Differential and Integral Calculus (Math 203)

BEN AMIRA Aymen

Department of Mathematics
College of Sciences
King Saud University

Chapter 5: Applications

Table of contents

Table of contents

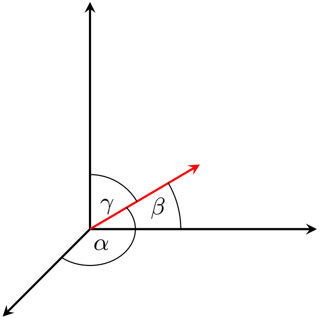
The Dot Product

Definition 1.1

In $\mathbb{R}^2$, *if* $u = (u_1, u_2)$ *and* $v = (v_1, v_2)$, *the dot product of* u *and* v *is the number* $\langle u, v\rangle = u_1v_1 + u_2v_2$.
In $\mathbb{R}^3$, *if* $u = (u_1, u_2, u_3)$ *and* $v = (v_1, v_2, v_3)$, *the dot product of* u *and* v *is the number* $\langle u, v\rangle = u_1v_1 + u_2v_2 + u_3v_3$.
The norm of a vector u *is* $\|u\| = \sqrt{\langle u, u\rangle}$.

Recall that if θ is the angle between the vectors $\overrightarrow{u}$ and $\overrightarrow{v}$, then

$$\langle u, v\rangle = \|u\| \; \|v\| \cos\theta.$$



The direction angles associated to a vector u are given by: $\cos\alpha = \frac{\langle u,i\rangle}{\|u\|}$, $\cos\beta = \frac{\langle u,j\rangle}{\|u\|}$, $\cos\gamma = \frac{\langle u,k\rangle}{\|u\|}$.

Definition 1.2

If $u_1 = (x_1, y_1, z_1)$ *and* $u_2 = (x_2, y_2, z_2)$, *then the cross product of* u_1 *and* u_2 *is the vector*

$$u_1 \wedge u_2 = \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} \overrightarrow{\mathbf{i}} + \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} \overrightarrow{\mathbf{j}} + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \overrightarrow{\mathbf{k}}.$$

Remark 1.1

1. The vector $u_1 \wedge u_2$ is orthogonal to the vectors u_1 and u_2 and its direction is given by the right-hand rule i.e. the determinant $|u_1, u_2, u_1 \wedge u_2|$ is non negative.
2. $|u_1 \wedge u_2|$ is the area of the parallelogram spanned by u_1 and u_2, i.e.,

$$|u_1 \wedge u_2| = |u_1|\,|u_2|\,\sin\theta$$

3. Two vectors u_1 and u_2 are parallel if and only if $u_1 \wedge u_2 = 0$.

Theorem 1.1 (Cross Product Properties)

Let u_1*,* u_2*, and* u_3 *be vectors and let* c *be a constant:*

1. $u_1 \wedge u_2 = -u_2 \wedge u_1$;
2. $(cu_1) \wedge u_2 = c(u_1 \wedge u_2) = u_1 \wedge (cu_2)$;
3. $u_1 \wedge (u_2 + u_3) = u_1 \wedge u_2 + u_1 \wedge u_3$;
4. $(u_1 + u_2) \wedge u_3 = u_1 \wedge u_3 + u_2 \wedge u_3$;
5. $u_1 \cdot (u_2 \wedge u_3) = (u_1 \wedge u_2) \cdot u_3$;
6. $u_1 \wedge (u_2 \wedge u_3) = (u_1 \cdot u_3)u_2 - (u_1 \cdot u_2)u_3$.

Scalar Triple Product

The scalar triple product of three vectors u_1, u_2, and u_3 is the determinant

$$\langle u_1, (u_2 \wedge u_3) \rangle = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}.$$

The volume of the parallelepiped formed by the vectors u_1, u_2, and u_3 is given by

$$|\langle u_1, (u_2 \wedge u_3) \rangle|.$$

Let f be a function defined on a domain $D \subset \mathbb{R}^2$. For $(x_0, y_0) \in D$, the partial derivatives of f with respect to x and y if they exist are defined by:

$$f_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h},$$

$$f_y(x_0, , y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}.$$

Consider a smooth scalar field $f : D \longrightarrow \mathbb{R}$. The partial derivatives of f in the point $\mathbf{r} = x\overrightarrow{\mathbf{i}} + y\overrightarrow{\mathbf{j}} + z\overrightarrow{\mathbf{k}} \in D$ when these limits exist:

$$\frac{\partial f}{\partial x}(\mathbf{r}) = \lim_{h\to 0} \frac{f(x+h, y, z) - f(x, y, z)}{h};$$

$$\frac{\partial f}{\partial y}(\mathbf{r}) = \lim_{h\to 0} \frac{f(x, y+h, z) - f(x, y, z)}{h};$$

$$\frac{\partial f}{\partial z}(\mathbf{r}) = \lim_{h\to 0} \frac{f(x, y, z+h) - f(x, y, z)}{h}.$$

The Directional Derivative

Let f be a function defined on a domain $D \subset \mathbb{R}^2$. For $(x_0, y_0) \in D$ and $u = (a, b)$ a unit vector in $\mathbb{R}^2$. The directional derivative of f in the direction of u at (x_0, y_0) if it exists is

$$
\begin{aligned}
D_u f(x_0, y_0) &= \lim_{h \to 0} \frac{f((x_0, y_0) + hu) - f(x_0, y_0)}{h} \\
&= \lim_{h \to 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h}.
\end{aligned}
$$

Example 1.1

1. If $u = (a, b)$, $D_u f(x_0, y_0)$ is the same as the derivative of $f(x_0 + at, y_0 + bt)$ at $t = 0$. We can compute this by the chain rule and get
$$D_u f(x_0, y_0) = af_x(x_0, y_0) + bf_y(x_0, y_0).$$
2. Find the directional derivative of $f(x, y) = xy^3 - x^2$ at $(1, 2)$ in the direction $u = (\frac{1}{2}, \frac{\sqrt{3}}{2})$
3. Find the directional derivative of $f(x, y) = x^2 \ln y$ at $(3, 1)$ in the direction of $u = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$.

Table of contents

Definition 2.1

A two-dimensional vector field is a function f that maps each point (x,y) in $\mathbb{R}^2$ to a two-dimensional vector $f(x,y) = (u(x,y), v(x,y))$.
We denote $f(x,y) = u(x,y)\overrightarrow{\mathbf{i}} + v(x,y)\overrightarrow{\mathbf{j}}$, where $\overrightarrow{\mathbf{i}} = (1,0)$ and $\overrightarrow{\mathbf{j}} = (0,1)$.
Similarly a three-dimensional vector field maps (x,y,z) to $f(x,y,z) = (u(x,y,z), v(x,y,z), w(x,y,z))$.
We denote $f(x,y,z) = u(x,y,z)\overrightarrow{\mathbf{i}} + v(x,y,z)\overrightarrow{\mathbf{j}} + w(x,y,z)\overrightarrow{\mathbf{k}}$, where $\overrightarrow{\mathbf{i}} = (1,0,0)$, $\overrightarrow{\mathbf{j}} = (0,1,0)$ and $\overrightarrow{\mathbf{k}} = (0,0,1)$.

