Chapter 7: Series

Table of contents

1. Convergent or Divergent Series
2. Positive-term Series
3. The Ratio Test and Root test
4. Alternating Series and Absolute convergence
5. Power Series
6. Power series representations of functions
7. Taylor and Maclaurin series

Table of contents

1. Convergent or Divergent Series
2. Positive-term Series
3. The Ratio Test and Root test
4. Alternating Series and Absolute convergence
5. Power Series
6. Power series representations of functions
7. Taylor and Maclaurin series

Convergent or Divergent Series

Definition 1.1 (Infinite Series)

Let $\{a_n\}$ *be an infinite sequence. An expression of the form*

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \ldots$$

is called an **infinite series** *or* **simply series**.

Definition 1.2 (Partial sum)

1. *The* **nth partial sum** *of the infinite series* $\sum_{n=1}^{\infty} a_n$ *is*

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_n$$

2. *The* **sequence of partial sums** *associated with the infinite series* $\sum_{n=1}^{\infty} a_n$ *is*

$$S_1, S_2, S_3, \ldots, S_n, \ldots$$

Convergent or Divergent Series

Definition 1.3

- *An infinite series* $\sum_{n=1}^{\infty} a_n$ *with sequence of partial sums* $\{S_n\}$ *is* **convergent** *(or* **converges***), if* $\lim_{n\to\infty} S_n = S$*, for some real number* S*. The series is* **divergent** *(or* **diverges***), if this limit does not exist.*
- *If the series* $\sum_{n=1}^{\infty} a_n$ *is a convergent infinite series and* $\lim_{n\to\infty} S_n = S$*, then* S *is called the* **sum of the series** *and we write*

$$S = \sum_{n=1}^{\infty} a_n$$

If the series diverges, it has no sum.

Example 1.1

Prove that the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1\times 2} + \frac{1}{2\times 3} + \cdots + \frac{1}{n(n+1)} + \ldots$$

converges and find its sum.

Convergent or Divergent Series

Solution

Let $a_n = \dfrac{1}{n(n+1)}$
The partial fraction decomposition of a_n is

$$a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$\begin{aligned} S_n &= a_1 + a_2 + a_3 + \cdots + a_n \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1} = \frac{n}{n+1} \end{aligned}$$

$\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} \dfrac{n}{n+1} = 1,$
the series converges and have the sum 1.

Example 1.2

Prove that the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} = \frac{1}{3} + \frac{1}{15} + \cdots + \frac{1}{4n^2 - 1} + \dots$$

converges and find its sum.

Convergent or Divergent Series

Solution

Let $a_n = \dfrac{1}{4n^2-1} = \dfrac{1}{(2n-1)(2n+1)}$
The partial fraction decomposition of a_n is

$$a_n = \frac{1}{n(n+1)} = \frac{1}{2(2n-1)} - \frac{1}{2(2n+1)} + \frac{1}{4n-2} - \frac{1}{4n+2}$$

$$\begin{aligned} S_n &= a_1 + a_2 + a_3 + \cdots + a_n \\ &= \left(\frac{1}{2} - \frac{1}{6}\right) + \left(\frac{1}{6} - \frac{1}{10}\right) + \left(\frac{1}{10} - \frac{1}{14}\right) + \cdots + \frac{1}{4n-2} - \frac{1}{4n+2} \\ &= \frac{1}{2} - \frac{1}{4n+2} = \frac{2n+2}{4n+2} \end{aligned}$$

$\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} \dfrac{2n+2}{4n+2} = \dfrac{1}{2}$,
the series converges and have the sum $\dfrac{1}{2}$.

Definition 1.4 (Harmonic series)

The Harmonic series is the series defined as follows

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \cdots + \frac{1}{n} + \ldots$$

Theorem 1.1

The Harmonic series diverge.

Definition 1.5 (Geometric series)

The Geometric series is the series defined as follows

$$\sum_{n=1}^{\infty} ar^n = ar + ar^2 + ar^3 + \cdots + ar^n + \dots$$

where a *and* r *are real numbers, and* $a \neq 0$.

Theorem 1.2

Let $a \neq 0$. *The geometric series* $\sum_{n=1}^{\infty} ar^n$

1. *converges and has the sum* $S = \dfrac{a}{1-r}$ *if* $|r| < 1$.
2. *diverges if* $|r| > 1$.

Convergent or Divergent Series

Example 1.3

Prove that the infinite series

$$\sum_{n=1}^{\infty} \frac{6}{10^n} = 0.6 + 0.06 + 0.006 + \cdots + \frac{6}{10^n} + \dots$$

converges and find its sum.

Solution

This is a Geometric series with $a = 6$ and $r = \dfrac{1}{10}$.
By Theorem 3.1, the series converges and the sum
$S = \dfrac{6}{1 - 0.1} = \dfrac{6}{0.9} = \dfrac{20}{3}$

Example 1.4

Prove that the infinite series

$$\sum_{n=1}^{\infty} \frac{2}{3^n} = \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \cdots + \frac{2}{3^n} + \dots$$

converges and find its sum.

