Chapter 4
Probability inequalities

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1. Markov's inequality

Often, for a random variable X that we are interested in, we want to know

"What is the probability that the value of the r.v., X, is 'far' from its expectation?"

A generic answer to this, which holds for any non-negative random variable, is given by Markov's inequality:

Theorem

For a nonnegative random variable, $X : \Omega \to \mathbb{R}$, where $X(s) \geq 0$ for all $s \in \Omega$, for any positive real number $a > 0$:

$$\mathbb{P}\left(X \geq a\right) \leq \frac{\mathbb{E}(X)}{a}$$

Let's look at a picture that illustrates the event that we are looking at.

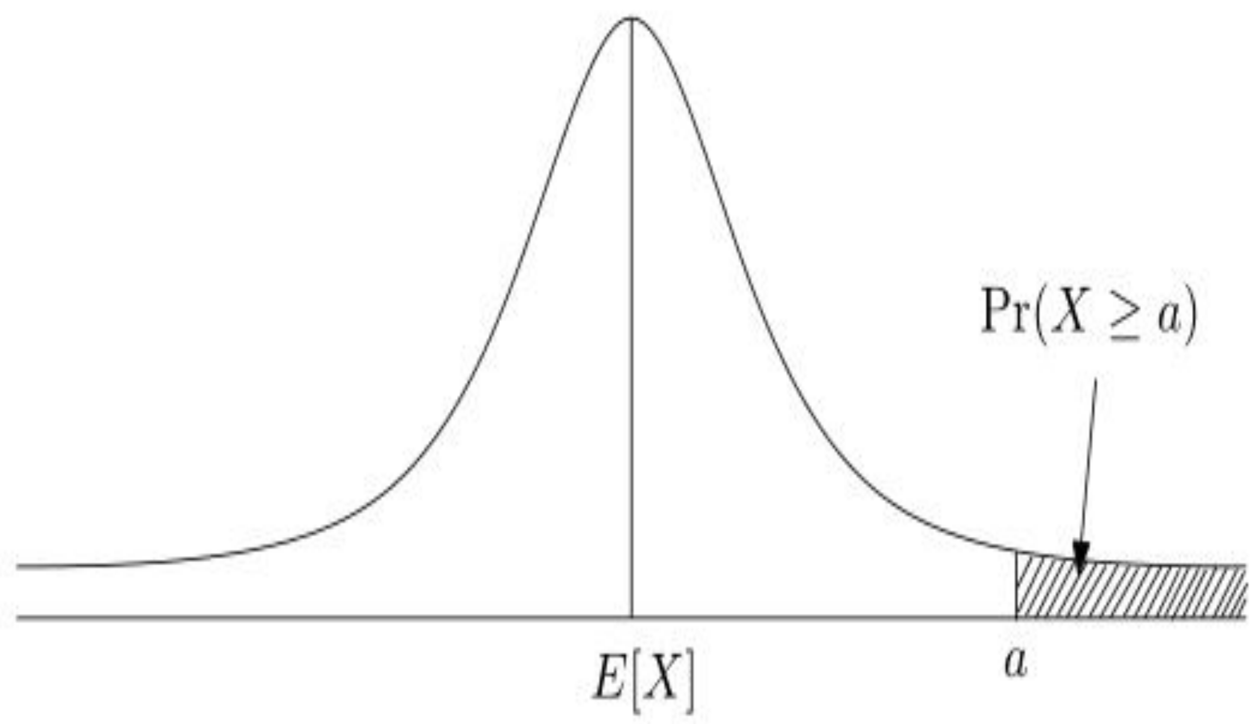

Figure : Markov's Inequality bounds the probability of the shaded region.

Example 1:

A biased coin, which lands heads with probability $\frac{1}{10}$ each time it is flipped, is flipped 200 times consecutively. Give an upper bound on the probability that it lands heads at least 120 times.