Example 2.1

The vector fields have many important significations, as they can be used to represent many physical quantities: gravity, electricity, magnetism or a velocity of fluid.

Let $r(t) = x(t)\overrightarrow{\mathbf{i}} + y(t)\overrightarrow{\mathbf{j}} + z(t)\overrightarrow{\mathbf{k}}$ be the position vector of an object. We can define various physical quantities associated with the object as follows:

velocity: $v(t) = r'(t) = \frac{dr}{dt} = x'(t)\overrightarrow{\mathbf{i}} + y'(t)\overrightarrow{\mathbf{j}} + z'(t)\overrightarrow{\mathbf{k}}$,

acceleration:

$a(t) = v'(t) = \frac{dv}{dt} = r^{''}(t) = \frac{d^2r}{dt^2} = x^{''}(t)\overrightarrow{\mathbf{i}} + y^{''}(t)\overrightarrow{\mathbf{j}} + z^{''}(t)\overrightarrow{\mathbf{k}}$, The norm $\|v(t)\|$ of the velocity vector is called the speed of the object.

Example 2.2

The gravitational force field between the Earth with mass M and a point particle with mass m is given by:

$$F(x, y, z) = -GmM \frac{x\overrightarrow{\mathbf{i}} + y\overrightarrow{\mathbf{j}} + z\overrightarrow{\mathbf{k}}}{(x^2 + y^2 + z^2)^{\frac{3}{2}}},$$

where G is the gravitational constant, and the (x, y, z) coordinates are chosen so that $(0, 0, 0)$ is the center of the Earth.

Gradient Fields

Let f be a scalar function of two variables, the gradient of f is defined by

$$\nabla f(x,y) = (\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y)).$$

If f is a scalar function of three variables, its gradient is a vector field on $\mathbb{R}^3$ given by

$$\nabla f(x,y,z) = (\frac{\partial f}{\partial x}(x,y,z), \frac{\partial f}{\partial y}(x,y,z), \frac{\partial f}{\partial z}(x,y,z)).$$

The operator ∇ will be denoted by:
$\nabla = \frac{\partial}{\partial x}\overrightarrow{\mathbf{i}} + \frac{\partial}{\partial y}\overrightarrow{\mathbf{j}} + \frac{\partial}{\partial z}\overrightarrow{\mathbf{k}}$ or $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ as a vector.

Remark 2.1

Let f be a function. The vector $\nabla f(x_0, y_0, z_0)$ is orthogonal to the level surface of f $S = \{(x, y, z) \in \mathbb{R}^3 : f(x, y, z) = C\}$ that contains (x_0, y_0, z_0).

Gradient Fields

Consider f and g two smooth scalar functions defined on a domain $D \subset \mathbb{R}^3$ and consider $F = (f_1, f_2, f_3)$ and $G = (g_1, g_2, g_3)$ two smooth vector fields.

$$\begin{aligned} \nabla(fg) &= \left(\frac{\partial(fg)}{\partial x}, \frac{\partial(fg)}{\partial y}, \frac{\partial(fg)}{\partial z}\right) \\ &= f\nabla(g) + g\nabla(f). \end{aligned}$$

$$\begin{aligned} \nabla(\langle F, G\rangle) &= \nabla(f_1g_1 + f_2g_2 + f_3g_3) \\ &= \nabla(f_1g_1) + \nabla(f_2g_2) + \nabla(f_3g_3) \\ &= f_1\nabla(g_1) + +f_2\nabla(g_2) + f_3\nabla(g_3) \\ & \quad g_1\nabla(f_1) + g_2\nabla(f_2) + g_3\nabla(f_3). \end{aligned}$$

Definition 2.2

A vector field F is called conservative, if F is the gradient of a function, $F = \nabla f$. In this case, the function f is called a potential of the vector field F.

For example the vector field

$$\begin{aligned} F &= \left(\frac{-x}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{-y}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{-z}{(x^2+y^2+z^2)^{\frac{3}{2}}} \right) \\ &= \nabla \frac{1}{\sqrt{x^2+y^2+z^2}}. \end{aligned}$$

Example 2.3 (The inverse square field)

Let $\mathbf{r}(x,y,z) = x\overrightarrow{\mathbf{i}} + y\overrightarrow{\mathbf{j}} + z\overrightarrow{\mathbf{k}}$ be the position vector of the point $M(x,y,z)$. The vector field $F(x,y,z) = \dfrac{c}{\|\mathbf{r}\|^3}\mathbf{r}(x,y,z)$ is called the inverse square field, where $c \in \mathbb{R}$.
The inverse field is conservative.

Test of Conservative

If $F = (P, Q) = \nabla f$. Then $P = \dfrac{\partial f}{\partial x}$ and $Q = \dfrac{\partial f}{\partial y}$, and provided that f is smooth, from Schwarz's Theorem, $\dfrac{\partial P}{\partial y} = \dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial Q}{\partial x}$. Hence, if $\dfrac{\partial P}{\partial y} \neq \dfrac{\partial Q}{\partial x}$, F is not conservative.

For a vector field $F = (P, Q, R)$, suppose that $(P, Q, R) = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$.
If z is constant, then $f(x, y, z)$ is a function of x and y, and by Schwarz's Theorem, $\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial Q}{\partial y}$. Likewise, if y is constant, then
$\frac{\partial P}{\partial z} = \frac{\partial^2 f}{\partial x \partial z} = \frac{\partial^2 f}{\partial z \partial x} = \frac{\partial R}{\partial x}$, and if x is constant, we get
$\frac{\partial Q}{\partial z} = \frac{\partial^2 f}{\partial y \partial z} = \frac{\partial^2 f}{\partial z \partial y} = \frac{\partial R}{\partial y}$.
Conversely, if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, $\frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}$, and $\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$ then F is conservative.

Example 2.4

1. The vector field $(1 + 3xy, 2x^2 - 3y^2)$ is not conservative because, $\dfrac{\partial(1 + 3xy)}{\partial y} = 3x$ and $\dfrac{\partial(2x^2 - 3y^2)}{\partial x} = 4x$.
2. The vector field $F = (y^2 z + y \cos x, 2xyz + \sin x - \sin y, xy^2)$ is conservative because, $F = \nabla(xy^2 z + y \sin x + \cos y)$.