Solution

This is a Geometric series with $a = 2$ and $r = \dfrac{1}{3}$.

By Theorem 3.2, the series converges and the sum $S = \dfrac{2}{1 - \frac{1}{3}} = \dfrac{2}{\frac{2}{3}} = 3$

Exercise 1.1

Determine whether the following series converges. If so, give the sum.

1. $$\sum_{n=1}^{\infty} \frac{5}{(5n+2)(5n+7)}.$$
2. $$\sum_{n=1}^{\infty} \frac{325}{1000^n} = 0.325 + 0.000325 + \cdots + \frac{325}{1000^n} + \dots$$

Convergent or Divergent Series

Theorem 1.3

If an infinite series $\sum_{n=1}^{\infty} a_n$ *converges, then* $\lim_{n\to\infty} a_n = 0$

Theorem 1.4

If $\lim_{n\to\infty} a_n \neq 0$, *then infinite series* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Example 1.5

Determine whether the following series converges or diverges

$$\sum_{n=1}^{\infty} \frac{n}{2n+1} = \frac{1}{3} + \frac{2}{5} + \frac{3}{7} \cdots + \frac{n}{2n+1} + \dots$$

converges and find its sum.

Solution

Since

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2} \neq 0$$

By theorem 3.4 the series diverges.

Convergent or Divergent Series

Theorem 1.5

If $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *are series such that* $a_j = b_j$ *for every* $j > k$, *with* k *is a positive integer, then both series converges or both series diverges.*

Theorem 1.6

For every positive integer k, *the series*

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots \quad \textit{and} \quad \sum_{n=k+1}^{\infty} a_n = a_{k+1} + a_{k+2} + \dots$$

either both converges or diverges.

Convergent or Divergent Series

Example 1.6

Prove that the infinite series

$$\sum_{n=5}^{\infty} \frac{1}{n(n+1)} = \frac{1}{5\times 6} + \frac{1}{6\times 7} + \cdots + \frac{1}{n(n+1)} + \ldots$$

converges and find its sum.

Solution

In example 3.1, we proved that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$, converges. So, by theorem 3.6, the series $\sum_{n=5}^{\infty} \frac{1}{n(n+1)}$ converges.

Convergent or Divergent Series

Theorem 1.7

If $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *are convergent series with sums* A *ans* B, *respectively, then*

1. $\sum_{n=1}^{\infty} a_n + b_n$ *converges and has sum* $A + B$.
2. $\sum_{n=1}^{\infty} a_n - b_n$ *converges and has sum* $A - B$.
3. $\sum_{n=1}^{\infty} ca_n$ *converges and has sum* cA, *for every real number* c.

Example 1.7

Prove that the infinite series

$$\sum_{n=1}^{\infty}\left(\frac{7}{n(n+1)}+\frac{2}{3^n}\right)$$

converges and find its sum.

Solution

$$\sum_{n=1}^{\infty}\left(\frac{7}{n(n+1)}+\frac{2}{3^n}\right)=7\sum_{n=1}^{\infty}\frac{1}{n(n+1)}+\sum_{n=1}^{\infty}\frac{2}{3^n}$$

From example 3.1, the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$ converges and

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=1.$$

From example 3.4, the series $\displaystyle\sum_{n=1}^{\infty}\frac{2}{3^n}$ converges and $\displaystyle\sum_{n=1}^{\infty}\frac{2}{3^n}=3.$

So the series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{7}{n(n+1)}+\frac{2}{3^n}\right)$ converges and

$$\sum_{n=1}^{\infty}\left(\frac{7}{n(n+1)}+\frac{2}{3^n}\right)=7*1+3=10$$

Convergent or Divergent Series

Theorem 1.8

If $\sum_{n=1}^{\infty} a_n$ *is a convergent series and* $\sum_{n=1}^{\infty} b_n$ *is a divergent series, then the series* $\sum_{n=1}^{\infty} a_n + b_n$ *is divergent.*

Example 1.8

Determine the convergence or divergence of the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{5^n} + \frac{1}{n} \right)$$

Solution

The series $\sum_{n=1}^{\infty} \frac{1}{5^n}$ is a geometric series with $r = \frac{1}{5}$, so it's convergent.

$\sum_{n=1}^{\infty} \frac{1}{n}$ is a the harmonis series, so it's divergent.

From theorem 3.8, the series $\sum_{n=1}^{\infty} \left(\frac{1}{5^n} + \frac{1}{n} \right)$ diverges.

