1 \cdot 4) + (2 \cdot 3) + (4 \cdot 5) &= 30 \\ (2 \cdot 4) + (6 \cdot 0) + (0 \cdot 2) &= 8 \\ (2 \cdot 1) - (6 \cdot 1) + (0 \cdot 7) &= -4 \\ (2 \cdot 3) + (6 \cdot 1) + (0 \cdot 2) &= 12 \end{aligned} \qquad AB = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}$$ ◀

Determining Whether a Product Is Defined



(3)

▶ **EXAMPLE 6**

Suppose that A, B, and C are matrices with the following sizes:

A	B	C
3×4	4×7	7×3

Then by (3), AB is defined and is a 3×7 matrix; BC is defined and is a 4×3 matrix; and CA is defined and is a 7×4 matrix. The products AC, CB, and BA are all undefined. ◀

THEOREM 1.4.1 Properties of Matrix Arithmetic

Assuming that the sizes of the matrices are such that the indicated operations can be performed, the following rules of matrix arithmetic are valid.

(*a*) $A + B = B + A$ **[Commutative law for matrix addition]**

(*b*) $A + (B + C) = (A + B) + C$ **[Associative law for matrix addition]**

(*c*) $A(BC) = (AB)C$ **[Associative law for matrix multiplication]**

(*d*) $A(B + C) = AB + AC$ **[Left distributive law]**

(*e*) $(B + C)A = BA + CA$ **[Right distributive law]**

(*f*) $A(B - C) = AB - AC$

(*g*) $(B - C)A = BA - CA$

(*h*) $a(B + C) = aB + aC$

(*i*) $a(B - C) = aB - aC$

(*j*) $(a + b)C = aC + bC$

(*k*) $(a - b)C = aC - bC$

(*l*) $a(bC) = (ab)C$

(*m*) $a(BC) = (aB)C = B(aC)$

▶ EXAMPLE 2 Order Matters in Matrix Multiplication

Consider the matrices

$$A = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$$

Multiplying gives

$$AB = \begin{bmatrix} -1 & -2 \\ 11 & 4 \end{bmatrix} \quad \text{and} \quad BA = \begin{bmatrix} 3 & 6 \\ -3 & 0 \end{bmatrix}$$

Thus, $AB \neq BA$. ◀

▶ EXAMPLE 3 Failure of the Cancellation Law

Consider the matrices

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}$$

We leave it for you to confirm that

$$AB = AC = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$$

Although $A \neq 0$, canceling A from both sides of the equation $AB = AC$ would lead to the incorrect conclusion that $B = C$. Thus, the cancellation law does not hold, in general, for matrix multiplication (though there may be particular cases where it is true).

EXAMPLE 4 A Zero Product with Nonzero Factors

Here are two matrices for which $AB = 0$, but $A \neq 0$ and $B \neq 0$:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 7 \\ 0 & 0 \end{bmatrix}$$

Partitioned Matrices

A matrix can be subdivided or ***partitioned*** into smaller matrices by inserting horizontal and vertical rules between selected rows and columns. For example, the following are three possible partitions of a general 3×4 matrix A—the first is a partition of A into four ***submatrices*** A_{11}, A_{12}, A_{21}, and A_{22}; the second is a partition of A into its row vectors $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$; and the third is a partition of A into its column vectors $\mathbf{c}_1$, $\mathbf{c}_2$, $\mathbf{c}_3$, and $\mathbf{c}_4$:

$$A = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$A = \left[\begin{array}{cccc} a_{11} & a_{12} & a_{13} & a_{14} \\ \hline a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right] = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{bmatrix}$$

$$A = \left[\begin{array}{c|c|c|c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right] = [\mathbf{c}_1 \quad \mathbf{c}_2 \quad \mathbf{c}_3 \quad \mathbf{c}_4]$$

DEFINITION 6 If $A_1, A_2, \ldots, A_r$ are matrices of the same size, and if $c_1, c_2, \ldots, c_r$ are scalars, then an expression of the form

$$c_1 A_1 + c_2 A_2 + \cdots + c_r A_r$$

is called a ***linear combination*** of $A_1, A_2, \ldots, A_r$ with ***coefficients*** $c_1, c_2, \ldots, c_r$.