Table of contents

Definition 3.1

The divergence of a vector field $F = (P, Q, R)$ *is*

$$\langle \nabla, F \rangle = \left\langle \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right), (P, Q, R) \right\rangle = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

The curl of a vector field

The curl of $F = (P, Q, R)$ is

$$\nabla \times F = \begin{vmatrix} \overrightarrow{\mathbf{i}} & \overrightarrow{\mathbf{j}} & \overrightarrow{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right).$$

If $F = P\,\overrightarrow{\mathbf{i}} + Q\,\overrightarrow{\mathbf{j}}$ is a two dimensional vector field, the curl $\nabla \times F$ can also be defined by regarding the $k-$component to be zero, i.e. $F = P\,\overrightarrow{\mathbf{i}} + Q\,\overrightarrow{\mathbf{j}} + 0\,\overrightarrow{\mathbf{k}}$, then $\mathrm{curl} F = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \overrightarrow{\mathbf{k}}$.

Theorem 3.1 (The Curl Test)

Given a vector field $F = (P, Q, R)$ is defined and continuously differentiable everywhere in $\mathbb{R}^3$ (or everywhere in $\mathbb{R}^2$ for vector fields in $\mathbb{R}^2$), then F is conservative if and $\mathrm{curl} F = 0$.

Here are two simple but useful facts about divergence and curl.

Theorem 3.2

$\langle \nabla, (\nabla \times F) \rangle = 0$. *In other words, the divergence of the curl is zero.*

Theorem 3.3

$\nabla \times (\nabla f) = 0$. *That is, the curl of a gradient is the zero vector.*

Exercise 3.1

A vector field F is said to be incompressible if $\langle \nabla, F \rangle = 0$.
Prove that any vector field of the form
$F(x, y, z) = (f(y, z), g(x, z), h(x, y))$ is incompressible.

Exercise 3.2

Find an f so that $\nabla f = (2x + y^2, 2y + x^2)$, or explain why there is no such f.

Exercise 3.3

Find an f so that $\nabla f = (x^3, -y^4)$, or explain why there is no such f.

Exercise 3.4

Find an f so that $\nabla f = (xe^y, ye^x)$, or explain why there is no such f.

Exercise 3.5

Find an f so that $\nabla f = (y\cos x, y\sin x)$, or explain why there is no such f.

Exercise 3.6

Find an f so that $\nabla f = (y\cos x, \sin x)$, or explain why there is no such f.

Exercise 3.7

Find an f so that $\nabla f = (x^2y^3, xy^4)$, or explain why there is no such f.

Exercise 3.8

Find an f so that $\nabla f = (yz, xz, xy)$, or explain why there is no such f.

Table of contents

Consider a plane curve given by the parametric equations

$$\gamma(t) = (x(t), y(t)), \quad t \in [a, b].$$

Definition 4.1

Let f be a continuous function on $\mathbb{R}^2$. If γ is continuously differentiable, the line integral of f on γ with respect to the arc length is defined by:

$$\int_a^b f \circ \gamma(t)\sqrt{(x'(t))^2 + (y'(t))^2}dt = \int_a^b f(x(t), y(t))\sqrt{(x'(t))^2 + (y'(t))^2}dt.$$

Remark 4.1

1. If $f = 1$, $\int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}dt$ is the length of γ.
   Note that $\sqrt{(x'(t))^2 + (y'(t))^2} = \|\gamma'(t)\|$. We denote
   $ds = \sqrt{(x'(t))^2 + (y'(t))^2}dt$.
2. The value of the line integral does not depend on the parametrization of the curve, provided that the curve is traversed exactly once as t increases from a to b.

Example 4.1

(Integrating along an arc of circle)

Consider the arc of circle C parametrized by $(\cos t, \sin t)$, with $t \in [0, \frac{\pi}{2}]$.

In this case $ds = \sqrt{\cos^2 t + \sin^2 t} dt = dt$

$$\begin{aligned}
\int_C (x + 4xy^2) ds &= \int_0^{\frac{\pi}{2}} (\cos t + 4 \cos t \sin^2 t) dt \\
&= \int_0^{\frac{\pi}{2}} \cos t (1 + 4 \sin^2 t) dt \\
&\overset{u=\cos t}{=} \int_0^1 (1 + 4u^2) du = \frac{7}{3}.
\end{aligned}$$

Definition 4.2

Let f *be a continuous function on* $\mathbb{R}^2$ *and let* γ *be piecewise-smooth curve, that is,* γ *is a union of a finite number of smooth curves* $\gamma_1, \ldots, \gamma_k$*, such that the initial point of* γ_{j+1} *is the terminal point of* γ_j*. Then we define the integral of a continuous function* f *along* γ *with respect to the arc length by:*

$$\int_{\gamma} f(x,y)ds = \sum_{j=1}^{k} \int_{\gamma_j} f(x,y)ds.$$

Definition 4.3 (Center of mass of a wire)

If $\rho(x, y)$ is the linear density at a point (x, y) of a thin wire shaped like a curve $\gamma\colon [a, b] \longrightarrow \mathbb{R}^2$. The mass of the thin is

$$m = \int_a^b \rho(\gamma(t))\|\gamma'(t)\|dt$$

and the center of mass of the thin

$$(x_0, y_0) = \left(\int_a^b x(t)\rho(\gamma(t))\|\gamma'(t)\|dt, \int_a^b y(t)\rho(\gamma(t))\|\gamma'(t)\|dt\right).$$

Example 4.2

A wire takes the shape of an arc of circle $(\cos t, \sin t)$, with $t \in [0, \pi]$. If the density of the thin is $\rho(x, y) = x^2 + y^2$. Then the mass of the thin is

$$m = \int_0^\pi dt = \pi$$

and the center of mass of the this $\left(\int_0^\pi \cos t dt, \int_0^\pi \sin t dt \right) = (0, 2)$.

Table of contents

Consider a space curve given by the parametric equations

$$\gamma(t) = (x(t), y(t), z(t)), \quad t \in [a, b].$$

Definition

Let f be a continuous function on $\mathbb{R}^3$. If γ is continuously differentiable, the line integral of f on γ with respect to the arc length is defined by:

$$\int_a^b f \circ \gamma(t) \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} dt = \int_a^b f(x(t), y(t), z(t)) \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} dt.$$

Remark 5.1

1. If $f = 1$, $\int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}dt$ is the length of γ.
   Note that $\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} = \|\gamma'(t)\|$ and we denote $ds = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}dt$.
2. The value of the line integral does not depend on the parametrization of the curve, provided that the curve is traversed exactly once as t increases from a to b.