Table of contents

1. Convergent or Divergent Series
2. Positive-term Series
3. The Ratio Test and Root test
4. Alternating Series and Absolute convergence
5. Power Series
6. Power series representations of functions
7. Taylor and Maclaurin series

Positive-term Series

Definition 2.1 (positive-term series)

A positive-term series, is a series $\sum_{n=1}^{\infty} a_n$ *such that* $a_n > 0$ *for every* n.

Theorem 2.1

If $\sum_{n=1}^{\infty} a_n$ *is a positive-term series and if there exists a number* M *such that*

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n < M, \text{ for every } n$$

then the series converges and has a sum $S \leq M$. *If no such* M *exists the series diverges.*

Theorem 2.2 (Integral test)

If $\sum_{n=1}^{\infty} a_n$ is a positive-term series, let $f(n) = a_n$ and let f be the function obtained by replacing n with x. If f is positive-valued, continuous and decreasing for every real number $x \geq 1$, then the series $\sum_{n=1}^{\infty} a_n$

1. *converges if $\int_1^{\infty} f(x)\, dx$ converges.*
2. *diverges if $\int_1^{\infty} f(x)\, dx$ diverges.*

Example 2.1

Use the integral test to prove that the Harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Solution

Since $a_n = \frac{1}{n}$, we let $f(n) = \frac{1}{n}$. Replacing n by x gives $f(x) = \frac{1}{x}$. For every $x \geq 1$, f is positive-valued, continuous and decreasing, we can apply then integral test.

$$\int_1^{\infty} \frac{1}{x}\, dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{x}\, dx = \lim_{t\to\infty} [\ln x]_1^t = \lim_{t\to\infty} [\ln t - \ln 1] = \infty.$$

The series diverges by theorem 4.2.

Positive-term Series

Definition 2.2 (p-series)

A **p-series**, *is a series of the form*

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \dots$$

where p is a positive real number.

Theorem 2.3 (p-series test)

The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$

1. *converges if $p > 1$.*
2. *diverges if $p \leq 1$.*

Example 2.2

Decide whether the following series converges or diverges?

1. $$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \ldots$$
2. $$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} + \ldots$$
3. $$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}} = 1 + \frac{1}{2^{\frac{3}{2}}} + \frac{1}{3^{\frac{3}{2}}} + \cdots + \frac{1}{n^p} + \ldots$$
4. $$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}} = 1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \cdots + \frac{1}{\sqrt[3]{n}} + \ldots$$

Theorem 2.4 (Basic Comparison Test)

Let $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *be positive-term series.*

1. *If the series* $\sum_{n=1}^{\infty} b_n$ *converges and* $a_n \leq b_n$ *for every positive integer* n, *the series* $\sum_{n=1}^{\infty} a_n$ *converges.*
2. *If the series* $\sum_{n=1}^{\infty} b_n$ *diverges and* $a_n \geq b_n$ *for every positive integer* n, *the series* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Example 2.3

Decide whether the following series converges or diverges?

1. $$\sum_{n=1}^{\infty} \frac{1}{2+5^n}.$$
2. $$\sum_{n=1}^{\infty} \frac{3}{\sqrt{n}-1}.$$

Solution

1. For every $n \geq 1$, $\dfrac{1}{2+5^n} < \dfrac{1}{5^n}$.

   Since the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5^n}$ converges, then the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2+5^n}$ converges.

2. For every $n \geq 1$, $\dfrac{1}{\sqrt{n}-1} > \dfrac{1}{\sqrt{n}}$, then $\dfrac{3}{\sqrt{n}-1} > \dfrac{1}{\sqrt{n}}$.

   Since the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges, then the series $\displaystyle\sum_{n=1}^{\infty} \frac{3}{\sqrt{n}-1}$ diverges.

Theorem 2.5 (Limit Comparison Test)

Let $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *be positive-term series. If there is a positive real number* c *such that*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0,$$

then either both series converges or both series diverges.

Example 2.4

Decide whether the following series converges or diverges?

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2+1}}$.
2. $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2+5n}{2^n(n^2+1)}$.

Solution

1. The n^{th} term of the series is $a_n = \dfrac{1}{\sqrt[3]{n^2+1}}$

   If we delete the number 1 from the radicand, we obtain $b_n = \dfrac{1}{\sqrt[3]{n^2}}$.

   $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{2}{3}}}$, which is a p-series with $p = \frac{2}{3}$, then its divergent.

   $$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt[3]{n^2}}{\sqrt[3]{n^2+1}} = \lim_{n\to\infty} \sqrt[3]{\frac{n^2}{n^2+1}} = 1 > 0.$$

   From theorem 4.5, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2+1}}$ diverges.

Solution

2. The n^{th} term of the series is $a_n = \dfrac{3n^2+5n}{2^n(n^2+1)}$
If we delete the least magnitude in the numerator and the denominator , we obtain $\dfrac{3n^2}{2^n n^2} = \dfrac{3}{2^n}$, we choose $b_n = \dfrac{3}{2^n}$ which is a geometric series with $r = \frac{1}{2}$, then its convergent.
$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(3n^2+5n)2^n}{2^n(n^2+1)} = \lim_{n\to\infty} \frac{3n^2+5n}{n^2+1} = 3 > 0.$$
From theorem 4.5, $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2+5n}{2^n(n^2+1)}$ converges.