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Answer: The number of heads is a binomially distributed r.v., X, with parameters $p = \frac{1}{10}$ and $n = 200$.
Thus, the expected number of heads is

$$\mathbb{E}(X) = np = \frac{1}{10}200 = 20$$

By Markov Inequality, the probability of at least 120 heads is

$$\mathbb{P}(X \geq 120) \leq \frac{\mathbb{E}(X)}{120} = \frac{20}{120} = \frac{1}{6}$$

Corollary

Let X be a discrete random variable and $h : \mathbb{R} \to \mathbb{R}$ be a nonnegative function. Then for any positive real number $a > 0$:

$$\mathbb{P}\left(h(X) \geq a\right) \leq \frac{\mathbb{E}(h(X))}{a}$$

In particular, we have

$$\mathbb{P}\left(|X| \geq a\right) \leq \frac{\mathbb{E}(|X|)}{a}$$

2. Chebyshev's inequality

Theorem

Let $X : \Omega \to \mathbb{R}$ *be any random variable, and let* $\alpha > 0$ *be any positive real number. Then:*

$$\mathbb{P}\left(|X - \mathbb{E}(X)| \geq \alpha\right) \leq \frac{V(X)}{\alpha^2} \qquad \textit{and} \qquad \mathbb{P}\left(|X - \mathbb{E}(X)| \leq \alpha\right) \geq 1 - \frac{V(X)}{\alpha^2}$$

Remark: For any random variable X and scalars $\alpha, t \in \mathbb{R}$ with $\alpha > 0$, convince yourself that

$$\mathbb{P}\left(|X - t| \geq \alpha\right) = \mathbb{P}\left((X - t)^2 \geq \alpha^2\right)$$

Again, let us look at a picture that illustrates Chebyshev's Inequality

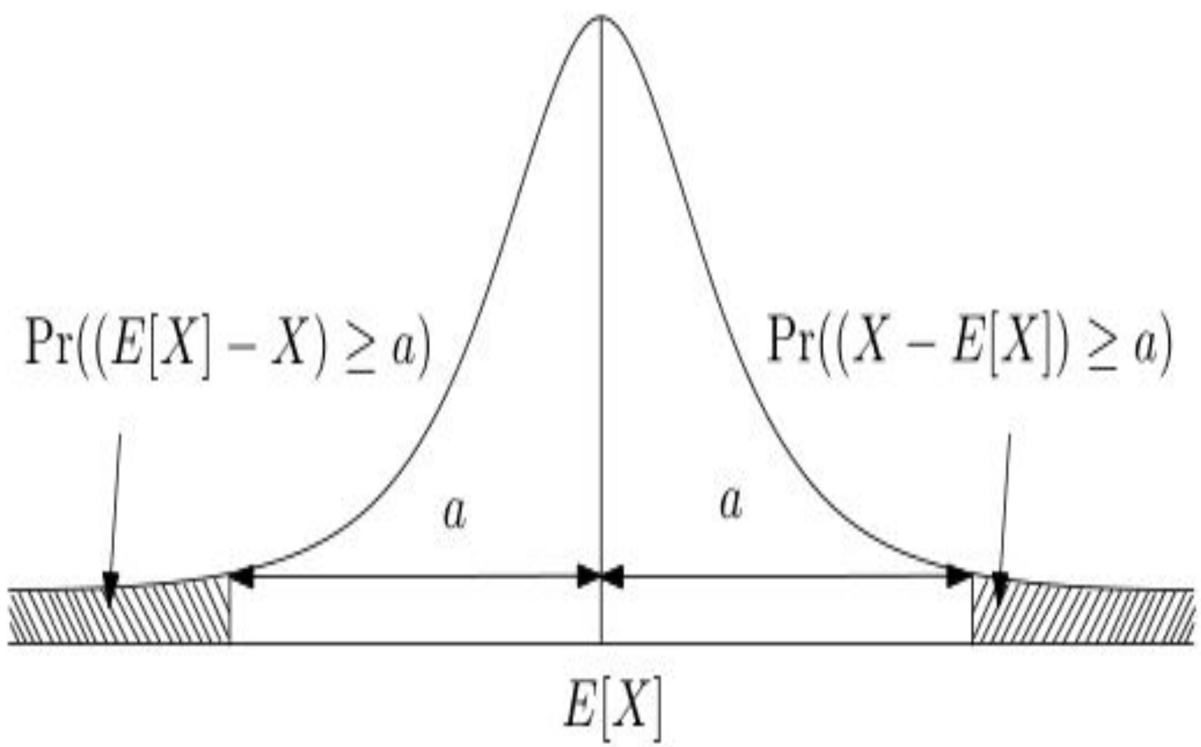

Figure : Chebyshev's Inequality bounds the probability of the shaded regions.