Example 5.1

Consider the curve γ parametrized by $\gamma(t) = (\cos t, \sin t, 1)$, with $t \in [0, \frac{\pi}{2}]$. In this case $ds = \sqrt{\cos^2 t + \sin^2 t}dt = dt$

$$\begin{aligned}
\int_C (2xz + 5xy^2 + z)ds &= \int_0^{\frac{\pi}{2}} (2\cos t + 5\cos t \sin^2 t + 1)dt \\
&= \frac{\pi}{2} + \int_0^{\frac{\pi}{2}} \cos t(2 + 5\sin^2 t)dt \\
&\overset{u=\sin t}{=} \frac{\pi}{2} + \int_0^1 (2 + 5u^2)du = \frac{\pi}{2} + \frac{11}{3}.
\end{aligned}$$

Definition 5.1

Let f be a continuous function on $\mathbb{R}^3$ and let γ be piecewise-smooth curve, that is, γ is a union of a finite number of smooth curves $\gamma_1, \ldots, \gamma_k$, such that the initial point of γ_{j+1} is the terminal point of γ_j. Then we define the integral of a continuous function f along γ with respect to the arc length as

$$\int_\gamma f(x,y,z)ds = \sum_{j=1}^{k} \int_{\gamma_j} f(x,y,z)ds.$$

Definition 5.2

Let f be a continuous function on $D \subset \mathbb{R}^3$ and let C be piecewise-smooth curve on D parametrized by $(x(t), y(t), z(t)),\ t \in [a, b]$:

1. *The line integral of $f(x, y, z)$ with respect to x along the oriented curve C is written $\int_C f(x, y, z)dx$ and defined by:*

$$\int_C f(x, y, z)dx = \int_a^b f(x(t), y(t), z(t))x'(t)dt$$

2. *The line integral of $f(x, y, z)$ with respect to y along the oriented curve C is written $\int_C f(x, y, z)dy$ and defined by:*

$$\int_C f(x, y, z)dy = \int_a^b f(x(t), y(t), z(t))y'(t)dt$$

3. The line integral of $f(x,y,z)$ with respect to z along the oriented curve C is written $\int_C f(x,y,z)dz$ and defined by:

$$\int_C f(x,y,z)dz = \int_a^b f(x(t),y(t),z(t))z'(t)dt$$

Work of a Force Field

If $F = (f, g, h)$ is a force field defined on a domain $D \subset \mathbb{R}^3$ and let C be piecewise-smooth curve on D parametrized by $(x(t), y(t), z(t)),\ t \in [a, b]$: The work of F along the curve C is defined by:

$$\begin{aligned} W &= \int_a^b f(x(t), y(t), z(t))x'(t)dt + \int_a^b g(x(t), y(t), z(t))y'(t)dt \\ &\quad + \int_a^b h(x(t), y(t), z(t))z'(t)dt \\ &= \int_a^b \langle F \circ C(t), C'(t)\rangle dt. \end{aligned}$$

$\int_a^b \langle F \circ C(t), C'(t)\rangle dt$ is denoted also $\int_C F(x, y, z).dr$

Table of contents

Definition 6.1

We say that the line integral $\int_C F.d\mathbf{r}$ *is independent of path in the domain* D *if the integral is the same for every path contained in* D *that has the same beginning and ending points.*

Theorem 6.1

Let $F=(f,g,h)$ be a continuous vector field defined on a connected region D and let C be a smooth parametric curve on D parameterized by $C(t)=(x(t),y(t),z(t))$, $t\in[a,b]$.
The integral

$$\begin{aligned}\int_C F.d\mathbf{r} \quad &= \quad \int_a^b f(x(t),y(t),z(t))x'(t)dt+\int_a^b f(x(t),y(t),z(t))x'(t)dt\\ &\int_a^b f(x(t),y(t),z(t))x'(t)dt\end{aligned}$$

is independent of the path if and only if F is conservative.

Independence of Path

Theorem 6.2 (Fundamental Theorem of Line Integrals)

Consider a smooth parametric curve C parameterized by a smooth vector function $C(t) = (x(t), y(t), z(t))$, $t \in [a, b]$. If f is a continuously differentiable function on a domain containing the curve C, then

$$\int_C \nabla f . d\mathbf{r} = f(C(b)) - f(C(a)).$$

In particular, if the curve is closed, (i.e. $C(b) = C(a)$), then

$$\int_C \nabla f . d\mathbf{r} = 0.$$

Example 6.1

Consider the vector field $F(x,y) = (2xy - 3, x^2 + 4y^3 + 5)$.

The line integral $\int_C F.d\mathbf{r}$ is independent of path. Then, evaluate the line integral for any curve C with initial point at $(-1, 2)$ and terminal point at $(2, 3)$.

$F = \nabla f$, $\frac{\partial f}{\partial x} = 2xy - 3$, $f = x^2y - 3x + g(y)$,

$\frac{\partial f}{\partial y} = x^2 + g'(y) = x^2 + 4y^3 + 5$. Then $f = x^2y - 3x + y^4 + 5y$.

$\int_C F.d\mathbf{r} = f(2,3) - f(-1,2) = 102 - 31 = 71$.

Conservative Vector Fields

Let $F(x, y) = (M(x, y), N(x, y))$, where we assume that $M(x, y)$ and $N(x, y)$ have continuous first partial derivatives on an open, simply-connected region $D \subset \mathbb{R}^2$. The following five statements are equivalent, meaning that for a given vector field, either all five statements are true or all five statements are false.

1. $F(x, y)$ is conservative on D.
2. $F(x, y)$ is a gradient field in D (i.e., $F(x, y) = \nabla f(x, y)$, for some potential function f, for all $(x, y) \in D$).
3. $\displaystyle\int_C F.d\mathbf{r}$ is independent of path in D.
4. $\displaystyle\int_C F.d\mathbf{r} = 0$ for every piecewise-smooth closed curve C lying in D.
5. $\dfrac{\partial M}{\partial y}(x, y) = \dfrac{\partial N}{\partial x}(x, y)$, for all $(x, y) \in D$.

Theorem 6.3

Consider a simple connected region D and let F be a vector field defined on D.

The following properties of a vector field F are equivalent:

1. F *is conservative.*
2. $\int_C F.d\mathbf{r}$ *is path-independent, (i.e. meaning that it only depends on the endpoints of the curve C.*
3. $\oint_C F.d\mathbf{r} = 0$ *around any closed smooth curve C in D.*

Table of contents

Green's Theorem

Theorem 7.1 (Green's Theorem)

Let γ be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by γ. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_{\gamma} P(x,y)dx + Q(x,y)dy = \int\int_{D} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy.$$

Remark 7.1

The notation $\oint_\gamma P(x,y)dx + Q(x,y)dy$ is sometimes used to indicate that the line integral is calculated using the positive orientation of the closed curve. The Green's Theorem can be written as

$$\int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int_{\partial D} P(x,y)dx + Q(x,y)dy$$

where ∂D is the positively oriented boundary curve of D.