Exercise 2.1

Decide whether the following series converges or diverges?

$$\sum_{n=1}^{\infty} \frac{8n+\sqrt{n}}{5+n^2+n^{\frac{7}{2}}}$$

Table of contents

1. Convergent or Divergent Series
2. Positive-term Series
3. The Ratio Test and Root test
4. Alternating Series and Absolute convergence
5. Power Series
6. Power series representations of functions
7. Taylor and Maclaurin series

The Ratio Test and Root test

Theorem 3.1

Let $\sum_{n=1}^{\infty} a_n$ *be positive-term series, and suppose that* $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$.

1. *If* $L < 1$, *the series converges.*
2. *If* $L > 1$, *the series diverges.*
3. *If* $L = 1$, *apply another test, the series may be convergent or divergent.*

Example 3.1

Decide whether the following series converges or diverges?

1. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n!}$.
2. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^2}$.

Solution

1. Applying theorem 5.1
$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{3^{n+1}n!}{3^n(n+1)!}=\lim_{n\to\infty}\frac{3}{n+1}=0<1,$$
the the series converges.
2. Applying theorem 5.1
$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{3^{n+1}n^2}{3^n(n+1)^2}=\lim_{n\to\infty}\frac{3n^2}{n^2+2n+1}=3>1,$$
the the series diverges.

Exercise 3.1

Decide whether the following series converges or diverges?

1. $$\sum_{n=1}^{\infty} \frac{n^n}{n!}$$
2. $$\sum_{n=1}^{\infty} n!$$
3. $$\sum_{n=1}^{\infty} \frac{1}{(n+1)!}$$

The Ratio Test and Root test

Theorem 3.2

Let $\sum_{n=1}^{\infty} a_n$ *be positive-term series, and suppose that* $\lim_{n\to\infty} \sqrt[n]{a_n} = L$.

1. *If* $L < 1$, *the series converges.*
2. *If* $L > 1$, *the series diverges.*
3. *If* $L = 1$, *apply another test, the series may be convergent or divergent.*

Example 3.2

Decide whether the following series converges or diverges?

$$\sum_{n=1}^{\infty} \frac{2^{3n+1}}{n^n}$$

The Ratio Test and Root test

Solution

Applying theorem 5.2

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{2^{3n+1}}{n^n}} = \lim_{n\to\infty} \frac{2^{3+\frac{1}{n}}}{n} = 0 < 1,$$

the the series converges.

Exercise 3.2

Decide whether the following series converges or diverges?

1. $$\sum_{n=1}^{\infty} \frac{5^n}{n^n}$$
2. $$\sum_{n=1}^{\infty} \left(\frac{8n^2 - 7}{n+1}\right)^n$$

Table of contents

1. Convergent or Divergent Series
2. Positive-term Series
3. The Ratio Test and Root test
4. Alternating Series and Absolute convergence
5. Power Series
6. Power series representations of functions
7. Taylor and Maclaurin series

Definition 4.1 (Alternating Series)

The alternating series is the series defined by

$$\sum_{n=1}^{\infty}(-1)^{n-1}a_n = a_1 - a_2 + \cdots + (-1)^{n-1}a_n + \dots$$

Theorem 4.1 (Alternating Series Test (AST))

The alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ *converges if the two following conditions are satisfied*

1. $a_k \geq a_{k+1} > 0$, *for every* k,
2. $\lim_{n\to\infty} a_n = 0$

Example 4.1

Determine whether the alternating series converges or diverges.

1. $$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2n}{4n^2-3}$$
2. $$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2n}{4n-3}$$

Solution

1. $a_n = \dfrac{2n}{4n^2-3}$
   - $a_k - a_{k+1} = \dfrac{2k}{4k^2-3} - \dfrac{2(k+1)}{4(k+1)^2-3} = \dfrac{8k^2+8k+6}{(4k^2-3)(4k^2+8k+1)} \geq 0$, so $a_k \geq a_{k+1}$
   - $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{2n}{4n^2-3} = 0$,

   From Theorem 6.1, the series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2n}{4n^2-3}$ converges.

2. $a_n = \dfrac{2n}{4n-3}$

   $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{2n}{4n-3} = \dfrac{1}{2}$, From Theorem 3.4, the series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2n}{4n^2-3}$ diverges.

Exercise 4.1

1. $$\sum_{n=1}^{\infty}(-1)^{n-1}n5^{-n}$$
2. $$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n+1}{n}$$

Alternating Series and Absolute convergence

Definition 4.2 (Absolute convergence)

The series $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent if the series*

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + \cdots + |a_n| + \dots$$

is convergent.