Example 2:

Let X be a random variable such that

$$\mathbb{E}(X) = 0, \qquad \mathbb{P}(-3 < X < 2) = \frac{1}{2}$$

Find a lower bound to its variance.

Answer: The lower bound can be derived thanks to Chebyshev's inequality:

$$\begin{aligned}
V(X) &\geq 2^2 \times \mathbb{P}\left(|X - \mathbb{E}(X)| \geq 2\right) \qquad \text{(by the Chebyshev's inequality)} \\
&\geq 4 \times \mathbb{P}\left(|X| \geq 2\right) \qquad \text{because } \mathbb{E}(X) = 0 \\
&\geq 4 \times (1 - \mathbb{P}(-2 < X < 2)) \qquad \text{(formula for the probability of a complem} \\
&\geq 4 \times (1 - \mathbb{P}(-3 < X < 2)) \qquad (\mathbb{P}(-3 < X < 2) \geq \mathbb{P}(-2 < X < 2)) \\
&\geq 4\left(1 - \frac{1}{2}\right) = 2
\end{aligned}$$

Thus, the lower bound is

$$V(X) \geq 2$$

Example 3:

1. Let $X \backsim Poisson(9)$. Give a lower bound for $\mathbb{P}(|X-\mu| \leq 5)$.
2.) Let $X \backsim N(100, 15)$. Give a lower bound for $\mathbb{P}(|X-\mu| \leq 20)$.

Answer:

1. For $X \backsim Poisson(9)$, $\mu = 9 = \sigma^2$ so $\sigma = 3$ then

$$\mathbb{P}(|X-\mu| \leq 5) = \mathbb{P}(|X-9| \leq 5) \geq 1 - \frac{\sigma^2}{5^2} = 1 - \frac{9}{25}$$

   Note: Using the pdf of $X \backsim Poisson(9)$ we obtain
   $\mathbb{P}(|X-\mu| \leq 5) = \mathbb{P}(4 \leq X \leq 14) \approx 0.9373$.

2. For $X \backsim N(100, 15)$, $\mu = 100$ and $\sigma = 15$ then

$$\mathbb{P}(|X-\mu| \leq 20) = \mathbb{P}(|X-100| \leq 20) \geq 1 - \frac{\sigma^2}{20^2} = 1 - \frac{15^2}{20^2} = 0.4375$$

One-Sided Chebyshev

Using the Markov Inequality, one can also show that for any random variable with mean μ and variance σ^2, and any positive number $\alpha > 0$, the following one-sided Chebyshev inequalities hold:

$$\begin{aligned} \mathbb{P}\left(X \geq \mu + \alpha\right) &\leq \frac{\sigma^2}{\alpha^2 + \sigma^2} \\ \mathbb{P}\left(X \leq \mu - \alpha\right) &\leq \frac{\sigma^2}{\alpha^2 + \sigma^2} \end{aligned}$$

Example 4: Roll a single fair die and let X be the outcome. Then, $\mathbb{E}(X) = 3.5$ and $V(X) = \frac{35}{12}$. Compute $p = \mathbb{P}(X \geq 6)$.

Answer: By using the one-sided Chebyshev inequality, we can obtain an even stronger bound on p:

$$p = \mathbb{P}(X \geq 6) = \mathbb{P}(X \geq 3.5 + 2.5) \leq \frac{\frac{35}{12}}{\frac{35}{12} + (2.5)^2} = \frac{7}{22} \approx 0.318$$