Example 7.1

Consider the curve defined by the boudary of the triangle Δ of vertices $(0,0),(1,0),(0,1)$. Use Green's Theorem to calculate a line integral $\int_\gamma x^2ydx + xy^2dy$.

$$\begin{aligned}\int_\gamma x^2ydx + xy^2dy &= \int_\Delta \left(y^2 - x^2\right) dxdy \\ &= \int_0^1 \left(\int_0^{1-x} (y^2 - x^2)dy\right) dx = 0.\end{aligned}$$

Example 7.2

Consider the curve defined by the circle C defined by $x^2 + y^2 = 9$. Use Green's Theorem to calculate a line integral $\int_C (3y - e^{\sin x})dx + (7x + \sqrt{y^4 + 1})dy$.

$$\begin{aligned} \int_C (3y - e^{\sin x})dx + (7x + \sqrt{y^4 + 1})dy &= \int_D (7 - 3)dxdy \\ &= 36\pi. \end{aligned}$$

Remark 7.2

Another application of Green's Theorem is in computing areas. Since the area of D is $\int\int_D dxdy$, we wish to choose P and Q so that $(\frac{\partial Q}{\partial x} - \frac{\partial Q}{\partial y}) = 1$. Hence the area of D id

$$A = \oint_{\partial D} xdy = -\oint_{\partial D} ydx = \frac{1}{2}\oint_{\partial D}(xdy - ydx).$$

For example the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. A paramatrization of the ellipse E is $x(t) = a\cos t$, $y(t) = b\sin t$.

$$A = \frac{1}{2}\oint_E (xdy - ydx) = \frac{1}{2}\int_0^{2\pi} ab\cos^2 t + ab\sin^2 tdt = \pi ab.$$

Use Green's Theorem to evaluate the line integral along the given positively oriented curve.

Exercise 7.1

$\int_C (xy^2dx + 2x^2ydy)$, where C is the triangle with vertices $(0, 0)$, $(2, 2)$, and $(2, 4)$.

Solution 1

$$\int_C (xy^2dx + 2x^2ydy) = \int_0^2 \int_x^{2x} (2xy)dydx = \int_0^2 3x^3dx = 12.$$

Exercise 7.2

$\int_C (\cos y dx + x^2 \sin y dy)$, where C is the rectangle with vertices $(0,0)$, $(5,0)$, and $(5,2)$.

Solution 2

$$\int_C (\cos y dx + x^2 \sin y dy) = \int_0^5 \int_0^2 (2x+1) \sin y dy dx = 30(1 - \cos 2).$$

Exercise 7.3

$\int_C (xe^{-2x}dx + (x^4 + 2x^2y^2)dy)$, where C is the boundary of the region between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Solution 3

$$\begin{aligned}\int_C (xe^{-2x}dx + (x^4 + 2x^2y^2)dy) &= \int_1^2 \int_0^{2\pi} (4r^3\cos^3\theta + 4r^3\cos\theta\sin^2\theta)rd \\ &= 4\int_1^2 r^4 \int_0^{2\pi} \cos\theta drd\theta = 0.\end{aligned}$$

Table of contents

Theorem 8.1 (Evaluation Theorem)

Consider a surface S *in* $\mathbb{R}^3$ *defined by* $z = g(x,y)$ *for* (x,y) *on a region* $R_{x,y} \subset \mathbb{R}^2$, *where* g *has continuous first partial derivatives, then*

$$\iint_S f(x,y,z)dS = \iint_{R_{x,y}} f(x,y,g(x,y))\sqrt{1+g_x^2+g_y^2}dA,$$

where $g_x = \dfrac{\partial g}{\partial x}$ *and* $g_y = \dfrac{\partial g}{\partial y}$.

Example 8.1

Evaluate the integral $\iint_S f(x,y,z)dS$, where $f(x,y,z) = x^2 + yz$ and S the upper half sphere $x^2 + y^2 + z^2 = R^2$.

$$\iint_S f(x,y,z)dS$$

$$\begin{aligned}
&= \iint_{D(0,R)} \left(x^2 + y\sqrt{R^2 - x^2 - y^2}\right) \sqrt{1 + \frac{x^2}{R^2 - x^2 - y^2} + \frac{y^2}{R^2 - x^2 - y^2}}\, dA \\
&= \int_0^{2\pi} \int_0^R \left(r^2 \cos^2\theta + r\sin\theta\sqrt{R^2 - r^2}\right) \frac{Rr}{\sqrt{R^2 - r^2}} dr d\theta \\
&= R \int_0^{2\pi} \int_0^R \frac{r^3}{\sqrt{R^2 - r^2}} \cos^2\theta dr d\theta = \frac{2\pi}{3} R^4.
\end{aligned}$$

Table of contents

Definition 9.1

A surface S is called orientable if a unit normal vector $\bar{n}$ can be defined at every non boundary point of S and $\bar{n}$ is continuous over the surface.
For a surface defined by $f(x, y, z) = c$,

$$\bar{n} = \pm \frac{\nabla f}{\|\nabla f\|}.$$

In particular if the surface is defined by $z = g(x, y)$, $\nabla f = (-g_x, -g_y, 1)$, $dS = \sqrt{1 + g_x^2 + g_y^2}$, $\bar{n} dS = \nabla f dA$.

Flux of a Vector Field

Consider $\bar{F}$ a vector field which can represents the velocity of some fluid in the space. The flux of the fluid across S measures how much fluid is passing through the surface S.
Consider the unit normal vector $\bar{n}$ to the surface at a point, the number $\bar{F}.\bar{n}$ represents the scalar projection of F onto the direction of $\bar{n}$. So it measures how fast the fluid is moving across the surface. Thus, the total flux across S is $\displaystyle\int_S \bar{F}.\bar{n}dS$.

Theorem 9.1

Let $\bar{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ be a continuous vector field defined on an oriented surface S defined by $z = g(x, y)$ on a region $R_{x,y}$. The surface integral of F over S (or the flux of F over S) is:

$$\int_S F.ndS = \iint_{R_{x,y}} (-Mg_x - Ng_y + P)dA$$

if the surface is oriented upward and

$$\int_S \bar{F}.\bar{n}dS = \iint_{R_{x,y}} (Mg_x + Ng_y - P)dA$$

if the surface is oriented downward.

Example 9.1

Compute the flux of the vector field $\bar{F}(x,y,z)=(x,y,0)$ over the portion of the paraboloid $z=x^2+y^2$ below $z=4$ (oriented with upward-pointing normal vectors).