Alternating Series and Absolute convergence

Example 4.2

Prove that the following alternating series is absolutely convergent.

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots + (-1)^{n-1}\frac{1}{n^2} + \dots$$

Solution

We have $a_n = (-1)^{n-1}\dfrac{1}{n^2}$, then

$\displaystyle\sum_{n=1}^{\infty}|a_n| = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots + \frac{1}{n^2} + \dots$, which a p-series with $p = 2$,

thus its convergent. Then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

Definition 4.3

The series $\sum_{n=1}^{\infty} a_n$ *is conditionally convergent if the series* $\sum_{n=1}^{\infty} a_n$ *is convergent and the series* $\sum_{n=1}^{\infty} |a_n|$ *is divergent.*

Theorem 4.2

If the series $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent the the series* $\sum_{n=1}^{\infty} a_n$ *is convergent*

Exercise 4.2

Determine whether the series is absolute convergent, conditionally convergent or divergent

1. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$
2. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{\sqrt{n}}$
3. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n}\frac{n}{n+1}$

Alternating Series and Absolute convergence

Theorem 4.3 (Absolute Ratio Test)

Let $\sum_{n=1}^{\infty} a_n$ *be a series of non-zero terms, and suppose* $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$.

1. *If* $L < 1$ *then the series* $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent.*
2. *If* $L > 1$ *or* $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$ *then the series* $\sum_{n=1}^{\infty} a_n$ *is divergent.*
3. *If* $L = 1$, *apply a different test; the series may be absolutely convergent, conditionally convergent, or divergent.*

Example 4.3

Determine whether the following series is absolutely convergent, conditionally convergent, or divergent:

$$\sum_{n=1}^{\infty}(-1)^n \frac{n^2+4}{2^n}$$

Solution

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2+4}{2^{n+1}}\frac{2^n}{n^2+4}\right|$$
$$= \lim_{n\to\infty}\frac{1}{2}\left(\frac{n^2+2n+5}{n^2+4}\right) = \frac{1}{2} < 1,$$

then, using theorem 6.3, the series is absolutely convergent.

Exercise 4.3

Determine whether the series is absolute convergent, conditionally convergent or divergent

1. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(-10)^n}{n!}$
2. $\sum_{n=1}^{\infty} (-1)^n \frac{n^4}{e^n}$

Table of contents

1. Convergent or Divergent Series
2. Positive-term Series
3. The Ratio Test and Root test
4. Alternating Series and Absolute convergence
5. Power Series
6. Power series representations of functions
7. Taylor and Maclaurin series

Definition 5.1 (Power Series)

Let x be a variable. **A power series in x** *is a series of the form*

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \ldots$$

where each a_k is a real number.

Remark 5.1

To find other values of x that produce convergent series, we often use the ratio test for absolute convergence, Theorem 4.3, as illustrated in the following examples.

Example 5.1

Find all values of x for which the following power series is absolutely convergent:

$$\sum_{n=0}^{\infty} \frac{n}{5^n} x^n = \frac{1}{5} x + \frac{2}{5^2} x^2 + \cdots + \frac{n}{5^n} x^n + \dots$$

Solution

If we let $u_n = \frac{n}{5^n} x^n$.

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)x^{n+1}}{5^{n+1}} \frac{5^n}{nx^n}\right|$$

$$= \lim_{n\to\infty} \left|\frac{(n+1)x}{5n}\right| = \lim_{n\to\infty} \left(\frac{n+1}{5n}\right)|x| = \frac{1}{5}|x|.$$

By the ratio test (Theorem 4.3), with $L = \dfrac{1}{5}|x|$, the series is absolutely convergent if the following equivalent inequalities are true:

$$L = \frac{1}{5}|x| < 1 \Longrightarrow |x| < 5 \Longrightarrow -5 < x < 5$$

Example 5.2

Find all values of x for which the following power series is absolutely convergent:

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + \frac{1}{1!} x + \frac{1}{2!} x^2 + \cdots + \frac{1}{n!} x^n + \ldots$$

Solution

If we let $u_n = \frac{1}{n!}x^n$.

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\frac{n!}{x^n}\right|$$
$$= \lim_{n\to\infty}\left|\frac{x}{n+1}\right| = \lim_{n\to\infty}\frac{1}{n+1}|x| = 0.$$

By the ratio test (Theorem 4.3), with $L = 0 < 1$, the power series is absolutely convergent for every real number x.

Example 5.3

Find all values of x for which the power series $\sum_{n=0}^{\infty} n!x^n$ is convergent.

Solution

If we let $u_n = n!x^n$, if $x \neq 0$.

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!x^{n+1}}{n!x^n}\right|$$

$$= \lim_{n\to\infty}|(n+1)x| = \lim_{n\to\infty}(n+1)|x| = \infty.$$

and, by the ratio test (Theorem 4.3), the series diverges. Hence, the power series is convergent only if $x = 0$.