To see how matrix products can be viewed as linear combinations, let A be an $m \times n$ matrix and $\mathbf{x}$ an $n \times 1$ column vector, say

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Then

$$A\mathbf{x} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

THEOREM 1.3.1 *If A is an $m \times n$ matrix, and if $\mathbf{x}$ is an $n \times 1$ column vector, then the product $A\mathbf{x}$ can be expressed as a linear combination of the column vectors of A in which the coefficients are the entries of $\mathbf{x}$.*

▶ EXAMPLE 8 Matrix Products as Linear Combinations

The matrix product

$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

can be written as the following linear combination of column vectors:

$$2\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} - 1\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

Column-Row Expansion

Partitioning provides yet another way to view matrix multiplication. Specifically, suppose that an $m \times r$ matrix A is partitioned into its r column vectors $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_r$ (each of size $m \times 1$) and an $r \times n$ matrix B is partitioned into its r row vectors $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_r$ (each of size $1 \times n$). Each term in the sum

$$\mathbf{c}_1\mathbf{r}_1 + \mathbf{c}_2\mathbf{r}_2 + \cdots + \mathbf{c}_r\mathbf{r}_r$$

has size $m \times n$ so the sum itself is an $m \times n$ matrix. We leave it as an exercise for you to verify that the entry in row i and column j of the sum is given by the expression on the right side of Formula (5), from which it follows that

$$AB = \mathbf{c}_1\mathbf{r}_1 + \mathbf{c}_2\mathbf{r}_2 + \cdots + \mathbf{c}_r\mathbf{r}_r \tag{11}$$

▶ EXAMPLE 10 Column-Row Expansion

Find the column-row expansion of the product

$$AB = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 4 \\ -3 & 5 & 1 \end{bmatrix}$$

Solution The column vectors of A and the row vectors of B are, respectively,

$$\mathbf{c}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}; \quad \mathbf{r}_1 = \begin{bmatrix} 2 & 0 & 4 \end{bmatrix}, \quad \mathbf{r}_2 = \begin{bmatrix} -3 & 5 & 1 \end{bmatrix}$$

Thus, it follows from (11) that the column-row expansion of AB is

$$\begin{aligned} AB &= \begin{bmatrix} 1 \\ 2 \end{bmatrix}\begin{bmatrix} 2 & 0 & 4 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \end{bmatrix}\begin{bmatrix} -3 & 5 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 & 4 \\ 4 & 0 & 8 \end{bmatrix} + \begin{bmatrix} -9 & 15 & 3 \\ 3 & -5 & -1 \end{bmatrix} \end{aligned} \tag{13}$$

As a check, we leave it for you to confirm that the product in (12) and the sum in (13) both yield

$$AB = \begin{bmatrix} -7 & 15 & 7 \\ 7 & -5 & 7 \end{bmatrix}$$ ◀

Remark:

Matrix multiplication has an important application to systems of linear equations. Consider a system of m linear equations in n unknowns:

$$\begin{array}{ccccccc} a_{11}x_1 & + & a_{12}x_2 & + \cdots + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + \cdots + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + \cdots + & a_{mn}x_n & = & b_m \end{array}$$

Is equivalent to

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

If we designate these matrices by A, $\mathbf{x}$, and $\mathbf{b}$, respectively, then we can replace the original system of m equations in n unknowns by the single matrix equation

$$A\mathbf{x} = \mathbf{b}$$

The matrix A in this equation is called the ***coefficient matrix*** of the system. The augmented matrix for the system is obtained by adjoining $\mathbf{b}$ to A as the last column; thus the augmented matrix is

$$[A \mid \mathbf{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Transpose of a Matrix

DEFINITION 7 If A is any $m \times n$ matrix, then the ***transpose of A***, denoted by A^T, is defined to be the $n \times m$ matrix that results by interchanging the rows and columns of A; that is, the first column of A^T is the first row of A, the second column of A^T is the second row of A, and so forth.