Solution First, observe that at any given point, the normal vectors for the paraboloid $z=x^2+y^2$ are $\pm(2x,2y,-1)$. For the normal vector to point upward, we need a positive $z-$component. In this case,

$$u=-(2x,2y,-1)=(-2x,-2y,1)$$

is such a normal vector. A unit vector pointing in the same direction as u is then

$$\bar{n}=\frac{1}{\sqrt{4x^2+4y^2+1}}(-2x,-2y,1).$$

We have $dS=\|u\|dA=\sqrt{4x^2+4y^2+1}dA$. Then

$$\begin{aligned}\iint_S \bar{F}.\bar{n}dS &= \iint_R (x,y,0).\frac{(-2x,-2y,1)}{\sqrt{4x^2+4y^2+1}}\sqrt{4x^2+4y^2+1}dA \\ &= \iint_R (x,y,0).(-2x,-2y,1)dA = \iint_R (-2x^2-2y^2)dA.\end{aligned}$$

The region $R_{x,y}$ is the disc $D(0,2)$, then

$$\iint_S \bar{F}.\bar{n}dS = \int_0^{2\pi}\int_0^2 -2r^3 dr d\theta = -16\pi.$$

Exercise 9.1

Evaluate $\int_D (2, -3, 4).\mathbf{n}dS$, where D is given by $z = x^2 + y^2$, $-1 \leq x \leq 1$, $-1 \leq y \leq 1$, oriented up.

Exercise 9.2

Evaluate $\int_D (x, y, 3).\mathbf{n}dS$, where D is given by $z = 3x - 5y$, $1 \le x \le 2, 0 \le y \le 2$, oriented up.

Exercise 9.3

Evaluate $\int_D (x, y, -2).\mathbf{n}dS$, where D is given by $z = 1 - x^2 - y^2$, $x^2 + y^2 \le 1$, oriented up.

Exercise 9.4

Evaluate $\int_D (xy, yz, zx).\mathbf{n}dS$, where D is given by $z = x + y^2 + 2$, $0 \le x \le 1, x \le y \le 1$, oriented up.

Exercise 9.5

Evaluate $\int_D (e^x, e^y, z).\mathbf{n}dS$, where D is given by $z = xy, 0 \leq x \leq 1, -x \leq y \leq x$, oriented up.

Exercise 9.6

Evaluate $\int_D (xz, yz, z).\mathbf{n}dS$, where D is given by $z = a^2 - x^2 - y^2$, $x^2 + y^2 \leq b^2$, oriented up.

Example 9.2

Compute the flux of $F = (x, y, z^4)$ across the cone $z = \sqrt{x^2 + y^2}$, $0 \leq z \leq 1$, in the downward direction.
We write the cone as a vector function: $\gamma = (v\cos u, v\sin u, v)$, $0 \leq u \leq 2\pi$ and $0 \leq v \leq 1$. Then $\gamma_u = (-v\sin u, v\cos u, 0)$, $\gamma_v = (\cos u, \sin u, 1)$, and $\gamma_u \times \gamma_v = (v\cos u, v\sin u, -v)$. The third coordinate $-v$ is negative, which is exactly what we desire, that is, the normal vector points down through the surface.

Then

$$
\begin{aligned}
& \int_0^{2\pi}\int_0^1 \langle (x,y,z^4),(v\cos u, v\sin u, -v)\rangle \, dv\, du \\
= & \int_0^{2\pi}\int_0^1 xv\cos u + yv\sin u - z^4 v \, dv\, du \\
= & \int_0^{2\pi}\int_0^1 v^2\cos^2 u + v^2 \sin^2 u - v^5 \, dv\, du \\
= & \int_0^{2\pi}\int_0^1 v^2 - v^5 \, dv\, du = \frac{\pi}{3}.
\end{aligned}
$$

Table of contents

Theorem 10.1 (The Divergence Theorem)

Let Q be a solid region bounded by a closed surface S oriented by a normal vector directed outward and if $\bar{F}$ is vector field C^1. Then

$$\iint_S \bar{\mathbf{F}}.\bar{\mathbf{n}}dS = \iiint_Q \nabla.\bar{F}dV = \iiint_Q div\bar{\mathbf{F}}dV.$$

Example 10.1

Use the Divergence Theorem to evaluate the surface integral $\iint\limits_S \mathbf{F} \cdot d\mathbf{S}$ of the vector field $\mathbf{F}(x, y, z) = \left(x^3, y^3, z^3\right)$, where S is the surface of a solid bounded by the cone $x^2 + y^2 - z^2 = 0$ and the plane $z = 1$.

Solution

Applying the Divergence Theorem, we can write:

$$\begin{aligned} I &= \iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_G (\nabla \cdot \mathbf{F})\, dV \\ &= \iiint_G \left[\frac{\partial}{\partial x}\left(x^3\right) + \frac{\partial}{\partial y}\left(y^3\right) + \frac{\partial}{\partial z}\left(z^3\right) \right] dxdydz \\ &= 3 \iiint_G \left(x^2 + y^2 + z^2\right) dxdydz. \end{aligned}$$

By changing to cylindrical coordinates, we have

$$
\begin{aligned}
I &= 3\iiint\limits_{G} \left(x^2+y^2+z^2\right) dxdydz \\
&= 3\int_0^{2\pi} d\varphi \int_0^1 \int_0^z \left(r^2+z^2\right) r dr dz = 6\pi \int_0^1 \left[\left(\frac{r^4}{4} + \frac{z^2 r^2}{2} \right)\Big|_{r=0}^{z} \right] dz \\
&= 6\pi \int_0^1 \frac{3z^4}{4} dz = \frac{9\pi}{2} \left[\left(\frac{z^5}{5} \right)\Big|_0^1 \right] = \frac{9\pi}{10}.
\end{aligned}
$$

Example 10.2

Evaluate the surface integral $\iint_S x^3 dydz + y^3 dxdz + z^3 dxdy$, where S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ that has upward orientation.

Solution

Using the Divergence Theorem, we can write:

$$\begin{aligned} I &= \iint\limits_S x^3 dydz + y^3 dxdz + z^3 dxdy = \iiint\limits_G \left(3x^2 + 3y^2 + 3z^2\right) dxdydz \\ &= 3 \iiint\limits_G \left(x^2 + y^2 + z^2\right) dxdydz. \end{aligned}$$

By changing to spherical coordinates, we have

$$\begin{aligned} I &= 3\iiint\limits_G \left(x^2+y^2+z^2\right)dxdydz = 3\iiint\limits_G r^2\cdot r^2\sin\theta drd\psi d\theta \\ &= 3\int_0^{2\pi} d\psi \int_0^{\pi}\sin\theta d\theta\int_0^a r^4 dr \\ &= 3\cdot 2\pi\cdot\left[(-\cos\theta)|_0^{\pi}\right]\cdot\left[\left(\frac{r^5}{5}\right)\Big|_0^a\right] = \frac{12\pi a^5}{5}. \end{aligned}$$

Example 10.3

Using the Divergence Theorem calculate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ of the vector field $\mathbf{F}(x, y, z) = (2xy, 8xz, 4yz)$, where is the surface of tetrahedron with vertices $A = (0, 0, 0)$, $B = (1, 0, 0)$, $C = (0, 1, 0)$, $D = (0, 0, 1)$.