Theorem 5.1

1. *If a power series* $\sum_{n=0}^{\infty} a_n x^n$ *converges for a nonzero number* c, *then it is absolutely convergent whenever* $|x| < |c|$.
2. *If a power series* $\sum_{n=0}^{\infty} a_n x^n$ *diverges for a nonzero number* d, *then it diverges whenever* $|x| > |d|$.

Theorem 5.2

If $\sum_{n=0}^{\infty} a_n x^n$ *a Power series, then exactly one of the following is true:*

1. *The series converges only if* $x = 0$.
2. *The series is absolutely convergent for every* x.
3. *There is a number* $r > 0$ *such that the series is absolutely convergent if* x *is in the open interval* $(-r, r)$ *(*$|x| < r$*) and divergent if* $x < -r$ *or* $x > r$ *(*$|x| > r$*).*

Remark 5.2

- The number r is called the **radius of convergence** of the series. Either convergence or divergence may occur at $-r$ or r, depending on the nature of the series.
- The totality of numbers for which a power series converges is called its **interval of convergence**. If the radius of convergence r is positive, then the interval of convergence is one of the following

$$(-r, r),\ (-r, r],\ [-r, r),\ [-r, r]$$

Example 5.4

Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} x^n$$

Solution

Note that the coefficient of x^0 is 0 and the summation begin with 1.

If we let $u_n = \dfrac{1}{\sqrt{n}} x^n$.

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{\sqrt{n+1}} \frac{\sqrt{n}}{x^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\sqrt{n}}{\sqrt{n+1}} x \right| = \lim_{n\to\infty} \sqrt{\frac{n}{n+1}} |x| = |x|.$$

By the ratio test (Theorem 4.3), with $L = |x|$, the series is absolutely convergent if the following equivalent inequalities are true:
$L = |x| < 1 \Longrightarrow -1 < x < 1$, then the radius of convergence is $r = 1$.
The case when $x = 1$, the power series will be a p-series with $p = \frac{1}{2}$, which is divergent.
The case when $x = -1$, the power series will be an alternating series $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$ which is convergent.
Thus the interval of convergence is $[-1, 1)$.

Definition 5.2

Let c be a real number and x be a variable. A **power series in $x - c$** *is a series of the form*

$$\sum_{n=1}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + a_n(x-c)^n + \ldots$$

where each a_k is a real number.

Theorem 5.3

If $\sum_{n=0}^{\infty} a_n(x-c)^n$ *a Power series, then exactly one of the following is true:*

1. *The series converges only if* $x - c = 0$*, that is* $x = c$.
2. *The series is absolutely convergent for every* x.
3. *There is a number* $r > 0$ *such that the series is absolutely convergent if* x *is in the open interval* $(c-r, c+r)$ $(|x-c| < r)$ *and divergent if* $x < c - r$ or $x > c + r$ $(|x-c| > r)$.

Example 5.5

Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty}(-1)^n\frac{1}{n+1}(x-3)^n$$

Solution

If we let $u_n=(-1)^n\dfrac{1}{n+1}(x-3)^n$.

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| &= \lim_{n\to\infty}\left|\frac{(x-3)^{n+1}}{n+2}\,\frac{n+1}{(x-3)^n}\right| \\ &= \lim_{n\to\infty}\left|\frac{n+1}{n+2}(x-3)\right| = \lim_{n\to\infty}\frac{n+1}{n+2}|x-3| = |x-3|.\end{aligned}$$

By the ratio test (Theorem 4.3), with $L = |x-3|$, the series is absolutely convergent if the following equivalent inequalities are true:
$L = |x-3| < 1 \Longrightarrow -1 < x-3 < 1 \Longrightarrow 2 < x < 4.$
The case when $x = 4$, the power series will be an alternating series
$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n+1}$ which is convergent.
The case when $x = 2$, the power series will be an harmonic series
$\sum_{n=1}^{\infty} \frac{1}{n+1}$ which is divergent.
Thus the interval of convergence is $(2, 4]$.

Table of contents

1. Convergent or Divergent Series
2. Positive-term Series
3. The Ratio Test and Root test
4. Alternating Series and Absolute convergence
5. Power Series
6. Power series representations of functions
7. Taylor and Maclaurin series

Definition 6.1

A power series $\sum a_n x^n$ *determines a function* f *whose domain is the interval of convergence of the series. Specifically, for each* x *in this interval, we let* $f(x)$ *equal the sum of the series, that is,*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \ldots$$

If a function f *is defined in this way, we say that* $\sum a_n x^n$ *is a* **power series representation for** $f(x)$ *(or of* $f(x)$*). We also use the phrase* f **is represented by the power series**.