▶ EXAMPLE 11 Some Transposes

The following are some examples of matrices and their transposes.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 5 & 6 \end{bmatrix}, \quad C = [1 \quad 3 \quad 5], \quad D = [4]$$

$$A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \\ a_{14} & a_{24} & a_{34} \end{bmatrix}, \quad B^T = \begin{bmatrix} 2 & 1 & 5 \\ 3 & 4 & 6 \end{bmatrix}, \quad C^T = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \quad D^T = [4]$$ ◀

Trace of a Matrix

DEFINITION 8 If A is a square matrix, then the ***trace of A***, denoted by $\operatorname{tr}(A)$, is defined to be the sum of the entries on the main diagonal of A. The trace of A is undefined if A is not a square matrix.

▶ **EXAMPLE 12 Trace**

The following are examples of matrices and their traces.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 2 & 7 & 0 \\ 3 & 5 & -8 & 4 \\ 1 & 2 & 7 & -3 \\ 4 & -2 & 1 & 0 \end{bmatrix}$$

$$\operatorname{tr}(A) = a_{11} + a_{22} + a_{33} \qquad \operatorname{tr}(B) = -1 + 5 + 7 + 0 = 11$$ ◀

Working with Proofs

35. Prove: If A and B are $n \times n$ matrices, then

$$\operatorname{tr}(A + B) = \operatorname{tr}(A) + \operatorname{tr}(B)$$

36. (a) Prove: If AB and BA are both defined, then AB and BA are square matrices.

(b) Prove: If A is an $m \times n$ matrix and $A(BA)$ is defined, then B is an $n \times m$ matrix.

Types of matrices:

2. **Square Matrix:** If $n = m$ that is number rows and columns are equal, then the matrix is square matrix.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix},\ 2x2 \text{ is a } \textbf{square matrix}$$

If number of rows and columns are not equal ($n \neq m$) then matrix is called **Rectangular matrix**.

$$B = \begin{bmatrix} 2 & 3 & 1 & 0 \\ 3 & 1 & 0 & 7 \\ 1 & 1 & -1 & 5 \end{bmatrix} \text{ is } 3x\,4 \text{ matrix}$$

3. **Row Matrix:** Matrix with only one row and can contain any number of columns

$$B = \begin{bmatrix} 1 & 2 & 4 & 3 \end{bmatrix},\ 1 \times 4 \text{ is a } \textbf{row matrix}$$

4. **Column Matrix:** Matrix with only one column and can contain any number of rows

$$C = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix},\ 4x1 \text{ is a } \textbf{column matrix}$$

5. **Zero Matrix:** A zero matrix is a matrix of nay order whose all entries are zero.

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},\ \text{is a } \textbf{zero matrix}.$$

THEOREM 1.4.2 Properties of Zero Matrices

If c is a scalar, and if the sizes of the matrices are such that the operations can be perfomed, then:

(*a*) $A + 0 = 0 + A = A$

(*b*) $A - 0 = A$

(*c*) $A - A = A + (-A) = 0$

(*d*) $0A = 0$

(*e*) *If* $cA = 0$, *then* $c = 0$ *or* $A = 0$.

6. **Diagonal Matrix:** A square matrix with all its non- diagonal entries are zero.

Examples.
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$B = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}$$

7. Unit Matrix: A diagonal matrix with all diagonal entries are one '1'

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

REMARK:

$$AI_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = A$$

and multiplying on the left by the 2×2 identity matrix yields

$$I_2 A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = A$$

Properties of the Transpose of a matrix

1. $(A^t)^t = A$
2. $(AB)^t = B^t A^t$
3. $(kA)^t = kA^t$, where k is a scalar.
4. $(A+B)^t = A^t + B^t$

2. Symmetric Matrix:
A square matrix is symmetric if $A^t = A$.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}, \quad A^t = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}, \quad A^t = A$$

3. Skew – symmetric Matrix :

A square matrix is skew symmetric if $A^t = -A$.

$$A = \begin{bmatrix} 0 & -2 & -3 \\ 2 & 0 & -4 \\ 3 & 4 & 0 \end{bmatrix}, \quad A^t = \begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}, \quad A^t = -A.$$