Solution

By Divergence Theorem,

$$\begin{aligned} I &= \iint\limits_S \mathbf{F} \cdot d\mathbf{S} = \iiint\limits_G (\nabla \cdot \mathbf{F})\, dV \\ &= \iiint\limits_G \left[\frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}(8xz) + \frac{\partial}{\partial z}(4yz)\right] dV \\ &= \iiint\limits_G (2y + 0 + 4y)\, dxdydz = 6 \iiint\limits_G y dxdydz. \end{aligned}$$

$$
\begin{aligned}
I &= 6\iiint\limits_G y\,dx\,dy\,dz = 6\int_0^1 dx\int_0^{1-x} dy\int_0^{1-x-y} y\,dz \\
&= 6\int_0^1 dx\int_0^{1-x}(1-x-y)\,y\,dy = 6\int_0^1 dx\int_0^{1-x}\left[y\,(1-x)-y^2\right]dy \\
&= 6\int_0^1\left[\left((1-x)\frac{y^2}{2}-\frac{y^3}{3}\right)\Big|_{y=0}^{1-x}\right]dx \\
&= 6\int_0^1\left[\frac{(1-x)^3}{2}-\frac{(1-x)^3}{3}\right]dx \\
&= 6\cdot\frac{1}{6}\int_0^1(1-x)^3dx = \frac{1}{4}.
\end{aligned}
$$

Example 10.4

Use the Divergence Theorem to evaluate the surface integral $\iint\limits_S \mathbf{F} \cdot d\mathbf{S}$ of the vector field $\mathbf{F}(x, y, z) = (x, y, z)$, where S is the surface of the solid bounded by the cylinder $x^2 + y^2 = a^2$ and the planes $z = -1$ and $z = 1$.

Solution

Using the Divergence Theorem, we can have:

$$\begin{aligned}\iint\limits_S \mathbf{F}\cdot d\mathbf{S} &= \iiint\limits_G (\nabla\cdot\mathbf{F})\, dV \\ &= \iiint\limits_G \left[\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)\right] dxdydz \\ &= \iiint\limits_G (1+1+1)\, dxdydz = 3\iiint\limits_G dxdydz.\end{aligned}$$

By switching to cylindrical coordinates, we have

$$\begin{aligned} I &= 3\iiint_G dxdydz = 3\int_{-1}^{1} dz \int_0^{2\pi} d\varphi \int_0^a r dr \\ &= 3 \cdot 2 \cdot 2\pi \cdot \left[\left(\frac{r^2}{2} \right) \Big|_0^a \right] = 6\pi a^2. \end{aligned}$$

Table of contents

Theorem 11.1 (Stokes's Theorem)

Let S be an oriented, piecewise-smooth surface with unit normal vector $\bar{\mathbf{n}}$, bounded by the simple closed, piecewise-smooth boundary curve C having positive orientation. Let $\mathbf{F}(x, y, z)$ be a vector field continuously differentiable in some open domain containing S. Then,

$$\oint_C \bar{\mathbf{F}}.d\bar{\mathbf{r}} = \oint_C \bar{\mathbf{F}}.\bar{T}ds = \iint_S \mathrm{curl} F.\bar{\mathbf{n}}dS.$$

$\bar{\mathbf{r}} = (x, y, z)$ is the position vector, $d\bar{\mathbf{r}} = (dx, dy, dz)$, the unit tangent vector to S at $\bar{\mathbf{r}} = (x, y, z)$ is

$$\bar{T} = \frac{dx}{ds}\overrightarrow{\mathbf{i}} + \frac{dy}{ds}\overrightarrow{\mathbf{j}} + \frac{dz}{ds}\overrightarrow{\mathbf{k}}.$$

Hence $d\bar{\mathbf{r}} = d\bar{T}ds$.

If the surface S is defined by $z = g(x, y)$ on a region $R_{x,y}$, then $\iint_S \mathrm{curl}F.\bar{\mathbf{n}}dS = \iint_{R_{x,y}} (-M_1 g_x - N_1 g_y + P_1)dA$, where $g_x = \dfrac{\partial g}{\partial x}$, $g_y = \dfrac{\partial g}{\partial y}$ and $\mathrm{curl}F = (M_1, N_1, P_1)$.

Example 11.1

Use Stoke's Theorem to evaluate the line integral $\oint_C (y+2z)\,dx + (x+2z)\,dy + (x+2y)\,dz$, where C is the curve formed by intersection of the sphere $x^2+y^2+z^2=1$ with the plane $x+2y+2z=0$.

Solution

Let S be the circle cut by the sphere from the plane. Find the coordinates of the unit normal vector $\bar{\mathbf{n}}$ to the surface S,

$$\bar{\mathbf{n}} = \frac{1\cdot\overrightarrow{\mathbf{i}} + 2\cdot\overrightarrow{\mathbf{j}} + 2\cdot\overrightarrow{\mathbf{k}}}{\sqrt{1^2+2^2+2^2}} = \frac{1}{3}\overrightarrow{\mathbf{i}} + \frac{2}{3}\overrightarrow{\mathbf{j}} + \frac{2}{3}\overrightarrow{\mathbf{k}}.$$

In this case $P = y+2z,\ \ Q = x+2z,\ \ R = x+2y$. Hence, the curl of the vector $\bar{\mathbf{F}}$ is

$$\begin{aligned}
\nabla \times \bar{\mathbf{F}} &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\overrightarrow{\mathbf{i}} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\overrightarrow{\mathbf{j}} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\overrightarrow{\mathbf{k}} \\
&= (2-2)\,\overrightarrow{\mathbf{i}} + (2-1)\,\overrightarrow{\mathbf{j}} + (1-1)\,\overrightarrow{\mathbf{k}} = \overrightarrow{\mathbf{j}}.
\end{aligned}$$

Using Stoke's Theorem, we have

$$\begin{aligned}\oint_C (y+2z)\,dx + (x+2z)\,dy + (x+2y)\,dz &= \iint_S \left(\nabla \times \bar{\mathbf{F}}\right)\cdot \bar{\mathbf{n}} dS \\ &= \iint_S \overrightarrow{\mathbf{j}} \cdot \left(\frac{1}{3}\overrightarrow{\mathbf{i}} + \frac{2}{3}\overrightarrow{\mathbf{j}} + \frac{2}{3}\overrightarrow{\mathbf{k}}\right. \\ &= \frac{2}{3}\iint_S dS.\end{aligned}$$

As the sphere $x^2+y^2+z^2=1$ is centered at the origin and the plane $x+2y+2z=0$ also passes through the origin, the cross section is the circle of radius 1. Hence the integral is

$$I = \frac{2}{3}\iint_S dS = \frac{2}{3}\cdot\pi\cdot 1^2 = \frac{2\pi}{3}.$$

Example 11.2

Use Stoke's Theorem to calculate the line integral

$$\oint_C y^3dx - x^3dy + z^3dz.$$

The curve C is the intersection of the cylinder $x^2 + y^2 = a^2$ and the plane $x + y + z = b$.