Power series representations of functions

Example 6.1

Find a function f that is represented by the power series

$$1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \dots$$

Solution

If $|x| < 1$, then the series is a geometric series which is convergent and has the sum

$$\frac{a}{1-r} = \frac{1}{1-(-x)} = \frac{1}{1+x}$$

Hence we may write

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \dots$$

This result is a power series representation for $f(x) = \dfrac{1}{1+x}$ on the interval $(-1, 1)$.

Power series representations of functions

Theorem 6.1

Suppose that a power series $\sum a_n x^n$ *has a radius of convergence* $r > 0$, *and let* f *be defined by*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \ldots$$

for every x *in the interval of convergence. If* $-r < x < r$. *then*

1. $f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1} + \cdots = \displaystyle\sum_{n=1}^{\infty} n a_n x^{n-1}$
2. $\displaystyle\int_0^x f(x)\,dx = a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} + \cdots + a_n \frac{x^{n+1}}{n+1} + \cdots = \sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n+1}$

The series obtained by differentiation or integration has the same radius of convergence as $\sum a_n x^n$.

Example 6.2

Use a power series representation for $\dfrac{1}{1+x}$ to obtain a power series representation for

$$\frac{1}{(1+x)^2}, \text{ if } |x| < 1$$

Solution

We have

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \ldots$$

If we differentiate each term of this series, then

$$\frac{-1}{(1+x)^2} = -1 + 2x - 3x^2 + \cdots + (-1)^n n x^{n-1} + \ldots$$

we may multiply both sides by -1, obtaining

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 + \cdots + (-1)^{n+1} n x^{n-1} + \ldots, \text{ if } |x| < 1$$

Example 6.3

Find a power series representation for

$$\ln(1+x), \text{ if } |x| < 1$$

Solution

If $|x| < 1$, then $\ln(1+x) = \int_0^x \frac{1}{1+t}\ dt$ We have

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \ldots$$

then $\ln(1+x) = \int_0^x \left[1 - t + t^2 - t^3 + \cdots + (-1)^n t^n + \ldots\right]\ dt$ we may integrate each term of the series as follows:

$$\ln(1+x) = \int_0^x 1\ dt - \int_0^x t\ dt + \int_0^x t^2\ dt + \cdots + (-1)^n \int_0^x t^n\ dt + \ldots$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^n \frac{x^{n+1}}{n+1} + \ldots \text{ if } |x| < 1$$

Example 6.4

Use the results of Example 1.3 to calculate $\ln(1.1)$ to five decimal places.

Solution

In Example 1.3, we found a series representation for $\ln(1+x)$ if $|x|<1$. Substituting 0.1 for x in that series gives us the alternating series

$$\begin{aligned}\ln(1.1) &= 0.1 - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} + \frac{(0.1)^4}{4} + \frac{(0.1)^5}{5} + \dots \\ &\approx 0.1 - 0.005 + 0.000333 - 0.000025 + 0.000002 + \dots\end{aligned}$$

If we sum the first four terms on the right and round off to five decimal places, we obtain $\ln(1.1) \approx 0.09531$.

Power series representations of functions

Example 6.5

Find a power series representation for $\tan^{-1} x$.

Power series representations of functions

Solution

We first observe that

$$\tan^{-1} x = \int_0^x \frac{1}{1+t^2}\, dt$$

We have $\dfrac{1}{1+t^2} = \dfrac{1}{1-(-t^2)}$, if $|t| < 1$, then $\dfrac{1}{1+t^2}$ is the sum of a geometric series with $a = 1$ and $r = -t^2$, thus

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + \cdots + (-1)^n t^{2n} + \ldots$$

we may integrate each term of the series from 0 to x to obtain

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \ldots,$$

when $|x| < 1$. It can be proved that this series representation is also valid when $|x| = 1$.

Power series representations of functions

Theorem 6.2

If x is any real number,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \ldots$$

Remark 6.1

To obtain a power series representation for e^{-x}, we need only substitute $-x$ for x:

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = 1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \cdots + \frac{(-x)^n}{n!} + \ldots$$

or

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + \frac{(-1)^n x^n}{n!} + \ldots$$

Example 6.6

Find the power series representations of the functions:

1. $f(x) = \cosh(x)$
2. $f(x) = \sinh(x)$

Power series representations of functions

Solution

1. We have $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$.

   Since $e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!} = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots + \dfrac{x^n}{n!} + \ldots$ and

   $e^{-x} = \sum_{n=0}^{\infty} \dfrac{(-1)^n x^n}{n!} = 1 - x + \dfrac{x^2}{2!} - \dfrac{x^3}{3!} + \cdots + \dfrac{(-1)^n x^n}{n!} + \ldots,$

   we find $e^x + e^{-x} = 2 + 2\dfrac{x^2}{2!} + 2\dfrac{x^4}{4!} + \cdots + 2\dfrac{x^{2n}}{2n!} + \ldots$,thus

   $\cosh(x) = 1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \cdots + \dfrac{x^{2n}}{2n!} + \ldots$

2. We have $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$.
we find $e^x - e^{-x} = 2x + 2\dfrac{x^3}{3!} + 2\dfrac{x^5}{5!} + \cdots + 2\dfrac{x^{2n+1}}{(2n+1)!} + \ldots$,thus
$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2n+1}}{(2n+1)!} + \ldots$$

Power series representations of functions

Example 6.7

Find a power series representation for $f(x) = xe^{-2x}$.