Solution

We suppose that S is the part of the plane cut by the cylinder. The curve C is oriented counterclockwise when viewed from the end of the normal vector $\bar{\mathbf{n}}$ which has coordinates

$$\bar{\mathbf{n}} = \frac{1 \cdot \overrightarrow{\mathbf{i}} + 1 \cdot \overrightarrow{\mathbf{j}} + 1 \cdot \overrightarrow{\mathbf{k}}}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{1}{\sqrt{3}}\overrightarrow{\mathbf{i}} + \frac{1}{\sqrt{3}}\overrightarrow{\mathbf{j}} + \frac{1}{\sqrt{3}}\overrightarrow{\mathbf{k}}.$$

As $P = y^3$, $Q = -x^3$, $R = z^3$, we can write:

$$\begin{aligned} \nabla \times \bar{\mathbf{F}} &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\overrightarrow{\mathbf{i}} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\overrightarrow{\mathbf{j}} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\overrightarrow{\mathbf{k}} \\ &= -3\left(x^2 + y^2\right)\overrightarrow{\mathbf{k}}. \end{aligned}$$

Applying Stoke's Theorem, we find:

$$\begin{aligned} I &= \oint_C y^3dx - x^3dy + z^3dz \\ &= \iint_S \left(\nabla \times \bar{\mathbf{F}}\right) \cdot \bar{\mathbf{n}} dS = \iint_S \left(\nabla \times \bar{\mathbf{F}}\right) \cdot \bar{\mathbf{n}} dS \\ &= = \iint_S \left(-3\left(x^2+y^2\right)\overrightarrow{\mathbf{k}}\right) \cdot \left(\frac{1}{\sqrt{3}}\overrightarrow{\mathbf{i}} + \frac{1}{\sqrt{3}}\overrightarrow{\mathbf{j}} + \frac{1}{\sqrt{3}}\overrightarrow{\mathbf{k}}\right) dS \\ &= -\sqrt{3}\iint_S \left(x^2+y^2\right) dS. \end{aligned}$$

We can express the surface integral in terms of the double integral:

$$\begin{aligned} I &= -\sqrt{3}\iint_S \left(x^2+y^2\right) dS \\ &= -\sqrt{3}\iint_{D(0,a)} \left(x^2+y^2\right)\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2} dxdy. \end{aligned}$$

The equation of the plane is $z = b - x - y$, so the square root in the integrand is equal to

$$\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1 + (-1)^2 + (-1)^2} = \sqrt{3}.$$

Hence,

$$I = -\sqrt{3} \iint_{D(0,a)} \left(x^2 + y^2\right) \sqrt{3} dxdy = -3 \iint_{D(x,y)} \left(x^2 + y^2\right) dxdy.$$

By changing to polar coordinates, we get

$$I = -3 \int_0^{2\pi} \int_0^a r^3 dr d\theta = -3 \cdot 2\pi \cdot \left.\frac{r^4}{4}\right|_0^a = -\frac{3\pi a^4}{2}.$$

Example 11.3

Use Stoke's Theorem to evaluate the line integral

$$\oint_C (x+z)\,dx + (x-y)\,dy + xdz.$$

The curve C is the ellipse defined by the equation $\frac{x^2}{4} + \frac{y^2}{9} = 1$, $z = 1$.

Solution

Let the surface S be the part of the plane $z = 1$ bounded by the ellipse. Obviously that the unit normal vector is $\mathbf{n} = \mathbf{k}$. Since $P = x + z$, $Q = x - y$, $R = x$, then the curl of the vector field $\bar{\mathbf{F}}$ is

$$
\begin{aligned}
\nabla \times \bar{\mathbf{F}} &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\overrightarrow{\mathbf{i}} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\overrightarrow{\mathbf{j}} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\overrightarrow{\mathbf{k}} \\
&= (1-0)\,\overrightarrow{\mathbf{k}} = \overrightarrow{\mathbf{k}}.
\end{aligned}
$$

By Stoke's Theorem,

$$
\begin{aligned}
\oint_C (x+z)\,dx + (x-y)\,dy + xdz &= \iint_S \left(\nabla \times \bar{\mathbf{F}}\right) \cdot \bar{\mathbf{n}} dS \\
&= \iint_S \left(\nabla \times \bar{\mathbf{F}}\right) \cdot \bar{\mathbf{n}} dS \\
&= \iint_S \overrightarrow{\mathbf{k}} \cdot \overrightarrow{\mathbf{k}}\, dS = \iint_S dS.
\end{aligned}
$$

The double integral in the latter formula is the area of the ellipse. Therefore, the integral is

$$
\iint_S dS = \pi \cdot 2 \cdot 3 = 6\pi.
$$

Example 11.4

Show that the line integral $\oint_C yzdx + xzdy + xydz$ is zero along any closed contour C.

Solution

Let S be a surface bounded by a closed curve C. Applying Stoke's formula, we identify that $P = yz, \;\; Q = xz, \;\; R = xy$.

Then

$$\nabla \times \bar{\mathbf{F}} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\overrightarrow{\mathbf{i}} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\overrightarrow{\mathbf{j}} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\overrightarrow{\mathbf{k}}$$
$$= (x - x)\overrightarrow{\mathbf{i}} + (y - y)\overrightarrow{\mathbf{j}} + (z - z)\overrightarrow{\mathbf{k}} = 0 \cdot \overrightarrow{\mathbf{i}} + 0 \cdot \overrightarrow{\mathbf{j}} + 0 \cdot \overrightarrow{\mathbf{k}} =$$

Hence, the line integral:

$$\oint_C yzdx + xzdy + xydz = \iint_S \left(\nabla \times \bar{\mathbf{F}}\right) \cdot \bar{\mathbf{n}} dS = \iint_S \mathbf{0} \cdot \bar{\mathbf{n}} dS = 0.$$