Solution

First we substitute $-2x$ for x in e^x representation and we have

$$e^{-2x} = 1 + (-2x) + \frac{(-2x)^2}{2!} + \frac{(-2x)^3}{3!} + \dots + \frac{(-2x)^n}{n!} + \dots$$

$$e^{-2x} = 1 - 2x + 4\frac{x^2}{2!} - 8\frac{x^3}{3!} + \dots + \frac{(-2)^n x^n}{n!} + \dots$$

Multiplying both sides by x gives us

$$xe^{-2x}x - 2x^2 + 4\frac{x^3}{2!} - 8\frac{x^4}{3!} + \dots + \frac{(-2)^n x^{n+1}}{n!} + \dots$$

$$f(x) = xe^{-2x} = \sum_{n=0}^{\infty} \frac{(-2)^n x^{n+1}}{n!}$$

Power series representations of functions

Example 6.8

Find a power series representation for $\displaystyle\int_0^x \frac{e^t - 1}{t}\, dt$.

Solution

Using the power series representation of e^x we have

$$e^t - 1 = t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots + \frac{t^n}{n!} + \ldots$$

then

$$\frac{e^t - 1}{t} = 1 + \frac{t}{2!} + \frac{t^2}{3!} + \cdots + \frac{t^{n-1}}{n!} + \ldots$$

we may integrate each term of the series from 0 to x to obtain

$$\int_0^x \frac{e^t - 1}{t}\, dt = x + \frac{x^2}{2 \times 2!} + \frac{x^3}{3 \times 3!} + \cdots + \frac{x^n}{n \times n!} + \ldots$$

Exercise 6.1

Find a power series representation for $f(x)$, $f'(c)$ and $\int_0^x f(t)\, dt$.

1. $f(x) = \dfrac{1}{3-2x}$.
2. $f(x) = \dfrac{x^3}{4-x^3}$.
3. $f(x) = \dfrac{x^2+1}{x-1}$.
4. $f(x) = x\ln(1-x)$.
5. $f(x) = x^2 e^{x^2}$

Exercise 6.2

Approximate the following integrals to four decimal places.

1. $\displaystyle\int_0^{0.1} e^{-x^2}\ dx$
2. $\displaystyle\int_0^{0.5} e^{-x^3}\ dx$
3. $\displaystyle\int_0^{\frac{1}{2}} \tan^{-1} x^2\ dx$

Table of contents

1. Convergent or Divergent Series
2. Positive-term Series
3. The Ratio Test and Root test
4. Alternating Series and Absolute convergence
5. Power Series
6. Power series representations of functions
7. Taylor and Maclaurin series

Taylor and Maclaurin series

In the preceding section, we considered power series representations for several special functions, including those where $f(x)$ has the form

$$\frac{1}{1+x},\ \ln(1+x),\ \tan^{-1}(x),\ e^x,\ \text{or },\ \cosh(x)$$

provided x is suitably restricted.
We now wish to consider the following two general questions.

Questions

1. If a function $f(x)$ has a power series representation

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \text{ or } f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

   what is the form of a_n?

2. What conditions are sufficient for a function f to have a power series representation?

Taylor and Maclaurin series

Theorem 7.1 (Maclaurin series for $f(x)$)

If a function f has a power series representation

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

with radius of convergence $r > 0$, then $f^{(k)}(0)$ exist for every positive integer k and $a_n = \dfrac{f^{(n)}(0)}{n!}$. Thus

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \ldots$$

or

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

Taylor and Maclaurin series

Theorem 7.2 (Taylor series for $f(x)$)

If a function f has a power series representation

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

with radius of convergence $r > 0$, then $f^{(k)}(c)$ exist for every positive integer k and $a_n = \dfrac{f^{(n)}(c)}{n!}$. Thus

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \ldots$$

or

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Exercise 7.1

Find Maclaurin series of

1. $\sin x$
2. $\cos x$
3. $x^2 \sin x$
4. e^x

Exercise 7.2

Find Taylor series of

1. $\sin x$, $x = \dfrac{\pi}{6}$
2. $\ln x$, $x = c$, $c > 0$

Exercise 7.3

Approximate the improper integral to four decimal places.

1. $\int_0^1 \sin x^2$
2. $\int_0^1 \frac{1-\cos x}{x^2}$